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NON-COMMUTATIVE AUTOMORPHISM GROUPS OF POSITIVE ENTROPY OF CALABI-YAU MANIFOLDS AND HYPERKÄHLER MANIFOLDS

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1. Introduction

This is a short summary of [Og3], which is grown up from my talk at the workshop celebrating the 60-th birthday of Professor Ushiki. First of all, I would like to thank to Professor Sumi for inviting me to the workshop and I would like to dedicate this short note to Professor Ushiki in this occasion.

Let us start mathematics. In his paper [We], Wehler gave two explicit examples of groups of biholomorphic automorphisms of K3 surfaces. His K3 surface automorphisms sometimes appear in papers concerning complex dynamics as interesting, handy examples. The aim of this note is to generalize them to higher dimensional manifolds, namely, even dimensional Calabi-Yau manifolds and hyperkähler manifolds.

Throughout this note, we work over the complex number field $\mathbf{C}$.

2. Calabi-Yau manifolds and compact hyperkähler manifolds

Let us recall the definition of Calabi-Yau manfolds (in the strict sense) and hyperkähler manifolds.

Definition 2.1. Let M be a d-dimensional compact Kähler manifold with trivial fundamental group. Then,

(1) M is called a *Calabi-Yau manifold* (in the strict sense) if M admits no non-zero global holomorphic i-form with $0 < i < d$ and admits a nowhere vanishing global holomorphic d-form.

(2) M is called a *hyperkähler manifold* if M admits an everywhere non-degenerate holomorphic 2-form σ_M and also any global holomorphic 2-form on M is a constant multiple of σ_M.

Any hyperkähler manifod is necessarily of even dimension. In dimension 2, Calabi-Yau manifolds and hyperkähler manifolds are the same and they are nothing but *K3 surfaces.*

According to the Bogomlov decomposition theorem [Be1], Calabi-Yau manifolds, hyperkähler manifolds and complex tori form building blocks of compact Kähler manifolds of vanishing first Chern class. So, both manifolds play very impotant roles in the classification of compact Kähler manifolds and projective manifolds.

It is therefore natural and meaningful to ask to what extent one can generalize properties of K3 surfaces to Calabi-Yau manifolds and/or hyperkähler manifolds. Since K3 surfaces

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enjoy so many interesting, beautiful properties, there are also many choices of such properties. Some of them, like Torelli type properties, are really important. This report, however, treats only one small (but pretty, I hope) aspect concerning automorphisms.

3. Wehler's K3 surfaces

In his short beautiful paper [We], Wehler gave two explicit examples of biholomorphic automorphisms of K3 surfaces. Let us review his examples.

First Example.

Let $\overline{W}$ be a *generic* complete intersection of hypersurfaces of bidegree $(1,1)$ and $(2,2)$ in $\mathbf{P}^2 \times \mathbf{P}^2$. Then $\overline{W}$ is a K3 surface by the adjunction formula and the Lefschetz theorem. Moreover, $\overline{W}$ is of Picard number 2. In fact, $\mathrm{Pic}\,(\mathbf{P}^2 \times \mathbf{P}^2) \simeq \mathrm{Pic}\,(\overline{W})$ under the natural restriction map. This is due to the Noether-Lefschetz theorem ([Vo], Theorem 3.33).

Let $p_i : \overline{W} \to \mathbf{P}^2$ ($i = 1,\ 2$) be the natural i-th projection. Then p_i is a *finite* double cover and the covering transformation ι_i associated to p_i acts on $\overline{W}$ as a *biholomorphic* autmorphism of $\overline{W}$. Among other things, Wehler proved the following:

Theorem 3.1.

$$\mathrm{Aut}\,(\overline{W}) = \langle \iota_1, \iota_2 \rangle = \langle \iota_1 \rangle * \langle \iota_2 \rangle \simeq \mathbf{Z}/2 * \mathbf{Z}/2\ .$$

In particular, the two involutions ι_1 and ι_2 have no relation.

The group $\langle \iota_1 \rangle * \langle \iota_2 \rangle$ is then non-commutative, but it has an abelian subgroup $\langle \iota_1\iota_2 \rangle \simeq \mathbf{Z}$ of index two. So, the group is the so called *almost abelian group* and it is not too much non-commutative.

Later, $\overline{W}$ is also studied from the view point of arithmetic by Silverman [Sil]. $\overline{W}$ is also interesting from the view of complex dynamics, because the automorphism $\iota_1\iota_2$ is of positive topological entropy.

Second Example.

Let W be a *generic* hypersurface of multi-degree $(2,2,2)$ in $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$. Then, for the same reason as in the first example, W is a K3 surface of Picard number 3, or more precisely,

$$\mathrm{Pic}\,(\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1) \simeq \mathrm{Pic}\,(W)$$

under the natural restriction map.

Let $\{1,2,3\} = \{i,j,k\}$ and $p_k : W \to \mathbf{P}^1 \times \mathbf{P}^1$ be the natural (i,j)-th projection. Then p_k is a *finite* double cover and the covering transformation ι_k associated to p_k acts on W as a *biholomorphic* autmorphism of W.

In the last two lines of the same paper [We], Wehler pointed out the following result without any proof:

Theorem 3.2.

$$\mathrm{Aut}\,(W) = \langle \iota_1, \iota_2, \iota_3 \rangle = \langle \iota_1 \rangle * \langle \iota_2 \rangle * \langle \iota_3 \rangle \simeq \mathbf{Z}/2 * \mathbf{Z}/2 * \mathbf{Z}/2\ .$$

In particular, the three involutions ι_k have no relation.

A full proof will be given in [Og3].

From group theoretical view point, the second example is more interesting, because the group $\mathbf{Z}/2*\mathbf{Z}/2*\mathbf{Z}/2$ contains a non-commutative free group $\mathbf{Z}*\mathbf{Z}$ of rank 2 as a subgroup, so that the group is really highly non-commutative. Also, $\mathbf{Z}/2*\mathbf{Z}/2*\mathbf{Z}/2$ is the simplest group among groups containing $\mathbf{Z}*\mathbf{Z}$, in the sense that it is generated by the smallest number of elements of the lowest order.

As before the element $\iota_1\iota_2\iota_3$ is of positive entropy. Some orbits of points of W with a specified equation are also described in a beautiful paper of McMullen [Mc].

One natural question, for which we shall give some answer, is the following:

Question 3.3. Can one construct higher dimensional examples of Calabi-Yau manifolds and/or hyperkähler manifolds admitting a faithful action of $\mathbf{Z}/2*\mathbf{Z}/2*\mathbf{Z}/2$?

4. Calabi-Yau examples - fake and right

Let us start from a fake example, which is nevertheless of its own interest from the view of birational geometry such as birational version of Morrison's cone conjecture [Ka].

Fake Example

Let $V := (\mathbf{P}^1)^{n+1}$ where $n \geq 3$. Then the generic hypersurface W_n of multi-degree $(2, 2, \dots, 2)$ on V is a Calabi-Yau manifold of dimension n (in the strict sense) with Picard number $n+1$. Or more precisely, $\mathrm{Pic}\,(V) \simeq \mathrm{Pic}\,(W_n)$ under the natural restriction map.

Let

$$\{1, 2, \dots, n+1\} = \{k, k_1, k_2, \dots, k_n\}$$

and $p_k : W_n \to (\mathbf{P}^1)^n$ be the natural projection to the product of $(k_1, k_2, \dots, k_n)$ factors of V. Then p_k is a double cover. Note that p_k is *not* a finite morphism. Let ι_k be the covering transformation associated to p_k. Then ι_k acts on W_n as a *birational* autmorphism.

Theorem 4.1. *In the group of birational automorphisms* $\mathrm{Bir}\,(W_n)$,

$$\langle \iota_1, \iota_2, \iota_3 \rangle = \langle \iota_1 \rangle * \langle \iota_2 \rangle * \langle \iota_3 \rangle \simeq \mathbf{Z}/2 * \mathbf{Z}/2 * \mathbf{Z}/2\ .$$

In particular, the three involutions ι_1, ι_2, ι_3 have no relation.

The case $n = 2$ is nothing but the second example of Wehler and this theorem may looks like a right answer to Question 3.3 for Calabi-Yau manifolds. However, contrary to the case of K3 surfaces (the case $n = 2$) and also contray to some expectations by some of those who are working in complex dynamics, one can show that the group $\mathrm{Aut}\,(W_n)$ of biholomorphic automorphisms of W_n is a *finite* group and

$$\mathrm{Aut}\,(W_n) \cap \langle \iota_1, \iota_2, \iota_3 \rangle = \{id\}$$

in $\mathrm{Bir}\,(W_n)$, whenever $n \geq 3$. So, Theorem 4.1 is not a right answer in the category of biholomorphic automorphisms. In other words, complex dynamics of birational automorphisms of W_n, if possible, will be interesting but not the complex dynamics of biholomorphic automorphisms of W_n when $n \geq 3$. See [Og3] for details and proof.

Right Example

Let S be an Enriques surface, that is, a compact complex surface whose universal cover is a K3 surface $\tilde{S}$. It is well-known that S is projective, $\pi_1(S) \simeq \mathbf{Z}/2$, and the Enriques surfaces form 10-dimensional family [BHPV]. So, there are lots of Enriques surfaces. In [OS], we found that the *universal cover* $M_n = M_n(S)$ of the Hilbert scheme $\mathrm{Hilb}^n(S)$ of points of length n on S is a $2n$-dimensional Calabi-Yau manifold (in the strict sense). The following theorem (see [Og3] for proof) gives one right answer to Question 3.3:

Theorem 4.2. *Let S be a generic Enriques surface. Then S admits a faithful biholomorphic action of $\mathbf{Z}/2 * \mathbf{Z}/2 * \mathbf{Z}/2$. Moreover, this group action lifts to the faithful biholomorphic action on $M_n = M_n(S)$ (without making any extension of the group). In particular, the Calabi-Yau manifold $M_n(S)$ has a biholomorphic automorphism subgroup isomorphic to $\mathbf{Z}/2 * \mathbf{Z}/2 * \mathbf{Z}/2$. Moreover, the product of the three involutions is of positive entropy.*

5. HYPERKÄHLER EXAMPLES

We shall give two constructions of compact hyperkähler manifolds with a group of biholomorphic automorphisms of Wehler type as in Question 3.3. The first one is in any dimension but the automorphisms comes directly from those of surfaces. The second one inspired by a result of Beauville [Be2] is only in dimension 4 but the automorphism does not come directly from automorphisms of surfaces. Note that Calabi-Yau manifolds of dimension ≥ 3 are alway projective but this is no longer true for hyperkähler manifolds. But it is shown by [Og1] that the bimeromorphic automorphism group of a non-projective hyperkähler manifold is always almsot abelian. So, a non-projective hyperkähler manifold never admits a group of automorphisms of Wehler type, $\mathbf{Z}/2 * \mathbf{Z}/2 * \mathbf{Z}/2$.

First Construction

As it is well-known, the Hilbert scheme $\mathrm{Hilb}^n(X)$ of points of length n on a K3 surface X is a $2n$-dimensional hyperkähler manifold. Let S be a generic Enriques surface and $\tilde{S}$ be the universal covering K3 surface of S. Recall from the first part of Theorem 4.2 that S admits a faithful holomorphic action of $\mathbf{Z}/2 * \mathbf{Z}/2 * \mathbf{Z}/2$.

Theorem 5.1. *Let S be a generic Enriques surface and $\tilde{S}$ be the universal covering K3 surface. Then the $2n$-dimensional hyperkähler manifold $\mathrm{Hilb}^n(\tilde{S})$ admits a faithful biholomorphic action of $\mathbf{Z}/2 * \mathbf{Z}/2 * \mathbf{Z}/2$ naturally induced by the action on S. Moreover, the product of the three involutions is of positive entropy.*

See [Og3] for details and proofs.

Second Construction

In his paper [Be2], Beauville found a very interesting involution on the Hilbert scheme of points of length 2 on a quartic K3 surface. Let us first recall his involution.

Let X be a smooth surface of degree 4 in $\mathbf{P}^3$. Then X is a K3 surface. Let ℓ be a general line in $\mathbf{P}^3$ passing through two general points of X, say p, q. Then ℓ meets S at four points, say $\{p, q, r, s\}$. The correspondence $\{p, q\} \mapsto \{r, s\}$ then defines a birational involution ι on $\mathrm{Hilb}^2(X)$, the Hilbert scheme of points of length 2 on X. As remarked before, $\mathrm{Hilb}^2(X)$ is a 4-dimensional hyperkähler manifold. Beauville further shows the following:

Theorem 5.2. *Assume that X is generic in the sense that X contains no line of $\mathbf{P}^3$. Then ι is a biholomorphic automorphism of $\mathrm{Hilb}^2(X)$.*

We call the involution in Theorem 5.2, *Beauville's involution* associated to the embedding $X \subset \mathbf{P}^3$. Then, in order to construct Wehler type automorphisms, it is natural to consider a K3 surface X with *three different* embeddings into $\mathbf{P}^3$ each of whose image contains no line of $\mathbf{P}^3$. In fact, this can be carried out as:

Theorem 5.3. *There is a K3 surface X admitting three different embeddings into $\mathbf{P}^3$, say f_1, f_2, f_3, each of whose image contains no line of $\mathbf{P}^3$ such that*

(1) Beauville's involutions ι_i of $\mathrm{Hilb}^2(X)$, associated to the embeddings f_i ($i = 1, 2, 3$) has no relation in $\mathrm{Aut}\,(\mathrm{Hilb}^2(X))$ and

(2) the product $\iota_1\iota_2\iota_3$ is of positive entropy.

In particular, this hyperkähler 4-fold $\mathrm{Hilb}^2(X)$ admits a faithful biholomorphic group action of $\mathbf{Z}/2 * \mathbf{Z}/2 * \mathbf{Z}/2$ with positive entropy elements.

In this construction, the involutions are not in the image of the natural group homomorphism $\mathrm{Aut}\,(X) \to \mathrm{Aut}\,(\mathrm{Hilb}^2(X))$. The construction of X is based on the Torelli type theorem for K3 surfaces and so far the defining equations of the three embeddings are not explicit. Again see [Og3] for details and proofs. See also [Og2] for a similar example of hyperkähler 4-folds admitting a faithful biholomorphic action of $\mathbf{Z}/2 * \mathbf{Z}/2$ (like the first example of Wehler).

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