数理解析研究所講究録
第1761巻 2011年 1-18

On a mathematical analysis of tsunami generation in shallow water due to seabed deformation

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1 Introduction

Tsunamis are known as one of disastrous phenomena of water waves and characterized by having very long wavelength. They are generated mainly by a sudden deformation of the seabed with a submarine earthquake. The motion of tsunamis can be modeled as an irrotational flow of an incompressible ideal fluid bounded from above by a free surface and from below by a moving bottom under the gravitational field. The model is usually called the full water wave problem. Because of complexities of the model, several simplified models have been proposed and used to simulate tsunamis. One of the most common models of tsunami propagation is the shallow water model under the assumptions that the initial displacement of the water surface is equal to the permanent shift of the seabed and that the initial velocity field is equal to zero. Namely, in numerical computations of tsunamis due to submarine earthquakes, one usually uses the shallow water equations

$$(1.1)\qquad \begin{cases} \eta_t + \nabla\cdot\big((h+\eta-b_1)u\big) = 0, \\ u_t + (u\cdot\nabla)u + g\nabla\eta = 0 \end{cases}$$

under the following particular initial conditions

$$(1.2)\qquad \eta|_{t=0} = b_1 - b_0, \qquad u|_{t=0} = 0,$$

where η is the variation of the water surface, u is the velocity of the water in the horizontal directions, g is the gravitational constant, h is the mean depth of the water, b_0 is the bottom topography before the submarine earthquake, and b_1 is that after the earthquake. The aim of this communication is to give a mathematically rigorous justification of this shallow water model starting from the full water wave problem, especially, the justification of the initial conditions (1.2).

In this communication two non-dimensional parameter δ and ε play an important role, where δ is the ratio of the water depth h to the wave length λ and ε is the ratio of the duration t_0 of the submarine earthquake to the period of tsunami $\lambda/\sqrt{gh}$. We note that $\sqrt{gh}$ is the propagation speed of linear shallow water waves and that the duration of the seabed deformation is very short compared to the period of tsunamis in general. Therefore, ε should be a small parameter. It is known that the shallow water equations (1.1) are derived from the full water wave problem in the limit $\delta \to +0$. The derivation goes back to G.B. Airy [1]. Then, K.O. Friedrichs [6] derived systematically the equations by using an expansion of the solution with respect to δ^2. See also H. Lamb [13] and J.J. Stoker [19]. A mathematically rigorous justification of the shallow water approximation for two-dimensional water waves over a flat bottom was given by L.V. Ovsjannikov [17, 18] under the periodic boundary condition with respect to the horizontal spatial variable, and then by T. Kano and T. Nishida [11] in a class of analytic functions. See also [12, 10]. The justification in Sobolev spaces was given by Y.A. Li [15] for two-dimensional water waves over a flat bottom and by B. Alvarez-Samaniego and D. Lannes [4] and the author [8] for three-dimensional water waves where non-flat bottoms were allowed. However, there is no rigorous result concerning the shallow water approximation in the case of moving bottom nor the justification of the initial conditions (1.2).

In this communication we will present that under appropriate conditions on the initial data and the bottom topography the solution of the full water wave problem can be approximated by the solution of the tsunami model (1.1) and (1.2) in the limit $\delta, \varepsilon \to +0$ under the restriction $\delta^2/\varepsilon \to +0$. This means that if the speed of seabed deformation is fast but not too fast, then the tsunami model would be a good approximation to the full water wave problem. Moreover, we also present that in the critical limit $\delta, \varepsilon \to +0$ and $\delta^2/\varepsilon \to \sigma$ with a positive constant σ the initial conditions (1.2) should be replaced by

$$\eta|_{t=0} = b_1 - b_0, \qquad u|_{t=0} = \nabla\biggl(\frac{1}{2}\int_0^{t_0} b_t(\cdot,t)^2 dt\biggr), \tag{1.3}$$

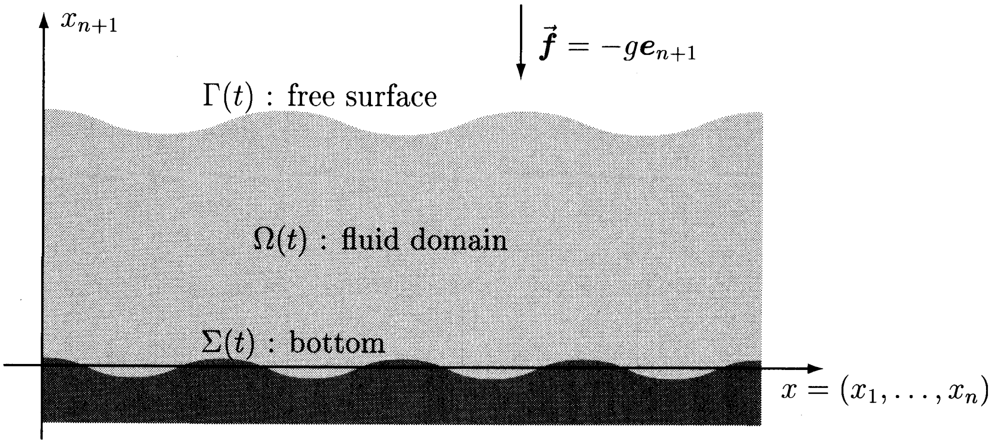
where $b = b(x,t)$ is a bottom topography during the deformation of the seabed. One of the hardest parts of the analysis is to derive a uniform bound of the solution with respect to small parameters δ and ε for the full water wave problem together with its derivatives, especially, for the time interval $0 \le t \le \varepsilon$ when the deformation of the seabed takes place. To this end, we adopt and extend the techniques used by the author [8].

2 Formulation of the Problem

We proceed to formulate the problem mathematically. Let $x = (x_1, x_2, \dots, x_n)$ be the horizontal spatial variables and x_{n+1} the vertical spatial variable. We denote by $X = (x, x_{n+1}) = (x_1, \dots, x_n, x_{n+1})$ the whole spatial variables. We will consider a water wave in $(n+1)$-dimensional space and assume that the domain $\Omega(t)$ occupied by the water at time t, the water surface $\Gamma(t)$, and the bottom $\Sigma(t)$ are of the forms

$$\begin{aligned}\Omega(t) &= \{X = (x, x_{n+1}) \in \mathbf{R}^{n+1}\,;\, b(x,t) < x_{n+1} < h + \eta(x,t)\},\\ \Gamma(t) &= \{X = (x, x_{n+1}) \in \mathbf{R}^{n+1}\,;\, x_{n+1} = h + \eta(x,t)\},\\ \Sigma(t) &= \{X = (x, x_{n+1}) \in \mathbf{R}^{n+1}\,;\, x_{n+1} = b(x,t)\},\end{aligned}$$

where h is the mean depth of the water. The shape of the fluid region is shown in the following illustration.



The functions b and η represent the bottom topography and the surface elevation, respectively. It is very important to predict the deformation process of the seabed, so that we have to analyze the behavior of this function b. However, in this communication we assume that b is a given function and we concentrate our attention on analyzing the behavior of the function η, namely, the water surface.

We assume that the water is incompressible and inviscid fluid, and that the flow is irrotational. Then, the motion of the water is described by the velocity potential $\Phi = \Phi(X,t)$ satisfying the equation

$$\Delta_X \Phi = 0 \quad \text{in} \quad \Omega(t), \tag{2.1}$$

where Δ_X is the Laplacian with respect to X, that is, $\Delta_X = \Delta + \partial_{n+1}^2$ and $\Delta = \partial_1^2 + \cdots + \partial_n^2$. The boundary conditions on the water surface are given by

$$\begin{cases} \eta_t + \nabla\Phi \cdot \nabla\eta - \partial_{n+1}\Phi = 0, \\ \Phi_t + \frac{1}{2}|\nabla_X \Phi|^2 + g\eta = 0 \quad \text{on} \quad \Gamma(t), \end{cases} \tag{2.2}$$

where $\nabla = (\partial_1, \ldots, \partial_n)^T$ and $\nabla_X = (\partial_1, \ldots, \partial_n, \partial_{n+1})^T$ are the gradients with respect to $x = (x_1, \ldots, x_n)$ and to $X = (x, x_{n+1})$, respectively, and g is the gravitational constant. The first equation is the kinematical condition and the second one is the restriction of

Bernoulli's law on the water surface. The kinematical boundary condition on the bottom is given by

$$b_t + \nabla\Phi\cdot\nabla b - \partial_{n+1}\Phi = 0 \quad \text{on} \quad \Sigma(t). \tag{2.3}$$

Finally, we impose the initial conditions

$$\eta = \eta_0, \quad \Phi = \Phi_0 \quad \text{at} \quad t = 0. \tag{2.4}$$

These are the basic equations for the full water wave problem.

Next, we rewrite the equations (2.1)–(2.3) in an appropriate non-dimensional form. Let λ be the typical wave length and h the mean depth. We introduce a non-dimensional parameter δ by $\delta = h/\lambda$ and rescale the independent and dependent variables by

$$x = \lambda\tilde{x}, \quad x_{n+1} = h\tilde{x}_{n+1}, \quad t = \frac{\lambda}{\sqrt{gh}}\tilde{t}, \quad \Phi = \lambda\sqrt{gh}\tilde{\Phi}, \quad \eta = h\tilde{\eta}, \quad b = h\tilde{b}. \tag{2.5}$$

Putting these into (2.1)–(2.3) and dropping the tilde sign in the notation we obtain

$$\delta^2\Delta\Phi + \partial_{n+1}^2\Phi = 0 \quad \text{in} \quad \Omega(t), \tag{2.6}$$

$$\begin{cases} \delta^2\big(\eta_t + \nabla\Phi\cdot\nabla\eta\big) - \partial_{n+1}\Phi = 0, \\ \delta^2\Big(\Phi_t + \dfrac{1}{2}|\nabla\Phi|^2 + \eta\Big) + \dfrac{1}{2}(\partial_{n+1}\Phi)^2 = 0 \quad \text{on} \quad \Gamma(t), \end{cases} \tag{2.7}$$

$$\delta^2\big(b_t + \nabla\Phi\cdot\nabla b\big) - \partial_{n+1}\Phi = 0 \quad \text{on} \quad \Sigma(t), \tag{2.8}$$

where

$$\begin{aligned} \Omega(t) &= \{X = (x, x_{n+1}) \in \mathbf{R}^{n+1}\,;\, b(x,t) < x_{n+1} < 1 + \eta(x,t)\}, \\ \Gamma(t) &= \{X = (x, x_{n+1}) \in \mathbf{R}^{n+1}\,;\, x_{n+1} = 1 + \eta(x,t)\}, \\ \Sigma(t) &= \{X = (x, x_{n+1}) \in \mathbf{R}^{n+1}\,;\, x_{n+1} = b(x,t)\}. \end{aligned}$$

Moreover, we assume that the seabed deforms only for time interval $[0, t_0]$ in the dimensional variable t, so that the function $b = b(x,t)$ which represents the bottom topography can be written in the form

$$b(x,t) = \beta(x, t/\varepsilon), \qquad \beta(x,\tau) = \begin{cases} b_0(x) & \text{for} \quad \tau \le 0, \\ b_1(x) & \text{for} \quad \tau \ge 1 \end{cases} \tag{2.9}$$

in the non-dimensional variables, where ε is a non-dimensional parameter defined by

$$\varepsilon = \frac{t_0}{\lambda/\sqrt{gh}}.$$

We note that in this non-dimensional time variable the bottom deforms only for the short time interval $0 \le t \le \varepsilon$ and it holds that $b_t = \varepsilon^{-1}\beta_\tau$. Since we are interested in asymptotic behavior of the solution when $\delta, \varepsilon \to +0$, we always assume $0 < \delta, \varepsilon \le 1$ in the following.

As in the usual way, we transform equivalently the initial value problem (2.6)–(2.8) and (2.4) to a problem on the water surface. To this end, we introduce a Dirichlet-to-Neumann map Λ^{DN} and a Neumann-to-Neumann map Λ^{NN} in the following way. In the definition the time t is arbitrarily fixed, so that we omit to write the dependence of t.

Definition 2.1 Under appropriate assumptions on η and b, for any functions ϕ on the water surface Γ and γ on the seabed Σ in some classes, the boundary value problem

$$\begin{cases} \Delta\Phi + \delta^{-2}\partial_{n+1}^2\Phi = 0 & \text{in } \Omega, \\ \Phi = \phi & \text{on } \Gamma, \\ \delta^{-2}\partial_{n+1}\Phi - \nabla b\cdot\nabla\Phi = \gamma & \text{on } \Sigma \end{cases} \tag{2.10}$$

has a unique solution Φ. Using the solution we define $\Lambda^{\mathrm{DN}}(\eta,b,\delta)$ and $\Lambda^{\mathrm{NN}}(\eta,b,\delta)$ by

$$\begin{aligned} \Lambda^{\mathrm{DN}}(\eta,b,\delta)\phi + \Lambda^{\mathrm{NN}}(\eta,b,\delta)\gamma &= \delta^{-2}(\partial_{n+1}\Phi)(\cdot,1+\eta(\cdot)) - \nabla\eta\cdot(\nabla\Phi)(\cdot,1+\eta(\cdot)) \\ &= (\delta^{-2}\partial_{n+1}\Phi - \nabla\eta\cdot\nabla\Phi)|_{\Gamma}. \end{aligned} \tag{2.11}$$

The solution Φ will be denoted by $(\phi,\gamma)^{\hbar}$.

We should remark that both of the maps $\Lambda^{\mathrm{DN}} = \Lambda^{\mathrm{DN}}(\eta,b,\delta)$ and $\Lambda^{\mathrm{NN}} = \Lambda^{\mathrm{NN}}(\eta,b,\delta)$ are linear operators acting on ϕ and γ, respectively. However, they depend also on the unknown function η and the dependence on η is strongly nonlinear.

Now, we introduce a new unknown function ϕ by

$$\phi(x,t) = \Phi(x,1+\eta(x,t),t) = \Phi|_{\Gamma(t)}, \tag{2.12}$$

which is the trace of the velocity potential on the water surface. Then, it holds that

$$\begin{cases} \phi_t = (\Phi_t + (\partial_{n+1}\Phi)\eta_t)|_{\Gamma(t)}, \\ \nabla\phi = (\nabla\Phi + (\partial_{n+1}\Phi)\nabla\eta)|_{\Gamma(t)}. \end{cases} \tag{2.13}$$

On the other hand, it follows from (2.6), (2.8), and (2.12) that Φ satisfies the boundary value problem (2.10) with γ replaced by $b_t = \varepsilon^{-1}\beta_\tau$, so that we have

$$\Lambda^{\mathrm{DN}}\phi + \varepsilon^{-1}\Lambda^{\mathrm{NN}}\beta_\tau = (\delta^{-2}\partial_{n+1}\Phi - \nabla\eta\cdot\nabla\Phi)|_{\Gamma(t)}. \tag{2.14}$$

These relations (2.13) and (2.14) imply that

$$\begin{cases} (\partial_{n+1}\Phi)|_{\Gamma(t)} = \delta^2(1+\delta^2|\nabla\eta|^2)^{-1}(\Lambda^{\mathrm{DN}}\phi + \varepsilon^{-1}\Lambda^{\mathrm{NN}}\beta_\tau + \nabla\eta\cdot\nabla\phi), \\ (\nabla\Phi)|_{\Gamma(t)} = \nabla\phi - \delta^2(1+\delta^2|\nabla\eta|^2)^{-1}(\Lambda^{\mathrm{DN}}\phi + \varepsilon^{-1}\Lambda^{\mathrm{NN}}\beta_\tau + \nabla\eta\cdot\nabla\phi)\nabla\eta, \\ \Phi_t|_{\Gamma(t)} = \phi_t - \delta^2(1+\delta^2|\nabla\eta|^2)^{-1}(\Lambda^{\mathrm{DN}}\phi + \varepsilon^{-1}\Lambda^{\mathrm{NN}}\beta_\tau + \nabla\eta\cdot\nabla\phi)\eta_t. \end{cases}$$

Putting these into (2.7) we see that η and ϕ satisfy the following initial value problem.

$$\begin{cases} \eta_t - \Lambda^{\mathrm{DN}}(\eta,b,\delta)\phi - \varepsilon^{-1}\Lambda^{\mathrm{NN}}(\eta,b,\delta)\beta_\tau = 0, \\ \phi_t + \eta + \frac{1}{2}|\nabla\phi|^2 \\ \quad -\frac{1}{2}\delta^2(1+\delta^2|\nabla\eta|^2)^{-1}\big(\Lambda^{\mathrm{DN}}(\eta,b,\delta)\phi + \varepsilon^{-1}\Lambda^{\mathrm{NN}}(\eta,b,\delta)\beta_\tau + \nabla\eta\cdot\nabla\phi\big)^2 = 0, \end{cases} \tag{2.15}$$

$$\eta = \eta_0, \quad \phi = \phi_0 \quad \text{at} \quad t = 0, \tag{2.16}$$

where the initial datum ϕ_0 is determined by $\phi_0 = \Phi_0(\cdot, 1 + \eta_0(\cdot))$. We will investigate this initial value problem (2.15) and (2.16) mathematically rigorously in this communication.

3 A shallow water approximation

We proceed to study formally asymptotic behavior of the solution $(\eta^{\delta,\varepsilon}, \phi^{\delta,\varepsilon})$ to the initial value problem (2.15) and (2.16) when $\delta, \varepsilon \to +0$ and derive the shallow water equations, whose solution approximates $(\eta^{\delta,\varepsilon}, \phi^{\delta,\varepsilon})$ in a suitable sense.

3.1 The case $\beta_\tau \equiv 0$

First, we consider the case $\beta_\tau \equiv 0$ where the seabed is fixed in time. It follows from the second equation in (2.15) that

$$\phi_t + \eta + \frac{1}{2}|\nabla\phi|^2 = O(\delta^2).$$

In order to derive an approximate equation to the first equation in (2.15) we need to expand the Dirichlet-to-Neumann map $\Lambda^{\rm DN} = \Lambda^{\rm DN}(\eta, b, \delta)$ with respect to δ^2. For a given function ϕ on Γ, we denote by Φ the solution of the boundary value problem

$$\begin{cases} \Delta\Phi + \delta^{-2}\partial_{n+1}^2\Phi = 0 & \text{in } \Omega, \\ \Phi = \phi & \text{on } \Gamma, \\ \delta^{-2}\partial_{n+1}\Phi - \nabla b \cdot \nabla\Phi = 0 & \text{on } \Sigma. \end{cases} \tag{3.1}$$

Here and in what follows, for simplicity we omit to write the dependence of the time t in the notation. By the first and the third equations in (3.1),

$$\begin{aligned} (\partial_{n+1}\Phi)(x, x_{n+1}, t) &= (\partial_{n+1}\Phi)(x, b(x), t) + \int_{b(x)}^{x_{n+1}} (\partial_{n+1}^2\Phi)(x, z, t)\mathrm{d}z \\ &= \delta^2\nabla b(x) \cdot \nabla\Phi(x, b(x), t) - \delta^2 \int_{b(x)}^{x_{n+1}} (\Delta\Phi)(x, z, t)\mathrm{d}z, \end{aligned} \tag{3.2}$$

which implies that $(\partial_{n+1}\Phi)(X, t) = O(\delta^2)$. Therefore,

$$\begin{aligned} \nabla\Phi(x, x_{n+1}, t) &= \nabla\Phi(x, 1 + \eta(x, t), t) + \int_{1+\eta(x,t)}^{x_{n+1}} (\nabla\partial_{n+1}\Phi)(x, z, t)\mathrm{d}z \\ &= \nabla\Phi(x, 1 + \eta(x, t), t) + O(\delta^2). \end{aligned}$$

Moreover, by the second equations in (3.1) it holds that

$$\begin{aligned} \nabla\phi(x, t) &= \nabla\Phi(x, 1 + \eta(x, t), t) + \nabla\eta(x)(\partial_{n+1}\Phi)(x, 1 + \eta(x), t) \\ &= \nabla\Phi(x, 1 + \eta(x, t), t) + O(\delta^2) \\ &= \nabla\Phi(X, t) + O(\delta^2). \end{aligned}$$

Similarly, we have

$$\Delta\phi(x,t) = \Delta\Phi(X,t) + O(\delta^2).$$

These relations and (3.2) imply that

$$\begin{aligned}(\partial_{n+1}\Phi)(x,1+\eta(x,t),t) &= \delta^2\nabla b(x)\cdot\nabla\phi(x,t) - \delta^2\int_{b(x)}^{1+\eta(x,t)}\Delta\phi(x,t)\mathrm{d}z + O(\delta^4)\\ &= -\delta^2(1+\eta(x,t))\Delta\phi(x,t) + \delta^2\nabla\cdot\big(b(x)\nabla\phi(x,t)\big) + O(\delta^4).\end{aligned}$$

Hence, by the definition (2.11) with $\gamma = 0$ we have

$$\big(\Lambda^{\mathrm{DN}}(\eta,b,\delta)\phi\big)(x,t) = -\nabla\cdot\big((1+\eta(x,t)-b(x))\nabla\phi(x,t)\big) + O(\delta^2). \tag{3.3}$$

This formal expansion of the operator $\Lambda^{\mathrm{DN}} = \Lambda^{\mathrm{DN}}(\eta,b,\delta)$ with respect to δ^2 can be justified mathematically by the following lemma.

Lemma 3.1 ([8]). *Let $M, c > 0$ and $s > n/2$. There exist positive constants C and δ_1 such that for any $\delta \in (0,\delta_1]$ and $\eta, b \in H^{s+2+1/2}(\mathbf{R}^n)$ satisfying*

$$\begin{cases} \|b\|_{s+2+1/2} + \|\eta\|_{s+2+1/2} \le M,\\ 1+\eta(x)-b(x) \ge c \quad \textit{for} \quad x\in\mathbf{R}^n,\end{cases}$$

we have

$$\|\Lambda^{\mathrm{DN}}(\eta,b,\delta)\phi + \nabla\cdot((1+\eta-b)\nabla\phi)\|_s \le C\delta^2\|\nabla\phi\|_{s+3}.$$

The first equation in (2.15) and (3.3) imply that

$$\eta_t + \nabla\cdot\big((1+\eta-b)\nabla\phi\big) = O(\delta^2).$$

To summarize, we have derived the partial differential equations

$$\begin{cases} \eta_t + \nabla\cdot\big((1+\eta-b)\nabla\phi\big) = O(\delta^2),\\ \phi_t + \eta + \frac{1}{2}|\nabla\phi|^2 = O(\delta^2),\end{cases}$$

which approximate the equations in (2.15) up to order δ^2. Letting $\delta \to 0$ in the above equations we obtain

$$\begin{cases} \eta^0_t + \nabla\cdot\big((1+\eta^0-b)\nabla\phi^0\big) = 0,\\ \phi^0_t + \eta^0 + \frac{1}{2}|\nabla\phi^0|^2 = 0.\end{cases}$$

Finally, putting $u^0 := \nabla\phi^0$ and taking the gradient of the second equation, we are led to the shallow water equations

$$\begin{cases} \eta^0_t + \nabla\cdot\big((1+\eta^0-b)u^0\big) = 0,\\ u^0_t + (u^0\cdot\nabla)u^0 + \nabla\eta^0 = 0.\end{cases}$$

Moreover, u^0 satisfies the irrotational condition

$$\operatorname{rot} u^0 = 0,$$

where $\operatorname{rot} u$ is the rotation of $u = (u_1,\dots,u_n)^T$ defined by $\operatorname{rot} u = (\partial_j u_i - \partial_i u_j)_{1\le i,j\le n}$.

3.2 The case $\beta_\tau \neq 0$

Next, we consider the general case where the seabed may deform with time. In order to derive approximate equations to (2.15) we need to expand the Neumann-to-Neumann map $\Lambda^{\mathrm{NN}} = \Lambda^{\mathrm{NN}}(\eta, b, \delta)$ with respect to δ^2. For a given function γ on Σ, we denote by Φ the solution of the boundary value problem

$$\begin{cases} \Delta\Phi + \delta^{-2}\partial_{n+1}^2\Phi = 0 & \text{in } \Omega, \\ \Phi = 0 & \text{on } \Gamma, \\ \delta^{-2}\partial_{n+1}\Phi - \nabla b \cdot \nabla\Phi = \gamma & \text{on } \Sigma. \end{cases}$$

Then, we see that

$$(3.4) \qquad \begin{aligned}(\partial_{n+1}\Phi)(x, x_{n+1}) &= (\partial_{n+1}\Phi)(x, b(x)) + \int_{b(x)}^{x_{n+1}} (\partial_{n+1}^2\Phi)(x,z)\mathrm{d}z \\ &= \delta^2\gamma(x) + \delta^2\nabla b(x)\cdot(\nabla\Phi)(x,b(x)) - \delta^2\int_{b(x)}^{x_{n+1}}(\Delta\Phi)(x,z)\mathrm{d}z,\end{aligned}$$

which implies that $(\partial_{n+1}\Phi)(X) = O(\delta^2)$ and that $(\nabla\partial_{n+1}\Phi)(X) = O(\delta^2)$. This and the relation

$$(3.5) \qquad (\nabla\Phi)(x, x_{n+1}) = (\nabla\Phi)(x, 1+\eta(x)) + \int_{1+\eta(x)}^{x_{n+1}} (\nabla\partial_{n+1}\Phi)(x,z)\mathrm{d}z$$

imply that $(\nabla\Phi)(X) = (\nabla\Phi)(x, 1+\eta(x)) + O(\delta^2)$. Differentiating the Dirichlet boundary condition $\Phi(x, 1+\eta(x)) = 0$ on Γ we obtain

$$(3.6) \qquad (\nabla\Phi)(x, 1+\eta(x)) = -(\partial_{n+1}\Phi)(x, 1+\eta(x))\nabla\eta(x),$$

which is $O(\delta^2)$. Therefore, we obtain $\nabla\Phi(X) = O(\delta^2)$ so that $\Delta\Phi(X) = O(\delta^2)$. It follows from these relation and (3.4) that $(\partial_{n+1}\Phi)(X) = \delta^2\gamma(x) + O(\delta^4)$, which together with (3.6) implies that $(\nabla\Phi)(x, 1+\eta(x)) = -\delta^2\gamma(x)\nabla\eta(x) + O(\delta^4)$. Thus, by (3.5) we obtain

$$(\nabla\Phi)(X) = -\delta^2\gamma(x)\nabla\eta(x) - \delta^2(1+\eta(x)-x_{n+1})\nabla\gamma(x) + O(\delta^4).$$

Particularly, it holds that

$$(\Delta\Phi)(X) = -\delta^2\nabla\cdot\big(\gamma(x)\nabla\eta(x)\big) - \delta^2\nabla\eta(x)\cdot\nabla\gamma(x) - \delta^2(1+\eta(x)-x_{n+1})\Delta\gamma(x) + O(\delta^4).$$

Therefore, by (3.4) we get

$$\begin{aligned}(\partial_{n+1}\Phi)(X) = \delta^2\gamma(x) &- \delta^4\nabla b(x)\cdot\big(\gamma(x)\nabla\eta(x) + (1+\eta(x)-b(x))\nabla\gamma(x)\big) \\ &+ \delta^4(x_{n+1}-b(x))\big(\nabla\cdot(\gamma(x)\nabla\eta(x)) + \nabla\eta(x)\cdot\nabla\gamma(x)\big) \\ &- \frac{1}{2}\delta^4\big((1+\eta(x)-x_{n+1})^2 - (1+\eta(x)-b(x))^2\big)\Delta\gamma(x) + O(\delta^6).\end{aligned}$$

Hence, by the definition (2.11) with $\phi = 0$ we have

$$\Lambda^{\text{NN}}(\eta, b, \delta)\beta = \gamma + \delta^2\nabla \cdot \Big((1+\eta-b)(\nabla\eta)\gamma + \frac{1}{2}(1+\eta-b)^2\nabla\gamma\Big) + O(\delta^4). \tag{3.7}$$

This formal expansion of the operator $\Lambda^{\text{NN}} = \Lambda^{\text{NN}}(\eta, b, \delta)$ with respect to δ^2 can be justified mathematically by the following lemma.

Lemma 3.2 ([9]). *Let $M, c > 0$ and $s > n/2 - 2$. There exist positive constants C and δ_1 such that for any $\delta \in (0, \delta_1]$ and $\eta, b \in H^{s+4}(\mathbf{R}^n)$ satisfying*

$$\begin{cases} \|b\|_{s+4} + \|\eta\|_{s+4} \le M, \\ 1 + \eta(x) - b(x) \ge c \quad \textit{for} \quad x \in \mathbf{R}^n, \end{cases}$$

we have

$$\Big\|\Lambda^{\text{NN}}(\eta, b, \delta)\gamma - \gamma - \delta^2\nabla \cdot \Big((1+\eta-b)(\nabla\eta)\gamma + \frac{1}{2}(1+\eta-b)^2\nabla\gamma\Big)\Big\|_s \le C\delta^4\|\gamma\|_{s+4}.$$

Here, we should remark that the approximation $\Lambda^{\text{NN}}\gamma = \gamma + O(\delta^2)$ is sufficient for a formal argument. However, in order to give a mathematically rigorous result we need to know the term of $O(\delta^2)$ of the map Λ^{NN}.

In view of (3.3) and (3.7), we see that the equations in (2.15) can be approximated by the ordinary differential equations

$$\begin{cases} \eta_t = \frac{1}{\varepsilon}\beta_\tau + \frac{1}{\varepsilon}O(\varepsilon + \delta^2), \\ \phi_t = \frac{1}{2}\big(\frac{\delta}{\varepsilon}\big)^2\beta_\tau^2 + \frac{1}{\varepsilon^2}O(\varepsilon^2 + \delta^4). \end{cases} \tag{3.8}$$

By resolving these equations under the initial conditions (2.16), we obtain

$$\begin{cases} \eta(x,t) = \eta_0(x) + \beta(x, t/\varepsilon) - b_0(x) + O(\varepsilon + \delta^2), \\ \phi(x,t) = \phi_0(x) + \dfrac{1}{2}\dfrac{\delta^2}{\varepsilon}\displaystyle\int_0^{t/\varepsilon} \beta_\tau(x,\tau)^2 \mathrm{d}\tau + \frac{1}{\varepsilon}O(\varepsilon^2 + \delta^4) \end{cases} \tag{3.9}$$

for the time interval $0 \le t \le \varepsilon$. Particularly, we get

$$\begin{cases} \eta(x,\varepsilon) = \eta_0(x) + \big(b_1(x) - b_0(x)\big) + O(\varepsilon + \delta^2), \\ \phi(x,\varepsilon) = \phi_0(x) + \dfrac{1}{2}\dfrac{\delta^2}{\varepsilon}\displaystyle\int_0^{1} \beta_\tau(x,\tau)^2 \mathrm{d}\tau + \frac{1}{\varepsilon}O(\varepsilon^2 + \delta^4). \end{cases} \tag{3.10}$$

As $\delta, \varepsilon \to +0$ these data converge only in the case when δ^2/ε also converges to some value σ. Therefore, in this communication we will consider asymptotic behavior of the solution $(\eta^{\delta,\varepsilon}, \phi^{\delta,\varepsilon})$ to the initial value problem (2.15) and (2.16) in the limit

$$\delta, \varepsilon \to +0, \quad \frac{\delta^2}{\varepsilon} \to \sigma. \tag{3.11}$$

On the other hand, noting that $\beta_\tau = 0$ and $b = b_1$ for $t > \varepsilon$, we see that the equations in (2.15) can be approximated by the partial differential equations

$$\begin{cases} \eta_t + \nabla \cdot \big((1+\eta-b_1)\nabla\phi\big) = O(\delta^2), \\ \phi_t + \eta + \frac{1}{2}|\nabla\phi|^2 = O(\delta^2) \end{cases} \tag{3.12}$$

for $t > \varepsilon$. Therefore, taking the limit (3.11) of (3.12) and (3.10) we obtain

$$\begin{cases} \eta_t^0 + \nabla \cdot \big((1+\eta^0-b_1)\nabla\phi^0\big) = 0, \\ \phi_t^0 + \eta^0 + \frac{1}{2}|\nabla\phi^0|^2 = 0 \end{cases}$$

with initial conditions

$$\eta^0 = \eta_0 + (b_1 - b_0), \quad \phi^0 = \phi_0 + \frac{\sigma}{2}\int_0^1 \beta_\tau(\cdot,\tau)^2 \mathrm{d}\tau \quad \text{at} \quad t = 0.$$

Finally, putting $u^0 := \nabla\phi^0$ and taking the gradient of the second equation, we are led to the shallow water equations

$$\begin{cases} \eta_t^0 + \nabla \cdot \big((1+\eta^0-b_1)u^0\big) = 0, \\ u_t^0 + (u^0\cdot\nabla)u^0 + \nabla\eta^0 = 0 \end{cases} \tag{3.13}$$

with initial conditions

$$\eta^0 = \eta_0 + (b_1 - b_0), \quad u^0 = \nabla\phi_0 + \nabla\Big(\frac{\sigma}{2}\int_0^1 \beta_\tau(\cdot,\tau)^2 \mathrm{d}\tau\Big) \quad \text{at} \quad t = 0. \tag{3.14}$$

Moreover, u^0 satisfies the irrotational condition

$$\operatorname{rot} u^0 = 0. \tag{3.15}$$

Here, we note that in the case $(\eta_0, \phi_0) = 0$ and $\sigma = 0$, if we rewrite (3.13) and (3.14) in the dimensional variables, then we obtain (1.1) and (1.2).

4 Main result

Before giving our main result we need to analyze a generalized Rayleigh–Taylor sign condition. It is known that the well-posedness of the initial value problem (2.1)–(2.4) for water waves may be broken unless a generalized Rayleigh–Taylor sign condition $-\partial p/\partial N \geq c_0 > 0$ on the water surface is satisfied, where p is the pressure and N is the unit outward normal to the water surface. For example, see J.T. Beale, T.Y. Hou, and J.S. Lowengrub [2]. S. Wu [20, 21] showed that this condition always holds for any smooth nonself-intersecting surface in the case of infinite depth. In the case with variable bottom, D. Lannes [14] gave a relation between this condition and the bottom topography. A. Constantin and W. Strauss [3] investigated the pressure of Stokes waves over a flat bottom and proved also that this condition holds for Stokes waves. We also mention the result by D.G. Ebin [5] where he considered a motion close to a rigid rotation of an incompressible ideal fluid surrounded by a free surface and showed that the corresponding initial value problem is ill-posed. In this case, a generalized Rayleigh–Taylor sign is not satisfied. One may think that the vorticity breaks the condition, but even in the irrotational case the condition does not hold in a certain situation. In fact, the author [7] considered an irrotational circulating flow of an incompressible ideal fluid around a rigid obstacle and showed that if the circulation is stranger than the gravity, then a generalized Rayleigh–Taylor sign is not satisfied and the problem is ill-posed. In the following we will consider this important condition in the limit (3.11).

In the dimensional variables we have so-called Bernoulli's law

$$\Phi_t + \frac{1}{2}|\nabla_X \Phi|^2 + \frac{1}{\rho}(p - p_0) + g(x_{n+1} - h) = 0 \quad \text{in} \quad \Omega(t), \tag{4.1}$$

where ρ is a constant density and p_0 is a constant atmospheric pressure. This equation is obtained by integrating the conservation of momentum, that is, the Euler equation $0 = \rho\big(\boldsymbol{v}_t + (\boldsymbol{v}\cdot\nabla_X)\boldsymbol{v}\big) + \nabla_X p + \rho g \boldsymbol{e}_{n+1} = \rho\nabla_X\big(\Phi_t + \frac{1}{2}|\nabla_X\Phi|^2 + \frac{1}{\rho}(p-p_0) + g(x_{n+1}-h)\big)$, where $\boldsymbol{v} = \nabla_X\Phi$ is the velocity and $\boldsymbol{e}_{n+1}$ is the unit vector in the vertical direction. We rescale the pressure p by $p = p_0 + \rho g h \tilde{p}$. Putting this and (2.5) into (4.1) and dropping the tilde sign in the notation we obtain

$$-p = \Phi_t + \frac{1}{2}\big(|\nabla\Phi|^2 + \delta^{-2}(\partial_{n+1}\Phi)^2\big) + (x_{n+1} - 1). \tag{4.2}$$

Moreover, in the non-dimensional variables the generalized Rayleigh–Taylor sign condition can be written in the form $a \geq c_0 > 0$, where

$$\begin{aligned} a &:= -(1+\delta^2|\nabla\eta|^2)^{-1}(\partial_{n+1}p - \delta^2\nabla\eta\cdot\nabla p)|_{\Gamma(t)} \\ &= -(\partial_{n+1}p)|_{\Gamma(t)} \\ &= 1 + \Big\{\partial_{n+1}\Big(\Phi_t + \frac{1}{2}\big(|\nabla\Phi|^2 + \delta^{-2}(\partial_{n+1}\Phi)^2\big)\Big)\Big\}\Big|_{\Gamma(t)} \\ &= 1 + \big(\partial_{n+1}\Phi_t + \nabla\Phi\cdot\nabla\partial_{n+1}\Phi - (\partial_{n+1}\Phi)\Delta\Phi\big)\big|_{\Gamma(t)}, \end{aligned} \tag{4.3}$$

where we used the relation $(\nabla Q)|_{\Gamma(t)} = \nabla(Q|_{\Gamma(t)}) - (\partial_{n+1}Q)|_{\Gamma(t)}\nabla\eta$, the boundary condition on the water surface (2.7), and scaled Laplace's equation (2.6).

We proceed to consider asymptotic behavior of this function a in the limit (3.11), so that we can assume $\delta^2 = O(\varepsilon)$. We note that Φ satisfies (2.6), (2.8), and (2.12), and that we have (2.9). Therefore, as in the same calculation in the previous section we see that

$$\nabla\Phi = \nabla\phi - \frac{\delta^2}{\varepsilon}\beta_\tau\nabla\eta - \frac{\delta^2}{\varepsilon}(1 + \eta - x_{n+1})\nabla\beta_\tau + O(\delta^2)$$

and that

$$
\begin{aligned}
\partial_{n+1}\Phi = \frac{\delta^2}{\varepsilon}\beta_\tau + \delta^2\nabla b\cdot\Big(\nabla\phi - \frac{\delta^2}{\varepsilon}\beta_\tau\nabla\eta - \frac{\delta^2}{\varepsilon}(1+\eta-b)\nabla\beta_\tau\Big) \\
- \delta^2(x_{n+1}-b)\Big(\nabla\cdot\Big(\nabla\phi - \frac{\delta^2}{\varepsilon}\beta_\tau\nabla\eta\Big) - \frac{\delta^2}{\varepsilon}\nabla\eta\cdot\nabla\beta_\tau\Big) \\
- \frac{\delta^2}{2}\frac{\delta^2}{\varepsilon}((1+\eta-x_{n+1})^2 - (1+\eta-b)^2)\Delta\beta_\tau + O(\delta^4).
\end{aligned}
$$

Here, it follows from (3.8) that $\eta_t = \frac{1}{\varepsilon}\beta_\tau + O(1)$, $\phi_t = \frac{1}{2}\left(\frac{\delta}{\varepsilon}\right)^2\beta_\tau^2 + O(1)$, and that $\nabla\phi_t - \frac{\delta^2}{\varepsilon}\beta_\tau\nabla\eta_t = O(1)$. Therefore,

$$
\begin{aligned}
&(\partial_{n+1}\Phi_t)|_{\Gamma(t)} \\
&= \left(\frac{\delta}{\varepsilon}\right)^2(1-\delta^2|\nabla\eta|^2)\beta_{\tau\tau} + \frac{\delta^2}{\varepsilon}\nabla\cdot\Big(\beta_\tau\Big((\nabla\phi - \frac{\delta^2}{\varepsilon}\beta_\tau\nabla\eta)\Big) - \left(\frac{\delta^2}{\varepsilon}\right)^2\beta_\tau\nabla\eta\cdot\nabla\beta_\tau \\
&\quad + \left(\frac{\delta^2}{\varepsilon}\right)^2\nabla\cdot\Big((1+\eta-b)\beta_{\tau\tau}\nabla\eta + \frac{1}{2}(1+\eta-b)^2\nabla\beta_{\tau\tau}\Big) + O(\delta^2).
\end{aligned}
$$

Putting these into (4.3) we obtain

$$
\begin{aligned}
a = 1 + \left(\frac{\delta}{\varepsilon}\right)^2(1-\delta^2|\nabla\eta|^2)\beta_{\tau\tau} + 2\frac{\delta^2}{\varepsilon}\Big(\nabla\phi - \frac{\delta^2}{\varepsilon}\beta_\tau\nabla\eta\Big)\cdot\nabla\beta_\tau \\
+ \left(\frac{\delta^2}{\varepsilon}\right)^2\nabla\cdot\Big((1+\eta-b)(\nabla\eta)\beta_{\tau\tau} + \frac{1}{2}(1+\eta-b)^2\nabla\beta_{\tau\tau}\Big) + O(\delta^2).
\end{aligned} \tag{4.4}
$$

On the other hand, in view of (3.9) and (3.11) we define an approximate solution $(\eta^{(0)}, \phi^{(0)})$ in the fast time scale $\tau = t/\varepsilon$ by

$$
\begin{cases}
\eta^{(0)}(x,\tau) := \eta_0(x) + \beta(x,\tau) - \beta(x,0), \\
\phi^{(0)}(x,\tau) := \phi_0(x) + \dfrac{\sigma}{2}\displaystyle\int_0^\tau \beta_\tau(x,\tilde\tau)^2 \mathrm{d}\tilde\tau.
\end{cases} \tag{4.5}
$$

Then, we have at least formally

$$
\eta(x,t) = \eta^{(0)}(x,t/\varepsilon) + O(\varepsilon), \quad \phi(x,t) = \phi^{(0)}(x,t/\varepsilon) + o(1)
$$

for $(x,t) \in \mathbf{R}^n \times [0,\varepsilon]$. Taking this and (4.4) into account we define a function $a^{(0)} = a^{(0)}(x,\tau)$ by

$$
\begin{aligned}
a^{(0)} := 2(\nabla\phi^{(0)} - \sigma\beta_\tau\nabla\eta^{(0)})\cdot\nabla\beta_\tau \\
+ \sigma\nabla\cdot\Big((1+\eta^{(0)}-\beta)(\nabla\eta^{(0)})\beta_{\tau\tau} + \frac{1}{2}(1+\eta^{(0)}-\beta)^2\nabla\beta_{\tau\tau}\Big),
\end{aligned}
$$

where $(\eta^{(0)}, \phi^{(0)})$ is the approximate solution defined in (4.5). We note that this function $a^{(0)}$ is explicitly written out in terms of the initial data (η_0, ϕ_0), the bottom topography β, and the constant σ in the limit (3.11). Then, by (4.4) we see that

$$
\begin{aligned}
a(x,t) = 1 + \left(\frac{\delta}{\varepsilon}\right)^2\big(1 - \delta^2(|\nabla\eta^{(0)}(x,t/\varepsilon)|^2 + C)\big)\beta_{\tau\tau}(x,t/\varepsilon) \\
+ \sigma\big(a^{(0)}(x,t/\varepsilon) + C\sigma\beta_{\tau\tau}(x,t/\varepsilon)\big) + o(1),
\end{aligned}
$$

where $C > 0$ is an arbitrary constant. Therefore, the generalized Rayleigh–Taylor sign condition is satisfied if the following conditions are fulfilled. The conditions depend on the relations between δ and ε.

Assumption 4.1 There exist constants $C, c > 0$ such that for any $(x, \tau) \in \mathbf{R}^n \times (0,1)$ the following conditions are satisfied.

(1) In the case $\delta/\varepsilon \to 0$: No conditions.
(2) In the case $\delta/\varepsilon \to \nu$: $1 + \nu^2 \beta_{\tau\tau}(x,\tau) \geq c$.
(3) In the case $\delta/\varepsilon \to \infty$ and $\delta^2/\varepsilon \to 0$: $\beta_{\tau\tau}(x,\tau) \geq 0$.
(4) In the case $\delta/\varepsilon \to \infty$ and $\delta^2/\varepsilon \to \sigma$: $\beta_{\tau\tau}(x,\tau) \geq 0$ and $1+\sigma\big(a^{(0)}+\sigma C\beta_{\tau\tau}\big)(x,\tau) \geq c$.

From a technical point of view, we also impose the following condition.

Assumption 4.2 For any $(x, \tau) \in \mathbf{R}^n \times (0,1)$ the following conditions are satisfied.

(1) In the case $\delta/\varepsilon \to \nu$: No conditions.
(2) In the case $\delta/\varepsilon \to \infty$: $\beta_{\tau\tau\tau}(x,\tau) \leq 0$.

The following theorem is one of the main results in this communication and asserts the existence of the solution to the initial value problem for the full water wave problem with uniform bounds of the solution independent of δ and ε on the time interval $[0, \varepsilon]$.

Theorem 4.1 ([9]). *Let $M_0, c_0 > 0$, $r > n/2$, and $s > (n+9)/2$. Under the Assumptions 4.1 and 4.2, there exist constants $C_0, \delta_0, \varepsilon_0, \gamma_0 > 0$ such that for any $\delta \in (0, \delta_0]$, $\varepsilon \in (0, \varepsilon_0]$, (η_0, ϕ_0), and b satisfying $|\delta^2/\varepsilon - \sigma| \leq \gamma_0$,* (2.9), *and*

$$\begin{cases} \|\beta(\tau)\|_{s+9/2} + \|\beta_\tau(\tau)\|_{s+5} + \|\beta_{\tau\tau}(\tau)\|_{s+1} + \|\beta_{\tau\tau\tau}(\tau)\|_{r+2} \leq M_0, \\ \|\nabla\phi_0\|_{s+3} + \|\eta_0\|_{s+4} \leq M_0, \quad 1 + \eta_0(x) - b_0(x) \geq c_0 \quad \textit{for} \quad (x,\tau) \in \mathbf{R}^n \times (0,1), \end{cases}$$

the initial value problem (2.15) *and* (2.16) *has a unique solution $(\eta, \phi) = (\eta^{\delta,\varepsilon}, \phi^{\delta,\varepsilon})$ on the time interval $[0, \varepsilon]$ satisfying*

$$\begin{cases} \|\eta^{\delta,\varepsilon}(t) - \eta^{(0)}(t/\varepsilon)\|_{s+2} + \|\phi^{\delta,\varepsilon}(t) - \phi^{(0)}(t/\varepsilon)\|_{s+2} \leq C_0(\varepsilon + |\delta^2/\varepsilon - \sigma|), \\ \|\eta^{\delta,\varepsilon}(t)\|_{s+3} + \|\nabla\phi^{\delta,\varepsilon}(t)\|_{s+2} \leq C_0, \\ 1 + \eta^{\delta,\varepsilon}(x,t) - b(x,t) \geq c_0/2 \qquad \textit{for} \qquad (x,t) \in \mathbf{R}^n \times [0,\varepsilon], \end{cases}$$

where $(\eta^{(0)}, \phi^{(0)})$ is the approximate solution in the fast time variable $\tau = t/\varepsilon$ defined by (4.5).

Once we obtain this kind of existence theorem of the solution with uniform bounds, combining the existence result obtained in [8] where the case of a fixed bottom was investigated, we can easily consider the limits $\delta, \varepsilon \to 0$ of the solution $(\eta^{\delta,\varepsilon}, \phi^{\delta,\varepsilon})$.

Theorem 4.2 ([9]). *Under the same hypothesis of Theorem* 4.1, *there exists a time* $T > 0$ *independent of* $\delta \in (0, \delta_0]$ *and* $\varepsilon \in (0, \varepsilon_0]$ *such that the solution* $(\eta^{\delta,\varepsilon}, \phi^{\delta,\varepsilon})$ *obtained in Theorem* 4.1 *can be extended to the time interval* $[0, T]$ *and satisfies*

$$\|\eta^{\delta,\varepsilon}(t) - \eta^0(t)\|_{s-1} + \|\nabla\phi^{\delta,\varepsilon}(t) - u^0(t)\|_{s-1} \le C_0(\varepsilon + |\delta^2/\varepsilon - \sigma|) \quad \text{for} \quad \varepsilon \le t \le T,$$

where (η^0, u^0) *is a unique solution of the shallow water equations* (3.13) *under the initial conditions* (3.14) *and* u^0 *satisfies the irrotational condition* (3.15).

This theorem ensures that the standard tsunami model (1.1) and (1.2) gives a good approximation in the scaling regime $\delta^2 \ll \varepsilon \ll 1$. Moreover, in the critical scaling regime $\delta^2 \simeq \varepsilon \ll 1$ we have to take into account the effect of the initial velocity field as (1.3).

5 Linearized equations and energy estimates

The most difficult part to give a mathematically rigorous justification of the tsunami model is to establish an existence theory for the initial value problems (2.15) and (2.16) together with uniform boundedness of the solution with respect to the small parameters δ and ε. Such uniform boundedness are obtained by the energy methods together with a precise analysis of the Dirichlet-to-Neumann map Λ^{DN} and the Neumann-to-Neumann map Λ^{NN}. In order to explain how to apply the method to our problem, we will consider linearized equations of (2.15) around an arbitrary flow (η, ϕ) and give a definition of an energy function for the linearized equations. Following D. Lannes [14], we linearize the equations in (2.15) around (η, ϕ). To this end, we need to calculate a variation of the Dirichlet-to-Neumann map $\Lambda^{\mathrm{DN}}(\eta, b, \delta)$ and the Neumann-to-Neumann map $\Lambda^{\mathrm{NN}}(\eta, b, \delta)$ with respect to η.

Lemma 5.1 ([9]). *The variation of the maps* $\Lambda^{\mathrm{DN}}(\eta, b, \delta)$ *and* $\Lambda^{\mathrm{NN}}(\eta, b, \delta)$ *with respect to* η *has the form*

$$\frac{\mathrm{d}}{\mathrm{d}h}\big(\Lambda^{\mathrm{DN}}(\eta + h\check{\eta}, b, \delta)\phi + \Lambda^{\mathrm{NN}}(\eta + h\check{\eta}, b, \delta)\gamma\big)\Big|_{h=0} = -\delta^2\Lambda^{\mathrm{DN}}(\eta, b, \delta)(Z\check{\eta}) - \nabla\cdot(v\check{\eta}),$$

where

$$\begin{cases} Z = (1 + \delta^2|\nabla\eta|^2)^{-1}(\Lambda^{\mathrm{DN}}(\eta, b, \delta)\phi + \Lambda^{\mathrm{NN}}(\eta, b, \delta)\gamma + \nabla\eta\cdot\nabla\phi), \\ v = \nabla\phi - \delta^2 Z\nabla\eta. \end{cases}$$

By this lemma, setting

$$\zeta := \check{\eta}, \qquad \psi := \check{\phi} - \delta^2 Z\check{\eta}$$

with variations $(\check{\eta}, \check{\phi})$ of (η, ϕ), we see that the linearized equations have the form

$$
\left\{\begin{array}{l}
\psi_t + v\cdot\nabla\psi + a\zeta = 0,\\
\zeta_t + \nabla\cdot(v\zeta) - \Lambda^{\mathrm{DN}}\psi = 0,
\end{array}\right. \tag{5.1}
$$

where $\Lambda^{\mathrm{DN}} = \Lambda^{\mathrm{DN}}(\eta, b, \delta)$ and

$$
\left\{\begin{array}{l}
Z = (1+\delta^2|\nabla\eta|^2)^{-1}(\Lambda^{\mathrm{DN}}(\eta,b,\delta)\phi + \varepsilon^{-1}\Lambda^{\mathrm{NN}}(\eta,b,\delta)\beta_\tau + \nabla\eta\cdot\nabla\phi),\\
v = \nabla\phi - \delta^2 Z\nabla\eta,\\
a = 1 + \delta^2(Z_t + v\cdot\nabla Z).
\end{array}\right. \tag{5.2}
$$

Here, we should remark that these functions Z and v are related to the velocity Potential Φ by $\delta^2 Z = (\partial_{n+1}\Phi)|_{\Gamma(t)}$ and $v = (\nabla\Phi)|_{\Gamma(t)}$, so that $\delta^2 Z$ and v represent the vertical and horizontal velocities on the water surface, respectively. Moreover, the function a can be written in terms of the pressure p in (4.2) as

$$
a = -(1+\delta^2|\nabla\eta|^2)^{-1}(\partial_{n+1}p - \delta^2\nabla\eta\cdot\nabla p)|_{\Gamma(t)}.
$$

Thus, the generalized Rayleigh–Taylor sign condition ensures the positivity of this function a as

$$
a(x,t) \geq c_0 > 0 \qquad \text{for} \quad x\in\mathbf{R}^n,\ 0\leq t\leq T.
$$

In order to define an energy function to the system (5.1), we need more information on the Dirichlet-to-Neumann map Λ^{DN}.

Introducing a $(n+1)\times(n+1)$ matrix I_δ by

$$
I_\delta = \begin{pmatrix} E_n & 0 \\ 0 & \delta^{-1} \end{pmatrix},
$$

where E_n is the $n\times n$ unit matrix, we can rewrite the boundary value problem (2.10) in Definition 2.1 with $\gamma = 0$ as the following form.

$$
\left\{\begin{array}{lll}
\nabla_X\cdot I_\delta^2\nabla_X\Phi = 0 & \text{in} & \Omega,\\
\Phi = \phi & \text{on} & \Gamma,\\
N\cdot I_\delta^2\nabla_X\Phi = 0 & \text{on} & \Sigma.
\end{array}\right.
$$

Lemma 5.2 *The Dirichlet-to-Neumann map $\Lambda^{\mathrm{DN}} = \Lambda^{\mathrm{DN}}(\eta,b,\delta)$ is symmetric in $L^2(\mathbf{R}^n)$, that is, for any $\phi,\psi\in H^1(\mathbf{R}^n)$ it holds that*

$$
(\Lambda^{\mathrm{DN}}\phi, \psi) = (\phi, \Lambda^{\mathrm{DN}}\psi).
$$

Proof. Set $\Phi := (\phi^{\hbar}, 0)$ and $\Psi := (\psi^{\hbar}, 0)$. By Green's formula we have

$$
\begin{aligned}
0 &= \int_\Omega \big((\nabla_X\cdot I_\delta^2\nabla_X\Phi)\Psi - \Phi(\nabla_X\cdot I_\delta^2\nabla_X\Psi)\big)\mathrm{d}X \\
&= \int_\Gamma \big((N\cdot I_\delta^2\nabla_X\Phi)\Psi - \Phi(N\cdot I_\delta^2\nabla_X\Psi)\big)\mathrm{d}S,
\end{aligned}
$$

where N is the unit outward normal to the boundary $\partial\Omega$. In the above calculation we used the boundary condition on the bottom Σ. Since $\Phi = \phi$, $\Psi = \psi$, $\sqrt{1+|\nabla\eta|^2}N\cdot I_\delta^2\nabla_X\Phi = \Lambda^{\mathrm{DN}}\phi$, $\sqrt{1+|\nabla\eta|^2}N\cdot I_\delta^2\nabla_X\Psi = \Lambda^{\mathrm{DN}}\psi$, and $\mathrm{d}S = \sqrt{1+|\nabla\eta|^2}\mathrm{d}x$ on Γ, we obtain the desired identity. □

Lemma 5.3 *For any $\phi \in H^1(\mathbf{R}^n)$, it holds that $(\Lambda^{\mathrm{DN}}\phi,\phi) = \|I_\delta\nabla_X\Phi\|^2_{L^2(\Omega)}$, where $\Phi = (\phi^\hbar, 0)$.*

Proof. By Green's formula we see that

$$0 = \int_\Omega (\nabla_X\cdot I_\delta^2\nabla_X\Phi)\Phi\,\mathrm{d}X = \int_{\partial\Omega}(N\cdot I_\delta^2\nabla_X\Phi)\Phi\,\mathrm{d}S - \int_\Omega |I_\delta\nabla_X\Phi|^2\,\mathrm{d}X.$$

This together with the boundary conditions yields the desired identity. □

These two lemmas imply that the Dirichlet-to-Neumann map Λ^{DN} is a positive operator in $L^2(\mathbf{R}^n)$. For simplicity, we first consider the linear equations (5.1) in the case $v = 0$, that is, the equations

$$\begin{cases} \psi_t + a\zeta = 0, \\ \zeta_t - \Lambda^{\mathrm{DN}}\psi = 0, \end{cases}$$

which can be written in the matrix form

$$\begin{pmatrix} \psi \\ \zeta \end{pmatrix}_t + \begin{pmatrix} 0 & a \\ -\Lambda^{\mathrm{DN}} & 0 \end{pmatrix}\begin{pmatrix} \psi \\ \zeta \end{pmatrix} = 0$$

or

$$\mathscr{A}_0 U_t + \mathscr{A}_1 U = 0,$$

where $U = (\psi,\zeta)^T$ and

$$\mathscr{A}_0 = \begin{pmatrix} \Lambda^{\mathrm{DN}} & 0 \\ 0 & a \end{pmatrix}, \qquad \mathscr{A}_1 = \begin{pmatrix} 0 & \Lambda^{\mathrm{DN}}a \\ -a\Lambda^{\mathrm{DN}} & 0 \end{pmatrix}.$$

Here, we note that $\mathscr{A}_0$ is positively definite and $\mathscr{A}_1$ is skew-symmetric, that is, $\mathscr{A}_1^* = -\mathscr{A}_1$. This means that the matrix operator $\mathscr{A}_0$ is a symmetrizer for the system (linearized-equation), so that the corresponding energy function is defined by

$$E(t) := (\mathscr{A}_0 U, U) = (\Lambda^{\mathrm{DN}}\psi,\psi) + (a\zeta,\zeta).$$

In fact, for any smooth solution (ψ,ζ) of the linearized equations (5.1) we see that

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = ([\partial_t,\Lambda^{\mathrm{DN}}]\psi,\psi) - 2(\Lambda^{\mathrm{DN}}\psi, v\cdot\nabla\psi) + (a_t\zeta,\zeta) + ((v\cdot\nabla a - a\nabla\cdot v)\zeta,\zeta).$$

Here, by (5.2) and Lemmas 3.1 and 3.2

$$a(x,t) = \frac{\delta^2}{\varepsilon^2}\beta_{\tau\tau}(x,t/\varepsilon) + O(1),$$

so that

$$a_t(x,t) = \frac{\delta^2}{\varepsilon^3}\beta_{\tau\tau\tau}(x,t/\varepsilon) + \frac{1}{\varepsilon}O(1) \leq \frac{1}{\varepsilon}O(1),$$

where we used Assumption 4.2. As a result, we can obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) \leq \frac{C}{\varepsilon}E(t),$$

so that Gronwall's inequality implies that $E(t) \leq \mathrm{e}^{Ct/\varepsilon}E(0) \leq \mathrm{e}^{C}E(0)$ for $0 \leq t \leq \varepsilon$. This is one of uniform estimates of the solution for the linearized equations.

In order to derive uniform estimates of the solution for the full nonlinear equations, we transform the equations to a qausilinear system of equations and apply the energy estimate. We refer to [9] for details.

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