Linear Fractional Recurrences
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We let $C^k$ denote complex Euclidean space, and we consider birational maps $f : C^k \rightarrow C^k$ of the form

$$f = f_{\alpha, \beta}(x_1, \ldots, x_k) = \left( x_2, \ldots, x_k, \frac{\alpha \cdot x}{\beta \cdot x} \right) \quad (1)$$

where $\alpha \cdot x = \sum \alpha_j x_j$ and $\beta \cdot x = \sum \beta_j x_j$. One feature of these maps is that they seem to be the simplest possible nonlinear maps. Something which has interested us is the question of periodicities: What are the constants $\alpha = (\alpha_1, \ldots, \alpha_k)$ and $\beta = (\beta_1, \ldots, \beta_k)$ for which $f_{\alpha, \beta}$ is periodic? By periodic we mean that $f^N = f \circ \cdots \circ f$ is the identity map for some $N$. We refer to [GL] and [KL] for further discussion. This question remains unsolved in general, but there is one observation we have made with Kyounghee Kim (see [BK2]):

**Theorem 1.** If $a = (-1)^{1/k}$ and

$$\beta = (a^{k-1}, 1, 0, \ldots, 0) \quad \text{and} \quad \alpha = (a^{k-2}/(1-a), 0, a^{k-2}, \ldots, a^2, a, 1) \quad (2)$$

then $f_{\alpha, \beta}$ is periodic with period $4k$.

Thus for each $k$, there are $k$ different maps of the form (1) which have period $4k$. We remark that the proof given in [BK2] is somewhat indirect. Namely, we consider $f_{\alpha, \beta}$ as a map of $P^k$. Then we construct a blowup space $\pi : X \rightarrow P^k$ and study the induced map $f_X := \pi^{-1} \circ f \circ \pi : X \rightarrow X$. We then determine the induced map $f_X^k$ on $H^{1,1}(X)$. We show that the eigenvalues of $f_X^k$ are roots of unity and that $f_X^k$ has period $4k$. After this, we show that $f^4k$ is the identity.

Let us recall the situation for dimension 2 (see [BK1]):

**Theorem 2.** If $k = 2$, then the only possible (nontrivial) periods for maps (1) are 6, 5, 8, 12, 18, and 30.

In this case, it is possible to enumerate all the possible values of $\alpha$ and $\beta$ and to verify directly that specific examples have the stated periods. The more difficult issue is to show that these are the only periodic possibilities.

We also consider the case of dimension 3 (see [BK3]):

**Theorem 3.** If $k = 3$, then the only possible (nontrivial) periods for maps (1) are 8 and 12.

The maps of period 12 which arise in Theorem 3 correspond to the maps in the case $k = 3$ in Theorem 1. The period 8 maps are given by:

$$f(x) = \left( x_2, x_3, \frac{1 + x_2 + x_3}{x_1} \right), \quad f(x) = \left( x_2, x_3, \frac{-1 - x_2 + x_3}{x_1} \right)$$

We note that the maps that had been observed earlier were the ones of period 8. The first of these was found by Lyness [Ly], and the second one is due to Csörnyei and Laczkovic [CL]. The behavior of the maps (1) is more complicated in dimension 3 than it was in dimension 2. One explanation for this is that the difficulties arise from blow-down and blow-up behaviors. In dimension 2, all such behavior is either a curve blowing down to a point or a point blowing
up to a curve. In dimension 3, a hypersurface can blow down either to a curve or to a point, and vice versa. Further, there can be blow-up behavior without blow-down behavior. For instance, we can have a birational map \( g : X \to Y \) and curves \( C \subset X \) and \( C' \subset Y \) such that \( g : X - C \to Y - C' \) is a biholomorphism, but each point of \( C \) blows up to \( C' \). The difference with dimension 2 is that the Jacobian of \( g \) is nonsingular (invertible) at each point of \( X - C \).

We know little about the case \( k \geq 4 \). In particular, we do not know whether there are nontrivial periods other than the ones given by the maps in Theorem 1 when \( k \geq 4 \).

One feature that has attracted us to the maps (1) is that the are in some sense the simplest nonlinear maps. Both \( f \) and its inverse have degree 2. That is, on \( \mathbb{P}^k \), the maps (1), as well as their inverses, are both written in terms of homogeneous polynomials of degree 2. In general, however, when \( k = 3 \) the inverse of a quadratic map can have degree 2, 3, or 4. The degree of a mapping, however, is not invariant under birational conjugacy. That is, if \( L \) is linear (and thus of degree 1), and if \( \varphi \) is birational, then \( \varphi^{-1} \circ L \circ \varphi \) can be nonlinear and have degree higher than one. We now define the dynamical degree, which is more natural as a dynamical invariant.

If \( f : X \to Y \) is a rational map, then there is a well-defined pullback on cohomology \( f^* : H^{p,q}(Y) \to H^{p,q}(X) \) (see [G]). Using this, we may define the dynamical degrees as follows. We then define the \( \ell \)-th dynamical degree as

\[
\delta_{\ell}(f) := \lim_{n \to \infty} \left( \left\| (f^n)^* \right\|_{H^{\ell,\ell}(X)} \right)^{1/n}
\]

(3)

Thus \( \delta_{\ell}(f) \) measures the exponential rate of growth of \( f \) on \( H^{\ell,\ell}(X) \), which, loosely speaking, corresponds to objects of codimension \( 2\ell \). \( \delta_k \) corresponds to the topological (mapping) degree of \( f \). If \( X = \mathbb{P}^k \), then \( H^{1,1}(\mathbb{P}^k, \mathbb{Z}) \cong \mathbb{Z} \), and \( f^*|_{H^{1,1}(\mathbb{P}^k)} = \text{deg} \), where \( \text{deg} \) denotes the usual degree in the representation of \( f \) in terms of homogeneous polynomials. That is, if \( H = \{ \sum c_j x_j = 0 \} \) is the class of a hyperplane in \( \mathbb{P}^k \), then \( f^*H = \{ \sum c_j f_j = 0 \} = (\text{deg})H \). \( \delta_k \) corresponds to the topological (mapping) degree of \( f \). The dynamical degree is an important measure of complexity for a rational dynamical system, and the quantity \( \delta_{\ell}(f) \) was shown to be an invariant of birational conjugacy by Dinh and Sibony [DS].

We note that our search for periodicities in the family (1) is essentially a process of eliminating the non-periodic maps. Our original approach was to find the \( \alpha \) and \( \beta \) for which \( \delta_1(f_{\alpha,\beta}) > 1 \). Obviously, if the degree growth is exponential, then the map is not periodic. With this approach, our study of the maps (1) quickly becomes an analysis of the critical maps; we will say that \( f_{\alpha,\beta} \) is critical if \( \beta_2 = \beta_3 = 0 \) and \( \beta_1 \alpha_2 \alpha_3 \neq 0 \).

**Theorem 4.** For a generic critical map, the first dynamical degree \( \delta_1(f_{\alpha,\beta}) \sim 1.32472 \), the largest root of \( x^3 - x - 1 \).

For \( 1 < \ell < k \), the dynamical degree \( \delta_{\ell} \) is not well understood. Of course, if \( f \) is in fact holomorphic, then \( \delta_{\ell} \) is the spectral radius of the map \( f^*|_{H^{\ell,\ell}(X)} \). However, when \( f \) is not holomorphic, a class \( \eta \in H^{\ell,\ell} \) might be carried by a cycle inside the indeterminacy locus, so the interpretation of \( f^*\eta \) is not obviously gotten by pulling back the cycle defining \( \eta \).

In the case of dimension 3, we have the Poincaré duality \( \langle \cdot, \cdot \rangle \) between \( H^{1,1}(X) \) and \( H^{2,2}(X) \) and thus an adjoint \( f_* \) acting on \( H^{2,2} \). That is, for \( \xi \in H^{1,1}(X) \) and \( \eta \in H^{2,2}(X) \), we have \( \langle f^*\eta, \xi \rangle = \langle \eta, f_*\xi \rangle \). Since \( f \) is birational, we also have the pullback of \( f^{-1} : X \to X \) acting on \( H^{1,1}(X) \). Thus the pullback \( (f^{-1})^*|_{H^{1,1}(X)} \) is equivalent under this duality to \( f^*|_{H^{2,2}} \). This gives us that \( \delta_2(f) = \delta_1(f^{-1}) \).
This leads to the question whether there is any family of rational maps for which it is possible to determine $\delta_\ell$ for $1 < \ell < k$. At present, the only general family for which $\delta_\ell$ is known is the family of monomial maps. That is, we let $A = (a_{i,j})$ be a $k \times k$ matrix with integer entries. (The interesting case here is when $A$ contains negative entries.) Then we define a rational map $g_A : C^k \to C^k$ by setting

$$
g_A(x_1, \ldots, x_k) = \left( \prod_j x_j^{a_{1,j}}, \ldots, \prod_j x_j^{a_{k,j}} \right)
$$

which, heuristically, is $g_A = e^{A \log x}$. A basic property is that iteration of the monomial map corresponds to matrix multiplication: $(g_A)^n = g_{A^n}$. As we noted above, $\delta_\ell$ is a birational invariant, so we can choose our space to work on. We choose to work on the manifold $X = (P^1)^k$, which is the compactification of $C^k$ obtained by taking the product of the compactifications of the factors $C$. It is evident that a basis for $H^{1,1}(X)$ is given by the coordinate hyperplanes $\{x_j = 0\}$. Further, a basis of $H^{p,p}(X)$ is given by $\{x_{i_1} = \cdots = x_{i_p} = 0\}$, where $1 \leq i_1 < \cdots < i_p \leq k$ consists of $p$ distinct indices. We also consider the following matrix operation: Given a matrix $M = (m_{i,j})$, we define $|M| = (|m_{i,j}|)$ to be the matrix obtained by taking absolute values of all the entries. J-L Lin [Li] has shown that $g_A^*|H_{p,p}$ is given by $|\wedge^p A|$.

**Proposition.** With respect to this basis, $g_A|H_{p,p}$ is given by $|\wedge^p A|$.

Working from this Proposition, Lin [Li] obtained the following result, which was also obtained independently using different methods by Favre and Wulcan [FW]:

**Theorem 5.** If $g_A$ is as in (4), then $\delta_p(g_A) = |\mu_1 \cdots \mu_p|$, where $|\mu_1| \geq |\mu_2| \geq \cdots \geq |\mu_p|$ are the eigenvalues of $A$.

The family (1) also leads us to automorphisms. To say $f$ is an automorphism means that there is a blowup space $\pi : X \to P^k$ (perhaps involving iterated blowups) such that the induced map $f_X := \pi^{-1} \circ f \circ \pi$ is an automorphism of $X$. De Fernex and Ein [dFE] have shown that if a map is periodic (in any dimension), then it is an automorphism in the sense above (see [dFE]).

[BK1] showed that looking inside the 2-dimensional version of family (1) reveals a number of rational surface automorphisms with positive entropy. When we go to higher dimension, we must be more careful. For a general manifold $X$ of dimension $k$, we follow Dolgachev and Ortland [DO] and say that $f : X \to X$ is a pseudo-automorphism if $f$ and $f^{-1}$ are local diffeomorphisms at all points away from the indeterminacy locus. In dimension 2, $f$ is a pseudo-automorphism if and only if it is an automorphism, but not in higher dimension. In [BK3] we find that the family (1) contains pseudo-automorphisms of positive entropy on spaces which are blowups of $P^3$:

**Theorem 6.** Suppose that $\alpha = (a, 0, \omega, 1)$ and $\beta = (0, 1, 0, 0)$ where $a \in C \setminus \{0\}$ and $\omega$ is a non-real cube root of the unity. Then there is a modification $\pi : Z \to P^3$ such that $f_Z$ is a pseudo-automorphism. The dynamical degrees $\delta_1(f) = \delta_2(f) \cong 1.28064 > 1$ are equal and are given by the largest root of $t^8 - t^5 - t^4 - t^3 + 1$. The entropy of $f_Z$ is the logarithm of the dynamical degree and is thus positive.

In addition, there is a sort of integrability for these maps:
Theorem 7. For the mappings in Theorem 1, there is a 1-parameter family of surfaces $S_c \subset Z$, $c \in C$ which have the invariance $fS_c = S_{\omega c}$. For generic $c$, $S_c$ is $K3$, and the restriction $f^3|_{S_c}$ is an automorphism. For generic $c$ and $c'$, the surfaces $S_c$ and $S_{c'}$ are biholomorphically inequivalent, and the automorphisms $f^3|_{S_c}$ and $f^3|_{S_{c'}}$ are not smoothly conjugate.

The surface $S_0$ is invariant, and the restriction $f_{S_0}$ is an automorphism which has the same entropy as $f$. This is smaller than the entropy of the automorphism constructed in [M, Theorem 1.2] and is thus the smallest known entropy for a projective $K3$ surface automorphism.

Let us write $f_c := f|_{S_c}$ for the restriction to $S_c$. The automorphisms of $K3$ surfaces were studied by Cantat [C]. In our case, it follows that there are positive, closed currents $\mu^\pm_c$ on $S_c$ such that $f_c^3 \cdot \mu^\pm_c = \delta^{\pm3} \mu^\pm_c$, and $\mu_c := \mu^+_c \wedge \mu^-_c$ is the unique measure of maximal entropy.

We let $\alpha^+ \in H^{1,1}(Z)$ denote the class which is expanded by $f_Z^3$. If $\alpha^+$ is nef, then by Diller and Guedj [DG] there is an invariant current $T^+$ in $\alpha^+$ which is invariant (expanded by $f_Z^3$) and which has the “attractor” property that for all smooth currents $\Xi^+$ in the class of $\alpha^+$, the normalized pullbacks $\delta^{-n} f_Z^n \Xi^+ \to T^+$. Inspired by Bayraktar [B], we can construct $Z$ such that $\alpha^+$ to be nef. Similarly, we have a corresponding current $T^+_Z$, and we may wedge these two currents to obtain an invariant $(0,2)$-current $T := T^+ \wedge T^-$, which satisfies $f^*T = T$. These currents have properties analogous to the bifurcation currents studied by Dujardin and Favre [DuF]. That is, their slices by the invariant $K3$ surfaces give the corresponding invariant currents/measures for $(f_c, S_c)$: $T^+_c = \mu^+_c$, and $T|_{S_c} = \mu_c$.

The following mappings have quadratic degree growth and complete integrability:

Theorem 8. Suppose that $\beta = (0, 1, 0, 0)$ and either $\alpha = (0, 0, 1, 0)$ or $\alpha = (0, 1, 1, 1)$ where $a \in C \setminus \{1\}$, $\omega \neq 1$, and $\omega^3 = 1$. Then the degree of $f^n$ grows quadratically in $n$. Further, there is a modification $\pi : Z \to P^3$ such that $f_Z$ is a pseudo-automorphism. There is a two-parameter family of surfaces $S_c$, $c = (c_1, c_2) \in C^2$ which are invariant under $f^3$. For generic $c$ and $c'$, $S_c$ is a smooth $K3$ surface, and $S_c \cap S_{c'}$ is a smooth elliptic curve.

For the mappings in Theorems 4 and 8, $f$ is reversible on the level of cohomology: $f_z^*$ is conjugate to $(f_z^{-1})^* = (f_z^*)^{-1}$. The identity $\delta_1(f) = \delta_2(f)$ for such maps is a consequence of the duality between $H^{1,1}$ and $H^{2,2}$, so they are not cohomologically hyperbolic, in the terminology of [G]. If any of the maps of Theorems 6 and 8 acts on $P^3$, then it is evident that the variety $R_0 = \{x_0 x_1 x_2 x_3 = 0\}$ is invariant. After the blow-up $\pi : Z \to P^3$, we have a divisor $R := \pi^{-1} R_0$ which now contains 8 components. In fact, $R$ is an invariant 8-cycle of surfaces under $f_Z$. The family of invariant $K3$ surfaces degenerates and becomes singular at a $R$. We have seen that $f_Z$ is a pseudo-automorphism and not an automorphism. This is a property of $f$ and not, somehow, a defect of our choice of a particular blowup space $Z$. In [BK3] we showed:

Theorem 7. Let $f$ be a map from Theorems 1 and 3. If $a \neq 1$, then the restriction $f^3|_{\{x_3 = 0\}}$ is not birationally equivalent to a surface automorphism. Thus there is no proper modification $\pi : W \to P^3$ such that the induced map $f_W$ is an automorphism.

References

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