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On bit-size estimates of triangular systems

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Abstract

When solving polynomial equations over an infinite field like \( \mathbb{Q} \), in an exact manner, that is without approximation, coefficients usually become very large. This survey presents some upper-bounds on the size of coefficients of some specific Gr"obner bases. They concern still quite limited families of such systems, but are among the first of this kind. A special emphasis is put on the elementary presentation of a main tool, height theory for measuring the complexity in term of space of a polynomial system.

1 Introduction

System of polynomial equations in several indeterminates arise nowadays in several contexts of concrete applications, that are in need for an efficient solving process. A standard method consists in computing Gr"obner bases. It is well-known that they can often require prohibitive memory size, limiting their computations drastically, although the several recent important improvements. The size of the coefficients of such Gr"obner bases can indeed be very large (and also can be the number of such coefficients!). It is widely experimentally observed that it is for lexicographic orders that this problem is the worth.

This suggests the following question:

How large can the coefficients can become when computing a lexicographic Gröbner basis of a given input polynomial system?

This is a quite general problem that has not been studied much when “large” means “number of digits” of integers or rational coefficients. If the coefficients are themselves parameters (polynomial or rational fractions in one or several indeterminates), then a few previous results exist, with not very satisfactory bounds. In both cases, the works [5, 3] gave new results or improvements on this matter. However, somewhat quite strong restrictive hypotheses are still in order.

Triangular sets We will call a polynomial system a triangular set any family \( T \) of polynomials \( T_1, \ldots, T_n \) that are in the following (“triangular”) shape:

\[
\begin{align*}
T_n(X_1, X_2, \ldots, X_{n-1}, X_n) &= X_n^{d_n} + \cdots \\
T_{n-1}(X_1, \ldots, X_{n-1}) &= X_{n-1}^{d_{n-1}} + \cdots \\
& \vdots \\
T_1(X_1) &= X_1^{d_1} + \cdots
\end{align*}
\]

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The degree $d_i$ is denoted $\deg_{X_i}(T_i)$. It is also required that $T$ is a reduced lexicographic Gröbner basis (necessarily for the monomial order $X_1 < \cdots < X_n$). This family is then a regular sequence, hence it generates a 0-dimensional ideal.

The method we have used to get upper-bounds in [5, 3] requires the classical algebra/geometry dictionary, so it is assumed that the output triangular system to be radical. The input polynomial system may not be radical, but then consists of as many polynomials as the number of indeterminates $n$, i.e. is a square system, and the solutions are those where the Jacobian determinant does not vanish. By the Jacobian criterion, these solutions are indeed simple. Let $V$ be this set of solutions over the algebraic closure $\overline{k}$ of the field of definition $k$.

**Assumption 1.** $V$ is finite, and there exists a triangular set $T_1(X_1), \ldots, T_n(X_1, \ldots, X_n)$ such that $V = Z(\langle T_1, \ldots, T_n \rangle)$.

Since we are with lexicographic orders, the elimination property fully holds. On the geometric side, this implies good properties under projections, and it is possible to rewrite the polynomials of a triangular set in a Lagrange interpolation formulation. Let us denote by $\pi_j$ the projection on the coordinate space spanned by $X_1, \ldots, X_j$. That is, given a point $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{C}^n$, $\pi_j(\gamma) = (\gamma_1, \ldots, \gamma_j)$. We also denote by $X_i$ the $i$-th coordinate function, defined by $X_i(\gamma) := \gamma_i$. The set $X_i(V)$ is therefore equal to \{ $b \in \mathbb{C}$ | $\exists (\gamma_1, \ldots, \gamma_n) \in V$ s.t. $X_i(\gamma) = \gamma_i = b$ \}, that is the projection of $V$ on the $X_i$-axis.

We now introduce the Lagrange basis built on the points in $\pi_{n-1}(V)$, in which we will rewrite the polynomial $T_n$ (it is similar for the other polynomial $T_\ell$, for $2 \leq \ell \leq n-1$, by considering the points in $\pi_{\ell-1}(V)$ instead). To $\alpha \in \pi_{n-1}(V)$ we associate the polynomial $E_\alpha(X_1, \ldots, X_{n-1})$ that verifies $E_\alpha(\alpha) = 1$ and $E_\alpha(\beta) = 0$ for any other point $\beta \in \pi_{n-1}(V)$.

$$E_\alpha(X_1, \ldots, X_{n-1}) := \prod_{i=1}^{n-1} \prod_{b \in X_i(V)}^{X_i-b \atop \alpha_i-b}.$$ (1)

For further need, we introduce also this alternative form:

$$F_\alpha(X_1, \ldots, X_{n-1}) := \prod_{i=1}^{n-1} \prod_{b \in X_i(V)}^{X_i-b}.$$ (2)

The expression of $T_n$ in the Lagrange basis $\{E_\alpha, \alpha \in \pi_{n-1}(V)\}$ is:

$$T_n(X_1, \ldots, X_n) = \sum_{\alpha \in \pi_{n-1}(V)} \left( \prod_{y \in \pi_{n-1}^{-1}(\alpha) \cap V} X_n - y \right) E_\alpha(X_1, \ldots, X_{n-1}).$$ (3)

When $n = 2$, Figure 1 shows a simple example.

**Primitive elements** Leaving Assumption 1, it is also possible to give a parametrization of the coordinates of $V$, parallel with a randomly chosen hyperplane. This is the data of a "randomly" chosen linear

---

1) The focus will be on $k = \mathbb{Q}$ in this survey, and for convenience we will use $\mathbb{C}$ instead of $\overline{\mathbb{Q}}$. In the original papers [5, 3], fields of rational functions are also considered, and in the Ph.D. thesis of the author, number and function fields are treated, only in dimension 0 though.
$q_3(Y) = (Y - 2)(Y - 3)$
$q_7(Y) = (Y - 4)(Y - 5)$
$q_9(Y) = (Y - 1)(Y - 6)$

$T_2(X, Y) = (q_3) \frac{(X-7)(X-9)}{(3-7)(3-9)} + (q_7)\frac{(X-3)(X-9)}{(7-3)(7-9)} + (q_9)\frac{(X-3)(X-7)}{(9-3)(9-7)}$

$N_2(X_1, X_2) = (q_3)(X-7)(X-9) + (q_7)(X-3)(X-9) + (q_9)(X-3)(X-7)$

Figure 1: Lagrange interpolation

form $\Delta(X_1, \ldots, X_n)$ defining $H$, that almost always will be separating for $V$ (that is $\Delta(\alpha) \neq \Delta(\beta)$ for all $\alpha \neq \beta$ in $V$) and a family of polynomials

\[
( q(T), \ X_1 - v_1(T), \ldots, X_n - v_n(T) ),
\]

such that the roots of $q$ are given by $\Delta(V)$, i.e. $q(t) = 0 \iff t = \Delta(\alpha)$ for a unique $\alpha$ in $V$. And, that $\alpha = (\alpha_1, \ldots, \alpha_n) = (v_1(t), \ldots, v_n(t))$. If $I(V)$ denotes the ideal of vanishing polynomials on $V$, we get:

\[
q(\Delta(X_1, \ldots, X_1)) \equiv 0 \mod I(V),
\]

Relying on a projection (over a line here), this representation admits also a description through Lagrange polynomials.

\[
v_i(T) = \sum_{\alpha \in V} \alpha_i \prod_{\beta \neq \alpha} \frac{T - \Delta(\alpha)}{\Delta(\beta) - \Delta(\alpha)}.
\]

$\vec{\delta}$ is the direction vector of $L$

\[
\vec{0}v = \Delta(\beta)\vec{\delta}
\]

\[
\vec{0}t = \Delta(\alpha)\vec{\delta}
\]

Figure 2: The orthogonal projection along $H$ of two points $\alpha$ and $\beta$ over $L$

Alternative representation: decrease of the coefficients size From the Lagrange formulas (6) and (3) we define 2 alternative representations.

Rational Univariate Representation: instead of considering the data of $(q(T), \ X_1 - v_1(T), \ldots, X_n - v_n(T) )$ like in (4), it consists of choosing rather $(q(T), \ q'(T)X_1 - w_1(T), \ldots, q'(T)X_n - w_n(T) )$,
where:

$$w_i(T) = \sum_{\alpha \in V, \beta \neq \alpha} \alpha_i \prod_{T - \Delta(\alpha)}. \quad (7)$$

It is easy to see that $w_i(T) \equiv q'(T)v_i(T) \mod q(T)$ (the mod $q(T)$ means taking the remainder of the Euclidean division by $q(T)$). The RUR is equivalent to the data of (4) since both parametrize the same set of points. It was introduced by Alonso et al. [1], as they remark the smaller coefficients $^2)$. Rouillier developed further the use of this representation, and renamed it Rational Univariate Representation [11]. By a different method, Giusti et al. [6] got an algorithm called geometric resolution to compute the same representation.

As for triangular sets, a similar transformation is possible and leads to the same nice decrease of the coefficients. Using the polynomial $F_\alpha$ in (2) instead of the $E_\alpha$ in (1):

$$N_n(X_1, \ldots, X_n) = \sum_{\alpha \in \pi_{n-1}(V)} \prod_{\substack{y \in \pi_{n-1}^{-1}(\alpha) \cap V \ \ y \neq \alpha}} X_n - y F_\alpha(X_1, \ldots, X_{n-1}). \quad (8)$$

**Statements of the results** There are two kinds of upper-bounds: the *intrinsic* one, depending only on quantities attached to the solutions $^3)$ and not of a particular system of equations. And the non-intrinsic ones, that depends on a specified system of equations (the input). The bounds from this last kind of measure are always deduced from the intrinsic ones, using a version of the "geometric-arithmetic" intersection theorem (the standard and the Arithmetic Bézout theorem (10)).

**In dimension 0.** Let $V$ be the set of solutions in an algebraic closure of a field $k$ of a given family of polynomials. We assume $V$ finite. We let $h_V$ be the height of $V$ (Cf. § 2.2 for a definition) and $d_V$ be its degree, that corresponds here to the cardinal of $V$. We get the following:

**Primitive element:** We let $\Delta$ be a separating linear form for $V$ and we let $(q, q'X_1 - w_1, \ldots, q'X_n - w_n)$ be the associated RUR (7). Then:

$$h(w_i) \leq h_V + d_V h(\Delta) + d_V \log(n + 2) + (n + 1) \log d_V$$

**Triangular set:** We assume that $V$ verifies Assumption 1:

$$h(T_n) \leq d_V (h_V + 5 \log(n + 3) + 4d_V)$$

**Alternative triangular polynomial systems:** With the same assumption as above, the polynomials $(N_1, \ldots, N_n)$ defined in Equation (8) verifies:

$$h(N_n) \leq h_V + 5d_V \log(n + 3) \quad (9)$$

The alternative representations have linear sized bounds with respect to the degree $\deg(V) := d_V$ and the height $h_V$, while the corresponding lexicographic Gröbner bases have quadratic sized bound. This better behavior is neatly observed experimentally.

**In positive dimension.** New results in this case are available in [3] but require several other notations to be stated. We have postponed this to § 4.

$^2)$ but was already known by Kronecker

$^3)$ seen as an algebraic variety
2 Height theory

This is a fundamental tool in our work. It comes from the theory of "Diophantine geometry" in mathematics. Several definitions have emerged, one of them developed mainly by Philippon, is using explicitly Elimination theory [9]. View our context, it is natural to choose this one. For any further details, we refer rather to the paper of Krick et al. [7] instead, because their presentation is closer to applications relevant to the community of "effective mathematicis".

2.1 Overview

There are two levels of measure with height theory: one concerning the algebraic numbers, the polynomials with coefficients of those, and one concerning algebraic varieties. In any case, the viewpoint lies in the parallel with the degree:

Polynomials We start with the definition of the height of a polynomial. The algebraic complexity is measured by the degree.

- total degree in $k[X_1, \ldots, X_m] - \{0\}$.
- extended to $k(Y_1, \ldots, Y_m)$ (taking maximum: if $\gcd(A, B) = 1$, $\text{tdeg}(\frac{A}{B}) := \max\{\text{tdeg}A, \text{tdeg}B\}$)
- extended to $k(Y_1, \ldots, Y_m)[X_1, \ldots, X_n]$ (reducing to the same denominator, then taking the maximum of each coefficient in $k(Y_1, \ldots, Y_m)$)

The height of a polynomial concerns its arithmetic complexity. We start by defining the height of a rational number.

- For $\frac{a}{b} \in \mathbb{Q}$ with $\gcd(a, b) = 1$, it is defined by $h(\frac{a}{b}) = \log \max\{|a|, |b|\}$.

  Formal definition: Let $p$ be a prime.

  $h_p(\frac{a}{b}) = \log \max\{1, p^{v_p(b) - v_p(a)}\}$, $h_p(\frac{a}{b}) \neq 0 \iff p \nmid a$, and $p\mid b$.

  $h_\infty(\frac{a}{b}) = \log \max\{|1, \frac{|a|}{|b|}\}$, $h_\infty(\frac{a}{b}) \neq 0 \iff |a| > |b|

  Height of a rational: $h(\frac{a}{b}) = \sum_{p \text{ prime}} h_p(\frac{a}{b}) + h_\infty(\frac{a}{b}) = \log \max\{|a|, |b|\}$.

- extended to $\mathbb{Q}[X_1, \ldots, X_n]$ in the following way: Let $F = \sum_{f \in N^n} f_1 X_1^{i_1} \cdots X_n^{i_n}$.

  Let $c = \text{LCM}\{\text{denom. of } f_1\}$. Then $cF \in \mathbb{Z}[X_1, \ldots, X_n]$, and define $h(F) = \log \max\{|c, h_\infty(cF)|\}$.

  Formal definition: For $v = p$, or $v = \infty$, let $h_v(F) = \log \max\{1, \max_{f \in N^n}\{h_v(f_1)\}\}$, then,

  height of a polynomial: $h(F) = \sum_{p \text{ prime}} h_p(F) + h_\infty(F)$.

Varieties The corresponding notion of height is more sophisticated to define. A convenient way to introduce it is the parallel with the degree seen as the algebraic complexity:

- $V$ equidimensional: generic number of intersection points with a linear space of complementary dimension.
- Additivity: if $V_1 \cap V_2 = \emptyset$, then $\deg(V_1) + \deg(V_2) = \deg(V_1 \cup V_2)$
• Affine version of Bézout theorem: \( \deg(V \cap W) \leq \deg(V) \deg(W). \)

• is well-defined for varieties defined over any field, not only \( \mathbb{Q} \) or number fields.

As for polynomials, the height of a variety is a measure of its arithmetic complexity. The following points are to be compared with the above ones that concerned the degree.

• \( V \) equidimensional: its height, \( h(V) \) is (almost) height of the Chow form (Cf. next paragraph).

• additive: \( h(V_1 \cup V_2) = h(V_1) + h(V_2). \)

• arithmetic Bézout theorem: if \( V \not\subset Z(f), \)

\[
h(V \cap Z(f)) \leq \deg(f)h_V + d_V h(f) + \log(n+1) \deg(f). \tag{10}
\]

• for varieties defined over \( \mathbb{Q} \) (more generally, over a number fields).

The detailed definitions are given in the two next paragraphs.

### 2.2 Chow form

This is a polynomial attached to a given variety \( V \) that contains all the information of it (but has many more variables).

Let \( V \subset \mathbb{C}^{m+n} \) be an equidimensional variety of dimension \( m \), \( \overline{V} \subset \mathbb{P}^{n+m}(\mathbb{C}) \) its projective closure with \( Y_0 \) as homogenizing new variable.

We introduce \( m+1 \) generic linear forms \( L_i, \ i = 0, \ldots, m, \) with generic coordinates represented by \( (n+m+1) (m+1) \) new variables: \( U_i = U_{i,0}, \ldots, U_{i,m+n} \)

\[
L_i^h = U_{i,0}Y_0 + U_{i,1}Y_1 + \cdots + U_{i,m}Y_m + U_{i,m+1}X_1 + \cdots + U_{i,m+n}X_n. \tag{11}
\]

The incidence variety \( W \) is by definition:

\[
W = \overline{V} \cap Z(L_0^h, \ldots, L_m^h) \subset \overline{V} \times \mathbb{P}^{m+n}(\mathbb{C}) \times \cdots \times \mathbb{P}^{m+n}(\mathbb{C})
\]

The Zariski closure of the projection of \( W \) on \( \mathbb{P}^{m+n}(\mathbb{C}) \times \cdots \times \mathbb{P}^{m+n}(\mathbb{C}) \) is a hypersurface. A Chow form is a square-free polynomial that defines this hypersurface. Here are simple remarks:

Fact 0: All Chow forms \( Ch_V \) are defined up to a constant factor.

Fact 1: The Chow forms are polynomials in \( (m+1) \) groups \( U_i \) of \( (m+n+1) \) variables, multi-homogeneous w.r.t. each groups.

If \( V \subset \mathbb{C}^{m+n} \) is defined over \( \mathbb{Z} \subset R \subset \mathbb{C} \), then:

\[
Ch_V(U_0, U_1, \ldots, U_m) \in R[U_0, \ldots, U_m]
\]

In this survey, \( R = \mathbb{Z} \).

Fact 2: If \( V \) is an irreducible variety, then \( Ch_V \) is an irreducible polynomial.

If \( V = V_1 \cap V_2 \), with \( \dim(V_1) = \dim(V_2) = m \), \( V_1 \cap V_2 = \emptyset \), then the product \( Ch_{V_1} Ch_{V_2} \) is a Chow form of \( V \).
A more geometric interpretation  The $m$ groups of $(m + n + 1)$ variables $U_1, \ldots, U_m$ are used to parametrize $m$ hyperplanes in $\mathbb{P}^{m+n}(\mathbb{C})$. For almost all values $U_i \leftarrow u_i = (u_{i,0}, \ldots, u_{i,m+n+1}) \in \mathbb{Q}^{m+n+1}$, let $H_i$ be the hyperplane defined by the linear form $L_i$ of (11) evaluated at $u_i$:

$$H_i := Z(L_i(u_{i,0}, u_{i,1}, \ldots, u_{i,m}, u_{i,m+1}, \ldots, u_{i,m+n})).$$

Remark: $H_i$ is an affine hyperplane, its projective closure is $\overline{H_i} := Z(L_i^h(u_i))$. By the property of the degree of $V$, then:

$$V_0 := V \cap H_1 \cdots \cap H_m$$

is finite, of cardinal $\leq d_V$, and for almost all choices of the evaluation points $\{u_{i,j}, 0 \leq j \leq m + n + 1, 1 \leq i \leq m\}$, \#$V_0$ is equal to $d_V$.

The first group of $m+n$ variables $U_{0,1}, \ldots, U_{0,m}$ of $U_0$ parametrizes the affine hyperplanes in $\mathbb{A}^{m+n}(\mathbb{Q})$ going though 0, leaving one variable $U_{0,0}$ free. Let $u_0' := (u_{0,1}, \ldots, u_{0,m+n}) \in \mathbb{Q}^{m+n}$.

homogeneous linear form  $L_0 := u_{0,1}Y_1 + \cdots + u_{0,m}Y_m + u_{0,m+1}X_1 + \cdots + u_{0,m+n}X_n$

Let $H_0 = Z(L_0)$ be the corresponding hyperplane going through 0. The Chow form of $V$ verifies the following property.

$$Ch_V(U_{0,0}, u_0', u_1, \ldots, u_{m+n}) = c \prod_{\alpha \in \mathbb{Q}^{m+n} \cap H_0} (U_{0,0} + L_0(\alpha)),$$

(12)

otherly said, the univariate polynomial $Ch_V(U_{0,0}, u_0', \ldots, u_{m+n})$ is a primitive element for $V_0$ w.r.t the hyperplane $H_0$.

A toy example in the plane  The circle has no point at infinity, so the projective hyperplanes (that are just lines here) will be represented w.l.o.g by affine ones.

Next we choose an intersection line $H_1$. The circle being of degree 2, there are 2 intersection points.
By definition, the first set of variables parametrize a homogeneous linear form, whose line $H_0$ is going through the origin.

This gives the primitive element representation of the intersection of $V$ with the $H_1$.

2.3 Mahler measures and height of varieties

The definitions are a bit more complicated, but we need only to manipulate the results.
Following some works of Nesterenko, Philippon in 1986 [9], defines a height of varieties using the Mahler measure of one of its Chow form:

$$f \in \mathbb{C}[X_1, \ldots, X_n], \quad m(f) := \int_0^1 \cdots \int_0^1 \log|f(e^{2i\pi t_1}, \ldots, e^{2i\pi t_n})|dt_1 \cdots dt_n.$$  

In 1 variable, let us write $f = a_d \prod_{i=1}^d (X - \alpha_i)$, then:

$$m(f) = \log|a_d| + \sum_{i=1}^d \max\{0, \log|\alpha_i|\} \quad \text{(Jensen's formula)}$$

showing a clear link with the traditional height of a polynomial $h_\infty(f)$ defined in § 2.1. We note the additivity of $m(\cdot)$:

$$m(fg) = m(f) + m(g).$$

This Mahler measure appeared to be not completely satisfactory, and later Philippon [10] modified it with the “$S^{m+1}_{n}$-Mahler measure”. We assume that $f$ is a polynomial in $(m+1)$ groups of $(m+n+1)$ variables, homogeneous for each group. Let $k = m + n$ be the dimension of the ambient space. Let $S^{m+1}_{k+1} = S_{k+1} \times \cdots \times S_{k+1}$, where $S_{k+1}$ is the unit sphere in $\mathbb{C}^{k+1}$. The $S^{m+1}_{k+1}$-Mahler measure is defined by:

$$m(f, S^{m+1}_{k+1}) = \int_{S^{m+1}_{k+1}} \log|f| \mu_{k+1}^{\wedge(m+1)}.$$  

Now if $V$ is an $m$-dimensional variety in $\mathbb{C}^{k}$, with $k = m + n$, then a Chow form of $V$ is a polynomial in $m+1$ groups of $n + m + 1$ variables, denoted $Ch_{V}(U_0, \ldots, U_m)$. It is possible to take its $S^{m+1}_{k+1}$-Mahler measure.

Remark: The following inequalities permit to link the above two definitions of Mahler measures with the height of a polynomial. Let $d = \deg(f)$.

$$0 \leq m(f) - m(f, S^{m+1}_{k+1}) \leq (m+1)d \sum_{j=1}^k \frac{1}{2j}$$

$$|m(f) - h_\infty(f)| \leq d \log(k+1)$$  

(13)  

(14)  

Here is how Philippon defined the height of a variety: $(V \subset \mathbb{C}^{k}, k = m + n$, equidimensional of dimension $m$ as usual):

$$h_{V} := \sum_{p \text{ primes}} h_{p}(Ch_{V}) + m(Ch_{V}, S^{m+1}_{k+1}) + (m+1)d_{V} \sum_{j=1}^k \frac{1}{2j}.$$  

(15)  

The last term $\sum_{j=1}^k \frac{1}{2j}$ permits to ensure that $h_{V} \geq 0$. Also,

$$h_{V \cup W} = h_{V} + h_{W}, \quad \text{if} \quad V \cap W = \emptyset, \quad \text{and} \quad \dim(V) = m = \dim(W).$$

Finally, we mention the arithmetic counterpart of the Bézout inequality, in a slightly different formulation than in Equation (10).

Arithmetic Bézout theorem (ABT): Let $V = Z(f_1, \ldots, f_s) \subset \mathbb{C}^{n}$, defined over $\mathbb{Q}$, with $h(f_j) \leq h$ and $tdeg(f_j) = d_j$, and $n_0 = \min\{n, s\}$:

$$h_{V} \leq \left(\prod_{j=1}^{n_0} d_j\right) \left(\sum_{j=1}^{n_0} \frac{1}{d_j}\right) h + (n + n_0) \log(n+1).$$  

(16)
3 The case of dimension 0

We assume here that \( V \subset \mathbb{C}^n \) is the finite set of solutions of a polynomial system \( f_1, \ldots, f_n \in \mathbb{Q}[X_1, \ldots, X_n] \).

3.1 Bounds for the RUR

Let \( \Delta(X_1, \ldots, X_n) = u_1X_1 + \cdots + u_nX_n \) be a separating linear form for \( V \) and let \( (q(T), q'(T)X_1 - w_1(T), \ldots, q'(T)X_n - w_n(T)) \) be the associated Rational Univariate Representation of \( V \). Let \( Ch_V \) be the monic Chow form of \( V \). Since \( V \) is 0-dimensional, there is only 1 group of \( n+1 \) variables \( U_0 = U_{0,0}, \ldots, U_{0,n} \). By Equation (12), we have:

\[
Ch_V(U_{0,0}, -u_1, \ldots, -u_n) = \prod_{\alpha \in V} U_{0,0} - u_1 \alpha_1 - \cdots - u_n \alpha_n, = \prod_{\alpha \in V} U_{0,0} - \Delta(\alpha),
\]

(17)

hence \( Ch_V(T, -u_1, \ldots, -u_n) = q(T) \) since they both vanish on \( \Delta(V) \) and have same degree, by Equation (5). This implies,

\[
\frac{\partial}{\partial U_{0,0}} Ch_V(U_{0,0}, -u_1, \ldots, -u_n) = q'(U_{0,0}).
\]

(18)

Let \( G(U_{0,1}, \ldots, U_{0,n}) := Ch_V(U_{0,1}X_1 + \cdots + U_{0,n}X_n, -U_{0,1}, \ldots, -U_{0,n}) \). From Equation (17), it arrives:

\[
G(u_1, \ldots, u_n) = Ch_V(\Delta, -u_1, \ldots, -u_n) \text{ vanishes on } V.
\]

(19)

This implies:

\[
\frac{\partial}{\partial U_{0,i}} G = \left( X_i \frac{\partial}{\partial U_{0,0}} Ch_V - \frac{\partial}{\partial U_{0,i}} Ch_V \right) (U_{0,1}X_1 + \cdots + U_{0,n}X_n, -U_{0,1}, \ldots, -U_{0,n}).
\]

If we perform the evaluation \( U_{0,i} = u_i \) in the above, it comes with Equation (18), that:

\[
X_i q'(u_1X_1 + \cdots + u_nX_n) = \frac{\partial}{\partial U_{0,i}} Ch_V(u_1X_1 + \cdots + u_nX_n, -u_1, \ldots, -u_n),
\]

vanishes on \( V \), that are equal modulo \( I(V) \). We use \( v_i(\Delta) \equiv X_i \mod I(V) \) (Cf. Equation (4)), multiply it by \( q' \) and perform the substitution \( T \leftrightarrow \Delta(X_1, \ldots, X_n) \):

\[
v_i(T)q'(T) - \frac{\partial}{\partial U_{0,i}} Ch_V(T, -u_1, \ldots, -u_n) \equiv 0 \mod q(T)
\]

By Equation (7) what follows it, \( w_i(T) = v_i(T)q'(T) \mod q(T) \). This gives:

\[
w_i(T) = \frac{\partial}{\partial U_{0,i}} Ch_V(T, -u_1, \ldots, -u_n),
\]

(20)

since both terms have same degree.

End of first step: This is the link between the Chow form and the polynomials occurring in the RUR.

Next step: Use height estimates. We start by two easy inequalities (21) and (22): if \( f \) is a univariate polynomial of degree \( d \), \( x \) a number such that \( |x| > 1 \), then:

\[
|f'(x)| \leq d^2 |x|^{d-1} \max\{|\text{coeff of } f|\}.
\]

(21)
If $F$ is an $(n+1)$-variate polynomial of total degree $d$, and $x_2, \ldots, x_{n+1}$ are complex numbers such that $|x_i| > 1$, then:
\[
\max\{|\text{coeff of } \frac{\partial F}{\partial X_1}(X_1, x_2, \ldots, x_{n+1})|\} \leq d^{n+1} \max_i |x_i|^d \max\{|\text{coeff of } F|\}
\]  
(22)

We apply it to Equation (20) and to Equation (18) to get:
\[
\max\{|\text{coeff of } q'|\} \quad \text{and} \quad \max\{|\text{coeff of } w_i|\} \leq d_{V}^{n+1} \max_i |u_i|^d \max \{|\text{coeff of } Ch_{V}|\}.
\]

By definition of the height:
\[
h_{\infty}(w_i) \quad \text{and} \quad h_{\infty}(q') \leq (n+1)\log d_{V} + d_{V}h_{\infty}(\Delta) + h_{\infty}(Ch_{V}).
\]  
(23)

Remark: When $v = p$, $p$ a prime, the estimates for $h_p(w_i)$ are easy to obtain, and the details are not given in this survey: $h_p(w_i) \leq h_p(Ch_V) + d_V h_p(\Delta)$.

3rd step: From height of Chow forms to height of varieties.

By property of Mahler measures (13) and (14), $h_{\infty}(Ch_{V}) \leq m(Ch_{V}, S_{n+1}) + d_{V} \log(n+2) + d_{V} \sum_{j=1}^{n+1} \frac{1}{2j}$, that implies:
\[
h_{\infty}(w_i) \leq m(Ch_{V}, S_{n+1}) + (n+1)\log(d_{V}) + d_{V}h_{\infty}(\Delta) + d_{V} \log(n+2) + d_{V} \sum_{j=1}^{n+1} \frac{1}{2j}.
\]

By definition of the height of a polynomial, it remains to sum over the absolute values $v$:
\[
h(w_i) \leq (\sum_{p \text{ prime}} h_p(Ch_V) + d_{V}h_p(\Delta)) + m(Ch_{V}, S_{n+1}) + (n+1)\log(d_{V}) + d_{V}h_{\infty}(\Delta) + d_{V} \log(n+2) + d_{V} \sum_{j=1}^{n+1} \frac{1}{2j}
\]

This is the intrinsic quantities that depends on the degree and the height of the variety $V$ and the separating linear form $\Delta$.

Last step: Use the Arithmetic Bézout Theorem to get non-intrinsic bounds, that depend on the input polynomial system, from the intrinsic ones obtained just above. If $V$ is the set of solutions of a polynomial system $(f_1, \ldots, f_n)$ with $d = \max_i \{\text{deg}(f_i)\}$ and $h = \max_i \{h(f_i)\}$, then from the standard Bézout inequality $d_{V} \leq d^n$, and from the ABT (16), $h_V \leq nd^{n-1}(h + 2d \log(n+1))$. Plugging these into the estimates of $h(w_i)$ obtained in Step 3 gives:
\[
h(w_i) \leq d^n(nh + h(\Delta) + 3n \log(n+2)) = O((nh + h(\Delta))d^n)
\]

Conclusion: we used 4 steps.

1. link between polynomials and the Chow forms
2. use height estimates
3. from height of Chow forms to height of variety. (use properties of Mahler measures)
4. Use of Arithmetic Bézout Theorem.
3.2 Bounds for triangular systems

We will treat only the easier case of the polynomials $N_1, \ldots, N_n$, when $n = 2$ and on an example (Cf. Figure 1). For the triangular sets $T_1, \ldots, T_n$, some extra complications occur to treat the denominators in Formula (1), when compared to Formula (2). This precisely explains the overhead quadratic behavior of their estimates, when compared to the linear behavior of the ones for the $N_1, \ldots, N_n$.

**Highlights of the proof on an example** This is not the purpose of this text to give the full details that require quite a lot of notations. Rather, the following might motivate to read the full proof in [5].

The example is the one of Figure 1. We have $\deg X(T_1) = d_1 = 3$ and $\deg Y(T_2) = d_2 = 2$ there. The monic Chow form of $V$ is: $Ch_V(U_{0,0}, U_{0,1}, U_{0,2}) = \prod_{\alpha \in V} U_{0,0} + \alpha_{1} U_{0,1} + \alpha_{2} U_{0,2}$. That implies:

$$Ch_V(X, -1, 0) = \prod_{\alpha \in V} X - \alpha_{1} = N_1(X)^2 = T_1(X)^2 \quad \text{(in general $T_1(X_1)^{d_2}$)}$$

Remains to treat $N_2(X, Y)$. For $\alpha_1 = 3, 7$ or 9, let $v_{\alpha_1} := \pi_1^{-1}(\alpha_1)$ the fiber over $\alpha_1$ of the projection of $V$ on the $X$-axis. By a classical property of varieties described by triangular sets ("equiprojectable"), $\# v_3 = \# v_7 = \# v_9 = \deg V(T_2) = 2$. By additivity of Chow forms$^4$ (§ 2.2, Fact 2):

$$v_3 \cup v_7 \cup v_9 = V \Rightarrow Ch_{v_3}Ch_{v_7}Ch_{v_9} = Ch_V. \quad (24)$$

We introduce the following subsets of $V$:

$$W_3 = v_3 \cup v_9 \ , \ W_7 = v_3 \cup v_9 \ , \ \text{and} \ W_9 = v_3 \cup v_9. \quad (25)$$

Since $Ch_{W_3} = \prod_{\alpha \in W_3} U_{0,0} + \alpha_{1} U_{0,1} + \alpha_{2} U_{0,2} \Rightarrow Ch_{W_3}(X, -1, 0) = \prod_{\alpha \in W_3} X - \alpha_{1} = (X - 7)^2(X - 9)^2$. Similarly, we can show that $Ch_{W_3}(Y, 0, -1) = q_3(Y) = (Y - 2)(Y - 3)$, and $Ch_{W_3}(Y, 0, -1) = q_7(Y) = (Y - 4)(Y - 5)$, and $Ch_{W_3}(Y, 0, -1) = q_9(Y) = (Y - 6)$. Then a look at the formula for $N_2$ in Figure 1 and the above shows that:

$$N_2(X, Y) = Ch_{v_3}(Y, 0, -1)Ch_{W_3}(X, -1, 0)^{1/2} + Ch_{v_7}(Y, 0, -1)Ch_{W_3}(X, -1, 0)^{1/2} + Ch_{v_9}(Y, 0, -1)Ch_{W_3}(X, -1, 0)^{1/2} \quad (26)$$

This ends the first step, which was intended to link the polynomial $N_1$ with the Chow form $Ch_V$.

**Second step:** We turn to the height estimation, starting by this simple result: If $f = \sum_{i \in \mathbb{N}^n} f_i X_1^{i_1} \ldots X_n^{i_n}$ is a polynomial, we denote by $d_f$ its total degree, and by $|f|_{\infty} := \max_{i \in \mathbb{N}^n} |f_i|$ the maximal absolute of its coefficients. According to the definitions of § 2.1, we have $h_{\infty}(f) = \max(1, \log |f|_{\infty})$. Let $g = \sum_{i \in \mathbb{N}^n} g_i X_1^{i_1} \ldots X_n^{i_n}$ be another polynomial, then:

$$h_{\infty}(f) + h_{\infty}(g) \leq h_{\infty}(fg) + 2(d_f + d_g) \log(n + 1). \quad (27)$$

Indeed, by Equation (14), \( |h_{\infty}(fg) - m(fg)| \leq (d_f + d_g) \log(n + 1) \), and by the additivity of $m(\cdot)$, comes $m(f) + m(g) \leq |h_{\infty}(fg)| + (d_f + d_g) \log(n + 1)$. Again, using Equation (14) gives $h_{\infty}(f) \leq m(f) + d_f \log(n + 1)$ and $h_{\infty}(g) \leq m(g) + d_g \log(n + 1)$, yielding the inequality (27).

\[^4]\text{these Chow forms are actually defined over a field extension of Q, that would require a special definition of height. This is not treated here.}\]
We can apply this result to \( f = g = \text{Ch}_{W_5}(X, -1, 0)^{1/2} \), that gives:

\[
2h_{\infty}(\text{Ch}_{W_5}(X, -1, 0)^{1/2}) \leq h_{\infty}(\text{Ch}_{W_5}(X, -1, 0)) + 4(3 - 1) \log(2) \quad \text{(in general, } (3 - 1) = d_1 - 1) \]

It is clear that the absolute value of the coefficients of \( \text{Ch}_{W_5}(X, -1, 0) \) are contained in those of \( \text{Ch}_{W_5} \), hence \( h_{\infty}(\text{Ch}_{W_5}(X, -1, 0)) \leq h_{\infty}(\text{Ch}_{W_5}) \) by definition of the height of a polynomial (Cf. § 2.1), this gives:

\[
h_{\infty}(\text{Ch}_{W_5}(X, -1, 0)^{1/2}) \leq \frac{1}{2} h_{\infty}(\text{Ch}_{W_5}) + 2(d_1 - 1) \log(2) \quad \text{(in general, } \frac{1}{2} = \frac{1}{d_2} \) \quad (28)

By Inequality (14), \( h_{\infty}(\text{Ch}_{W_5}) \leq m(\text{Ch}_{W_5}) + 2(3 - 1) \log(4) \), which is equal in general to \( m(\text{Ch}_{W_5}) + d_2(d_1 - 1) \log(n + 2) \), since \( n = 2 \) and the Chow form is a polynomial in \( n + 1 \) variables. Plugging this with \( d_2 = 2 \) in (28), we obtain:

\[
h_{\infty}(\text{Ch}_{W_5}(X, -1, 0)^{1/2}) \leq \frac{1}{2} m(\text{Ch}_{W_5}) + (d_1 - 1)(2 \log(2) + \log(n + 2)). \quad (29)
\]

A similar inequality holds for \( W_7 \) and \( W_9 \).

On the other hand, easier calculations than above, that we do not do, give:

\[
h_{\infty}(\text{Ch}_{v_3}(Y, 0, -1)) \leq m(\text{Ch}_{v_3}) + d_2 \log(n + 2). \quad (30)
\]

Next, for 2 polynomials \( f \) and \( g \) in \( n \) variables, the inequality \(|fg|_{\infty} \leq (d_f + d_g)^n |f|_{\infty} |g|_{\infty} \) holds. This translates in terms of heights to \( h_{\infty}(fg) \leq h_{\infty}(f) + h_{\infty}(g) + \log(n)(d_f + d_g) \). Follows the first inequality below, since \( d_1 - 1 = 2 = \deg_X(\text{Ch}_{W_5}(X, -1, 0)^{1/2}) \) and \( d_2 = 2 = \deg_Y(\text{Ch}_{v_3}(Y, 0, -1)) \):

\[
h_{\infty}(\text{Ch}_{W_5}(X, -1, 0)^{1/2}\text{Ch}_{v_3}(Y, 0, -1)) \leq h_{\infty}(\text{Ch}_{W_5}(X, -1, 0)^{1/2}) + h_{\infty}(\text{Ch}_{v_3}(Y, 0, -1)) + (d_1 - 1 + d_2) \log(n). \quad (31)
\]

With Equation (30) and (29), it becomes after a few simplifications:

\[
h_{\infty}(\text{Ch}_{W_5}(X, -1, 0)^{1/2}\text{Ch}_{v_3}(Y, 0, -1)) \leq \frac{1}{2} m(\text{Ch}_{W_5}) + m(\text{Ch}_{v_3}) + (2d_2 + 3d_1) \log(n + 2) + (d_1 - 1) \log(n).
\]

Recall that by Equalities (24) and (25), \( V = W_3 \cup v_3 \). Also the positivity and additivity of the Mahler measure implies: \( \frac{1}{2} m(\text{Ch}_{W_5}) + m(\text{Ch}_{v_3}) \) lem(Ch_\mathcal{V}). Replaced in the equation above,

\[
h_{\infty}(\text{Ch}_{W_3}(X, -1, 0)^{1/2}\text{Ch}_{v_3}(Y, 0, -1)) \leq m(\text{Ch}_V) + (2d_2 + 3d_1) \log(n + 2).
\]

A similar inequality holds for \( W_7, v_7 \) and for \( W_9, v_9 \). It remains to add the height of these 3 terms in the interpolation formula (26) of \( N_2(X, Y) \). It is easy to see that:

\[
h_{\infty}(N_2) \leq \max_{i=3,7,9} \{h_{\infty}(\text{Ch}_{W_i}(X, -1, 0)^{1/2}\text{Ch}_{v_i}(Y, 0, -1))\} + \log(3) \quad \text{(in general, } 3 = d_1) \]

\[
\leq m(\text{Ch}_V) + (2d_2 + 3d_1) \log(n + 2) + \log(d_1). \quad \text{(when } n = 2, d_1 = \prod_{j=1}^{n-1} d_j \) \quad (32)
\]

3rd step: We use the definition of a height of a variety (15), relying on the Mahler measure. First the work done in 2nd step concerns exclusively the component \( h_{\infty}(\cdot) \) of the height (Cf. § 2.1 for definitions). The \( p \)-adic components \( h_p(\cdot) \) were not treated, but it is easier and we refer to the original paper for a proof:

\[
h_p(N_2) \leq h_p(\text{Ch}_V).
\]
Finally, \[
    h(N_2) = \sum_{p \text{ prime}} h_p(N_2) + h_\infty(N_2),
\]
\[
    \leq \sum_p h_p(CH_V) + m(CH_V, S_{n+1}) + \deg(V)(\log(n+2) + \sum_{j=1}^n \frac{1}{2j})
    + (d_1 + d_2) \log(n+1) + \log(d_1)
\]

After simplifications, such as \(d_1 + d_2 \leq d_1 d_2 = d_V\) (assuming \(\max\{d_1, d_2\} > 1\)) and using the definition of the height of a variety \(15\)
\[
    h(N_2) \leq h_V + 5d_V \log(n+2)
\]
(true in general)

This is the upper-bound presented in the introduction.

4th step. Using intrinsic bounds to get extrinsic ones through the Arithmetic Bezout theorem. If \(V = Z((f_1, \ldots, f_n))\), with \(\max_{i} \{\text{tdeg}(f_i)\} = d\) and \(\max_{i} \{h(f_i)\} = h\), then \(d_V \leq d^n\) by the standard Bezout theorem, by the ABT \(16\)
\[
    h_V \leq nd^{n-1}(h + 2d\log(n+1))
\]

Using these inequalities in the upper-bound for \(h(N_2)\) just obtained above gives:
\[
    h(N_2) \leq d^n(nh + 7\log(n+2)) = O(nhd^n).
\]

4 Toward the positive dimension

We assume here that \(V\) is of positive dimension \(m > 0\), equidimensional. For convenience we introduce the variables \(Y = Y_1, \ldots, Y_m\) along with the usual \(X = X_1, \ldots, X_n\) ones. We make the assumption:

Assumption 2 The projection of all irreducible components \(V\) on the \(Y\)-space is dense (for the Zariski topology).

Let \(K = \mathbb{Q}(Y)\). Under Assumption 2, the variety \(V^* \subset \overline{K}^n\) defined by the same defining equations of \(V\) after scalar extension from \(\mathbb{Q}[Y, X_1, \ldots, X_n]\) to \(\mathbb{Q}(Y)[X_1, \ldots, X_n]\) is of dimension 0. Similarly, we assume that \(V^*\) verifies Assumption 1 that is can be defined by a triangular set \(T_1, \ldots, T_n\) over the field \(K\). We define as in Equation \(8\), the corresponding polynomials \(N_1, \ldots, N_n\). How large the integer coefficients of the \(T_i\)’s and \(N_i\)’s can be in this case?

For \(1 \leq \ell \leq n\), let us write \(N_\ell\) as
\[
    N_\ell = \sum_i \frac{\gamma_{i,\ell}}{\varphi_{i,\ell}} X_1^{i_1} \cdots X_\ell^{i_\ell} + \frac{\gamma_\ell}{\varphi_\ell} X_\ell^{d_\ell}
\]
and \(T_\ell\) as
\[
    T_\ell = \sum_i \frac{\beta_{i,\ell}}{\alpha_{i,\ell}} X_1^{i_1} \cdots X_\ell^{i_\ell} + X_\ell^{d_\ell},
\]
where:
- all multi-indices \(i = (i_1, \ldots, i_\ell)\) satisfy \(i_r < d_r\) for \(r \leq \ell\);
- all polynomials \(\gamma_{i,\ell}, \varphi_{i,\ell}, \gamma_\ell\) and \(\varphi_\ell\), and \(\beta_{i,\ell}, \alpha_{i,\ell}\), are in \(\mathbb{Z}[Y]\);
in $\mathbb{Z}[Y]$, the equalities $\gcd(\gamma_{i,\ell}, \varphi_{i,\ell}) = \gcd(\gamma_{\ell}, \varphi_{\ell}) = \gcd(\beta_{i,\ell}, \alpha_{i,\ell}) = \pm 1$ hold.

Then, all polynomials $\gamma_{\ell}$ and $\gamma_{i,\ell}$, $\varphi_{i,\ell}$ and $\varphi_{\ell}$, as well as the LCM of all $\varphi_{i,\ell}$ and $\varphi_{\ell}$, have degree bounded by $d_{\ell}$ and height bounded by

$$\mathcal{H}_{\ell} \leq 2h_{\ell} + ((4m + 2)d_{\ell} + 4m) \log(d_{\ell}) + ((10m + 16)d_{\ell} + 5(2m + 2)m) \log(m + \ell + 3).$$

All polynomials $\beta_{i,\ell}$ and $\alpha_{i,\ell}$, as well as the LCM of all $\alpha_{i,\ell}$, have degree bounded by $2d_{\ell}^2$ and height bounded by

$$\mathcal{H}_{\ell}' \leq 4d_{\ell}^2 h_{\ell} + 3d_{\ell}^2 + 4(2m + 1)d_{\ell} + m(d_{\ell} + 1)) \log(d_{\ell} + 1) + ((20m + 22)d_{\ell}^2 + 5(d_{\ell} + \ell + m)) \log(m + \ell + 3).$$

Here again, the polynomials $N_i$'s enjoy a better behavior in term of complexity bounds (linear, versus quadratic for the polynomials $T_i$'s), also experimentally observed.

**Strategy of proof**  Regarding Assumption 2 made above, the idea is naturally to reduce to dimension 0 by *specialization* of the variables $Y$, use the upper-bounds that were just proven in § 2 and then lift back the variables $Y$ while evaluating the growth of the coefficients of the overall process.

While the strategy is simple, several technical difficulties arise. We introduce some new notations in order to detail them more. Let $f_1, \ldots, f_n \in \mathbb{Q}[Y, X]$ an input polynomial system defining $V$. Let $y = (y_1, \ldots, y_m)$ be a point in $\mathbb{Q}^m$. For convenience, the polynomials $f_{i,y}$ will denote the evaluation of the variables $Y$ at the pint $y$: $f_{i,y} := f_i(y_1, \ldots, Y_m, X_1, \ldots, X_n)$. We say that $y = (y_1, \ldots, y_m)$ is a "good" specialization point if:

- the polynomials in $K$ at the denominators of the polynomials in $T$ do not vanish at $y$; that is the polynomials $T_{i,y}$ are well-defined.
- the triangular set $(Y_1 - y_1, \ldots, Y_m - y_m, T_{1,y}(X_1), T_{2,y}(X_1, X_2), \ldots, T_{n,y}(X_1, \ldots, X_n))$ is the reduced Gröbner basis of the variety $V_y := V \cap Z((Y_1 - y_1, \ldots, Y_m - y_m))$

It is easy to show that there are reasonably "small" enough good specialization points $\bar{y}$ (Cf. [3, Prop. 8]). Using the bounds in the case of dimension 0, upper-bounds on the heights of the $T_{i,y}$ and $N_{i,y}$ can be deduced, but in term of the *monic* Chow form $\overline{Ch_y}$ of $V_y$, that lies in $K[U_0]$. However, interesting Chow forms of $V$ lie in $\mathbb{Z}[U_0, \ldots, U_m]$, and it is necessary to link both Chow forms (Cf. § 6 of [3]). A particularly nice case is when $V$ verifies the following:

**Assumption 3** The degree of the projection fiber is equal to the degree of the variety.

It is the case for the example of the figures of § 2.2. The circle is a degree 2 variety and in each fiber of its projection on the $Y$-axis, there are generically 2 points.

Then given a Chow form of $V$, $Ch_V \in \mathbb{Z}[U_0, \ldots, U_m]$ if we perform the following specialization in $Ch_V$:

$$\begin{bmatrix}
U_0 \\
U_1 \\
\vdots \\
U_m \\
\end{bmatrix} \rightarrow \\
\begin{bmatrix}
U_0 \\
Y_1 \\
\vdots \\
Y_m \\
\end{bmatrix} \rightarrow \\
\begin{bmatrix}
0 & \cdots & 0 & U_1 & \cdots & U_n \\
-1 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & -1 & 0 & \cdots & 0 \\
\end{bmatrix}$$

the resulting polynomial that is in indeed in $\mathbb{Z}[Y, U]$, is a Chow form of $V'$. Unfortunately, considering varieties verifying Assumption 2, that are the subject of this study, Assumption 3 is not automatically
verified (Cf. Example following Prop. 2 in [3]). To circumvent this problem, a generic linear change of the coordinates $Y$ is necessary (the full details can not be explained briefly, Cf § 5 in [3]).

This permits to deduce an estimates on $\overline{Ch_Y}$ in function of $Ch_Y$, required to pursue the computations of the upper-bound on the height of $N_{i_Y}$ in function of the variety $V$.

5 One application and some remarks

Besides the understanding of the triangular representations of algebraic varieties, the bounds presented in this article find a natural application in the context of modular methods.

Probability estimates for modular computations Let us say that this consists here roughly in performing the computations modulo a prime $p$ instead of over $\mathbb{Q}$, aiming at staying with reasonably small coefficients. For polynomial system solving with rational coefficients, large numbers is typically a strong bottleneck.

Of course, a prime $p$ must guarantee compatibility between the reduction modulo $p$ of the resulting polynomial system computed over $\mathbb{Q}$, and the resulting one computed modulo $p$. Such primes are called compatible primes in [8]. There are a lot of compatible primes, but the question is rather, Are there a lot of small compatible primes ?

Without bounds like the ones written here, that give an idea of the size of the ouptut, no quantification of the choice of compatible primes is possible. It is convenient here to make a parallel with a classical problem in linear algebra: the inversion of a non-singular matrix. This operation typically increases the coefficients, and modular computations are routinely implemented. Then, the Hadamard's inequality give a quantification in the choice of compatible primes for the inversion operation. The bounds given here play the same role as does the Hadamard's inequality in linear algebra, but in polynomial systems.

Concluding remarks The bounds provided are the first one polynomial w.r.t. the degree and the height of the variety. The hypotheses required are quite strong, but they constitute a first step toward hopefully a whole generality. Nonetheless, using triangular decomposition permits to lever Assumption 1 up. Indeed, the equiprojectable decomposition [4] of $V$ permits to reduce the general case of a 0-dimensional variety to the ones verifying Assumption 1. We mention that in 2 variables, similar results for the lexicographic Gröbner bases of 0-dimensional varieties that do not verify Assumption 1 have been achieved [2].

How tight are the bounds ? We have no answer to this question. Still, we believe that the growth rate of the degree and the height in these bounds is tight.

All the results are based on the Lagrange interpolation allowed by the elimination property hold by lexicographic orders. Somehow, this property is hold also by the degree lexicographic monomial orders. It would be interesting, and quite challenging, to obtain upper-bounds on the coefficients of degree lexicographic Gröbner bases. They are indeed widely used in practice due to a better computational efficiency.
References


