<table>
<thead>
<tr>
<th>Title</th>
<th>Postcritical sets and saddle basic sets for Axiom A polynomial skew products on $\mathbb{C}^2$ (Research on Complex Dynamics and Related Fields)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Nakane, Shizuo</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2011), 1762: 114-119</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2011-09</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/171369">http://hdl.handle.net/2433/171369</a></td>
</tr>
<tr>
<td>Right</td>
<td></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
Postcritical sets and saddle basic sets for Axiom A polynomial skew products on $\mathbb{C}^2$

東京工芸大学 中根静男 (Shizuo Nakane)

Tokyo Polytechnic University

1 Introduction

We consider Axiom A regular polynomial skew products on $\mathbb{C}^2$. It is of the form: \( f(z, w) = (p(z), q(z, w)) \), where \( p(z) = z^d + \cdots \) and \( q_z(w) = q(z, w) = w^d + \cdots \) are polynomials of degree \( d \geq 2 \). Then its k-th iterate is expressed by:

\[
f^{ok}(z, w) = (p^{ok}(z), q_{p^{k-1}(z)}(w)) =: (p^{ok}(z), Q_{z}^{ok}(w)).
\]

Hence it preserves the family of fibers \( \{z\} \times \mathbb{C} \) and this makes it possible to study its dynamics more precisely. Let \( K \) be the set of points with bounded orbits and set \( K_z := \{w \in \mathbb{C}; (z, w) \in K\} \) and \( K_{J_p} := K \cap (J_p \times \mathbb{C}) \). The fiber Julia set \( J_z \) is the boundary of \( K_z \).

Let \( \Omega \) be the set of non-wandering points for \( f \). Then \( f \) is said to be Axiom A if \( \Omega \) is compact, hyperbolic and periodic points are dense in \( \Omega \). For polynomial skew products, Jonsson [J2] has shown that \( f \) is Axiom A if and only if the following three conditions are satisfied:

(a) \( p \) is hyperbolic,
(b) \( f \) is vertically expanding over \( J_p \),
(c) \( f \) is vertically expanding over \( A_p := \{\text{attracting periodic points of } p\} \).

Here \( f \) is vertically expanding over \( Z \subset \mathbb{C} \) with \( p(Z) \subset Z \) if there exist \( \lambda > 1 \) and \( C > 0 \) such that \( |(Q_{z}^k)'(w)| \geq C\lambda^k \) holds for any \( z \in Z, w \in J_z \) and \( k \geq 0 \).

We are interested in the dynamics of \( f \) on \( J_p \times \mathbb{C} \) because the dynamics outside \( J_p \times \mathbb{C} \) is fairly simple. Consider the critical set

\[
C_{J_p} = \{(z, w) \in J_p \times \mathbb{C}; q_z'(w) = 0\}
\]

over the base Julia set \( J_p \). Let \( \mu \) be the ergodic measure of maximal entropy for \( f \) (see Fornaess and Sibony [FS1]). Its support \( J_2 \) is called the second Julia set of \( f \). Let \( D_{J_p} := \overline{\bigcup_{n \geq 1} f^n(C_{J_p})} \) be the postcritical set of \( C_{J_p} \). Jonsson [J2] has shown that

(d) \( J_2 = \bigcup_{z \in J_p} \{z\} \times \{z\} \),
(e) the condition (b) \( \iff D_{J_p} \cap J_2 = \emptyset \),
(f) \( J_2 \) is the closure of the set of repelling periodic points of \( f \).
By the Birkhoff ergodic theorem, $\mu$-a.e. $x$ has a dense orbit in $J_2$. Especially, $J_2 = \text{supp}\mu$ is transitive. Hence $J_2$ coincides with the basic set of unstable dimension two. See also [FS2].

For any subset $X$ in $\mathbb{C}^2$, its accumulation set is defined by

$$A(X) = \cap_{N \geq 0} \cup_{n \geq N} f^m(X).$$

DeMarco & Hruska [DH1] defined the pointwise and component-wise accumulation sets of $C_{J_p}$ respectively by

$$A_{pt}(C_{J_p}) = \bigcup_{x \in C_{J_p}} A(x)$$

and

$$A_{cc}(C_{J_p}) = \overline{\bigcup_{C \in C(C_{J_p})} A(C)},$$

where $C(C_{J_p})$ denotes the collection of connected components of $C_{J_p}$. It follows from the definition that

$$A_{pt}(C_{J_p}) \subset A_{cc}(C_{J_p}) \subset A(C_{J_p}).$$

It also follows that $A_{pt}(C_{J_p}) = A_{cc}(C_{J_p})$ if $J_p$ is a Cantor set and $A_{cc}(C_{J_p}) = A(C_{J_p})$ if $J_p$ is connected.

Let $\Lambda$ be the closure of the set of saddle periodic points in $J_p \times \mathbb{C}$. It decomposes into a disjoint union of saddle basic sets: $\Lambda = \bigcup_{i=1}^m \Lambda_i$. Put $\Lambda_z = \{w \in \mathbb{C}; (z, w) \in \Lambda\}$. The stable and unstable sets of $\Lambda$, the local stable and local unstable manifolds of $x \in \Lambda$ and $\hat{x} \in \Lambda$ are respectively defined by

$$W^s(\Lambda) = \{y \in \mathbb{C}^2; f^{ok}(y) \rightarrow \Lambda\},$$

$$W^u(\Lambda) = \{y \in \mathbb{C}^2; \exists \text{ backward orbit } \hat{y} = (y_{-k}) \text{ tending to } \Lambda\},$$

$$W^s_\delta(x) = \{y \in \mathbb{C}^2; ||f^{ok}(y) - f^{ok}(x)|| < \delta, \forall k \geq 0\},$$

$$W^s_\delta(\hat{x}) = \{y \in \mathbb{C}^2; \exists \hat{y} \text{ s.t. } ||y_{-k} - x_{-k}|| < \delta, \forall k \geq 0\}.$$  

On $\Lambda$, $f$ is contracting in the fiber direction and

$$W^s_\delta(x) \subset \{z\} \times \mathbb{C}, \ x = (z, w) \in \Lambda.$$

**Theorem A.** ([DH1])

$$A_{pt}(C_{J_p}) = \Lambda, \quad A(C_{J_p}) = W^u(\Lambda) \cap (J_p \times \mathbb{C}).$$

**Theorem B.** ([DH1, DH2])

$$A_{cc}(C_{J_p}) = A_{pt}(C_{J_p}) \Rightarrow \forall C \in C(C_{J_p}), \ C \cap K = \emptyset \text{ or } C \subset K. \quad (1)$$
Theorem C. ([DH1, DH2])

\[ A(C_{J_{p}}) = A_{pt}(C_{J_{p}}) \iff \text{the map } z \mapsto \Lambda_{z} \text{ is continuous in } J_{p}. \]  \hspace{1cm} (2)

Under the assumption \( W^{u}(\Lambda) \cap W^{s}(\Lambda) = \Lambda \),

\[ A(C_{J_{p}}) = A_{pt}(C_{J_{p}}) \iff \text{the map } z \mapsto K_{z} \text{ is continuous in } J_{p}. \]  \hspace{1cm} (3)

As for the assumption in the above theorem, we have

**Lemma 1.** \( W^{u}(\Lambda) \cap W^{s}(\Lambda) = \Lambda \iff W^{u}(\Lambda_{i}) \cap W^{s}(\Lambda_{j}) = \emptyset \) if \( i \neq j \).

Sumi [S] gives an example of Axiom A polynomial skew product which does not satisfy the condition in the above lemma. See the last section. It is also (incorrectly) described as Example 5.10 in [DH1]. See also [DH2].

We define a relation \( \succ \) among saddle basic sets by

\[ \Lambda_{i} \succ \Lambda_{j} \iff (W^{u}(\Lambda_{i}) \setminus \Lambda_{i}) \cap (W^{s}(\Lambda_{j}) \setminus \Lambda_{j}) \neq \emptyset. \]

A cycle is a chain of basic sets:

\[ \Lambda_{i_{1}} \succ \Lambda_{i_{2}} \succ \cdots \succ \Lambda_{i_{n}} = \Lambda_{i_{1}}. \]

For Axiom A open endomorphisms, there is no trivial cycle because \( W^{u}(\Lambda_{i}) \cap W^{s}(\Lambda_{i}) = \Lambda_{i} \) holds for any \( i \). See [J2], Proposition A.4. Jonsson has also shown that, for Axiom A polynomial skew products on \( \mathbb{C}^{2} \), the non-wandering set \( \Omega \) coincides with the chain recurrent set \( \mathcal{R} \). This leads to the following lemma.

**Lemma 2.** ([J2], Corollary 8.14) Axiom A polynomial skew products on \( \mathbb{C}^{2} \) have no cycles.

Set \( \Lambda_{0} := \emptyset, W^{s}(\Lambda_{0}) := (J_{p} \times \mathbb{C}) \setminus K \) and \( C_{i} := C_{J_{p}} \cap W^{s}(\Lambda_{i}) \) \( (0 \leq i \leq m) \). If we consider in \( \mathbb{P}^{2} \), \( \Lambda_{0} \) should be the superattracting fixed point \{[0 : 1 : 0]\}.

We will give characterizations of the equalities \( A_{cc}(C_{J_{p}}) = A_{pt}(C_{J_{p}}) \) and \( A_{pt}(C_{J_{p}}) = A(C_{J_{p}}) \) in terms of \( C_{i} \).

**Lemma 3.** \( C_{J_{p}} = \bigcup_{i=0}^{m} C_{i} \).

Note that \( A(C_{i}) \supset A_{pt}(C_{i}) = \Lambda_{i} \) for any \( i \geq 0 \).

The author would like to thank Hiroki Sumi for helpful discussion on his example.
2 Results

Theorem 1. \[ A_{cc}(C_{J_{p}}) = A_{pt}(C_{J_{p}}) \iff \forall C \in C(C_{J_{p}}), \ 0 \leq \exists i \leq m \ such \ that \ C \subset C_{i}. \] (4)

In terms of $C_{i}$, the condition in (1) is expressed by

\[ \forall C \in C(C_{J_{p}}), \ C \subset C_{0} \ or \ C \subset \bigcup_{i=1}^{m}C_{i}. \]

Hence, if $m = 1$, that is, $\Lambda$ itself is a basic set, then the condition in (4) coincides with that in (1). In general, the condition in (4) is stronger than that in (1).

We have another characterization of $A_{pt}(C_{J_{p}}) = A(C_{J_{p}})$ in terms of $C_{i}$.

Theorem 2. For any $i \geq 0$, we have \[ A(C_{i}) = \Lambda_{i} \iff C_{i} \ is \ closed. \] (5)

Consequently we have \[ A_{pt}(C_{J_{p}}) = A(C_{J_{p}}) \iff C_{i} \ is \ closed \ for \ any \ i \geq 0. \]

As for the condition in (3), we have

Theorem 3. The following three conditions are equivalent to each other.

(a) $C_{0}$ is closed,
(b) $A(C_{J_{p}}) = W^{u}(\Lambda) \cap W^{s}(\Lambda)$,
(c) the map $z \mapsto K_{z}$ is continuous in $J_{p}$.

As a corollary, we get the following.

Corollary 1. $W^{u}(\Lambda) \cap W^{s}(\Lambda) = \Lambda \iff C_{i} \ is \ closed \ for \ any \ i \geq 1.$

As for the condition in (2), we have the following.

Theorem 4. For each $j \geq 1$,

\[ C_{j} \ is \ open \ in \ C_{J_{p}} \iff W^{u}(\Lambda_{j}) \cap (J_{p} \times \mathbb{C}) = \Lambda_{j} \iff z \mapsto \Lambda_{j,z} \ is \ continuous \ in \ J_{p}. \]

Consequently,

\[ \forall j \geq 1, C_{j} \ is \ open \ in \ C_{J_{p}} \iff W^{u}(\Lambda) \cap (J_{p} \times \mathbb{C}) = \Lambda \iff z \mapsto \Lambda_{z} \ is \ continuous \ in \ J_{p}. \]

Recall that $C_{0} = C_{J_{p}} \setminus K$ is always open in $C_{J_{p}}$. 

3 Sumi's example

Sumi [S] considers the following example.

\[ f(z, w) = \left( (z^2 - R)^{on}, \frac{w + \sqrt{R}}{2\sqrt{R}}t_{n,\epsilon}(w) \right). \]

Here \( R \gg 1, 0 < \epsilon \ll 1, n \) is even and large, and

\[ t_{n,\epsilon}(w) = ((w - \epsilon)^2 - 1 + \epsilon)^{on} - w^{2n}. \]

Let \( \alpha < 0 \) and \( \beta > 0 \) be the fixed points of \( z^2 - R \). It satisfies the following.

- \( J_p \) is a Cantor set in \( \mathbb{D}(-\sqrt{R}, r) \cup \mathbb{D}(\sqrt{R}, r) \) for some \( r \).
- \( J_\alpha \) is a quasicircle, while \( J_\beta \) is a basilica.
- \( \Lambda = \Lambda_1 \cup \Lambda_2 \), where \( \Lambda_1 \subset \{ \beta \} \times \mathbb{C} \) is a single point.
- \( C_{J_p} \subset K \), i.e. \( C_0 = \emptyset \), hence \( z \mapsto K_z \) is continuous in \( J_p \).
- \( C_1 \subset \{ \beta \} \times \mathbb{C} \) is a finite set.
- \( C_2 = C_{J_p} \setminus C_1 \) is open in \( C_{J_p} \) and \( \overline{C_2} \supset C_1 \).
- \( W^u(\Lambda_1) \cap W^s(\Lambda_2) \setminus \Lambda \neq \emptyset \), i.e. \( \Lambda_1 \succ \Lambda_2 \).
- The map \( z \mapsto \Lambda_{2,z} \) is continuous in \( J_p \).
- \( A_{pt}(C_{J_p}) = A_{cc}(C_{J_p}) \neq A(C_{J_p}) \).
References


