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Dynamics of Generalized Chebyshev maps of $\mathbf{C}^2$

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1 Introduction

In this note we study generalized Chebyshev maps of $\mathbf{C}^2$. They are specific polynomial endomorphisms of $\mathbf{C}^2$ that have connections with the complex Lie algebras of type A_2. We consider stochastic properties of generalized Chebyshev map of $\mathbf{C}^2$.

The theory in higher dimensional complex dynamics is developed using mostly pluripotential theory. Measures of maximal entropy are constructed and topological entropies and Lyapunov exponents are computed or estimated in many situations. The topological entropy of a holomorphic endomorphisms of algebraic degree d on $\mathbf{P}^k$ is equal to $k\log d$. The Lyapunov exponents of quadratic maps z^2+c on $\mathbf{C}$ are computed and they are equal to $\log 2$ if c lies in Mandelbrot set. In both cases, the stochastic invariants do not vary even if parameters change.

One of the most famous theories in the real one-dimensional dynamics is the kneading theory on the quadratic maps $g(x)=ax(1-x)$. The kneading invariants varies if the parameter a changes. We know generalized Chebyshev maps are extension of the quadratic maps. So we attempt to construct a kneading theory on generalized Chebyshev maps of $\mathbf{C}^2$.

2 Stochastic properties of the Green measure

We consider c-Chebyshev maps of degree 2 (see [U])

$$f_c(x,y)=(x^2-2cy,y^2-2cx).$$

The maps have some relations to the complex Lie algebra A_2.
The Green's function of $f_c(x,y)$ is given by

$$G(x,y):=\lim_{n\to\infty}\frac{1}{2^n}\log(\| f_c^n(x,y)\|^2+1)^{\frac{1}{2}}$$

This function gives the super-exponential rate at which the orbit of $(x, y) \in \mathbf{C}^2$ approaches the line at infinity. That function is Hölder-continuous on every compact subset of $\{(x, y, c) \in \mathbf{C}^2 \times \mathbf{C}\}$ and plurisubharmonic on $\mathbf{C}^2$. The Green's current is the positive closed (1,1)-current given by

$$T := \frac{1}{2\pi} dd^c G.$$

The first Julia set is given by $J_1 := supp(T)$ and

$$J_1(f_c) = \mathbf{C}^2 \setminus \{Fatou\ \ set\ \ of\ \ f_c\}.$$

Set $\mu := T \wedge T$. Then μ is known to be a maximal entropy measure.
The second Julia set is defined by $J_2 := supp(\mu)$.

As for the topological entropy the following result is known (see [DS]).

Theorem 2.1. *(Gromov-Misurewicz-Przytycky) Let f be a holomorphic endomorphism of algebraic degree d on $\mathbf{P}^k$. Then the topological entropy $h_t(f)$ of f is equal to $k \log d$, i.e. to the logarithm of the maximal dynamical degree.*

Hence the topological entropy of our c-Chebyshev map $f_c(x, y)$ is equal to $2 \log 2$ that does not vary if the parameter c changes.

Bedford and Jonsson[BJ]defined the critical measure (in our case) by

$$\mu_c := [Crit(f_c)] \wedge T = \frac{1}{2\pi} dd^c H \wedge T,$$

where $H = \log \mid \det Df_c \mid$
Then μ_c is a well defined positive measure and the mass of μ_c is equal to the degree of the critical locus (= 4 in our case). Note that the mass of μ_c does not vary even if the parameter c changes.

The sum of the Lyapunov exponents of$f$ with respect to μ is the number

$$\Lambda(f) = \lim_{n \to \infty} \frac{1}{n} \log \mid detDf^n(x) \mid, \quad \text{for} \quad \mu - a.e.\ \ x \in \mathbf{P}^2.$$

Bedford and Jonsson [BJ] show

$$\Lambda(f) = \Lambda(f_\Pi) + \log d + \int G \mu_c.$$

Using this equality and a result in [A], we can show the following result concerning the sum of the Lyapunov exponents $\Lambda(f_c)$.

Proposition 2.2. *(1) If $c = 1$, then $\Lambda(f_1) = 2\log 2$.*
(2) If $c > 1$, then $\Lambda(f_c) > 2\log 2$.

Proof. (1) Bedford and Jonsson [BJ](Theorem 7.4) and Andrei[A](Theorem 4.4) tell us that if

$$Crit(f_c) \cap W^s(J_\Pi, f_c) = \phi,$$

then $\Lambda(f_c) = 2\log 2$. So it suffices to prove that

$$Crit(f_1) \cap W^s(J_\Pi, f_1) = \phi.$$

Assume that $c = 1$. From [U], we know that we can write $(x, y) \in \mathbf{C}^2$ as

$$x = t_1 + t_2 + t_3, \quad y = t_1t_2 + t_1t_3 + t_2t_3, \quad t_1t_2t_3 = 1$$

and if $(x, y) \in Crit(f_1)$, then

$$(t_1 + t_2 + t_3) \times (t_1t_2 + t_1t_3 + t_2t_3) = 1. \qquad (2.1)$$

On the other hand if $(x, y) \in W^s(J_\Pi, f_1)$, then we can express t_1, t_2 and t_3 as

$$t_1 = re^{i\sigma}, t_2 = \frac{1}{r}e^{-i\tau}, t_3 = e^{i(\tau-\sigma)}, \quad r > 1. \qquad (2.2)$$

From (2.1) and (2.2) we have an equation

$$-2 = r^2e^{(\sigma+\tau)i} + \frac{1}{r^2}e^{-(\sigma+\tau)i} + re^{(2\sigma-\tau)i} + \frac{1}{r}e^{-(2\sigma-\tau)i} + re^{(2\tau-\sigma)i} + \frac{1}{r}e^{-(2\tau-\sigma)i}. \qquad (2.3)$$

By comparing the real part and the imagenary part of (2.3), we can conclude that there is no solution of (2.3) satisfying $r > 1$.

$$\text{Then} \qquad Crit(f_1) \cap W^s(J_\Pi, f_1) = \phi.$$

(2) From [U], we can parametrize $Crit(f_c) = \{(ct, c/t) : t \in \mathbf{C}\backslash\{0\}\}$ by the t-plane. Set $(u_n(t), v_n(t)) := f_c^n(ct, c/t)$. We consider the circle S^1 in $Crit(f_c)$ defined by

$$S^1 = \{(ce^{i\theta}, ce^{-i\theta}) : 0 \leq \theta < 2\pi\}.$$

In the proof of Proposition 3.1 in [U], we show that if $c > 1$, $\{f_c^n(ce^{i\theta}, ce^{-i\theta})\}$ converges to the line at infinity. Since $v_n(t) = u_n(1/t)$,

$$\overline{u_n(e^{i\theta})} = v_n(e^{i\theta})$$

and so

$$\lim_{n\to\infty} \frac{| u_n(e^{i\theta} |}{| v_n(e^{i\theta}) |} = 1.$$

Clearly

$$J_\Pi = \{[x : y : 0] : \quad | x |=| y |\}.$$

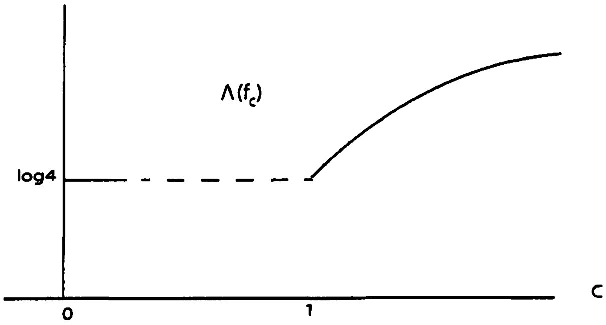
Hence

$$S^1 \subset W^s(J_\Pi, f_c)$$

and so

$$Crit(f_c) \cap W^s(J_\Pi, f_c) \neq \phi.$$

Then from Theorem 4.4 in [A], we can see that $\Lambda(f_c) > 2\log 2$. □



This graph of the sum of the Lyapunov exponents may be constant in the interval [0,1].

If we restrict our maps $f_c(x, y)$ to the diagonal line $\{x = y\}$, we have quadratic maps

$$q_c(z) = z^2 - 2cz \quad on \quad \mathbf{C}.$$

If we restrict the maps $q_c(z)$ to the real line, we have real one-dimensional quadratic maps $p_c(z_1)$. For the maps $p_c(z_1)$ the kneading theory is well-known. It presents some invariants that vary if the parameter c moves along the interval [0,1]. Hence we will reach a quantity which varies if c moves along the interval [0,1].

3 Slices of critical measures and computer experiments

We begin with the critical measures. By direct calculation, we have

$$\int_{\mathbf{C}^2} \mu_c = \int_{\mathbf{C}^2} [Crit(f_c)] \wedge T = \int_{Crit(f_c)} T = 2.$$

Instead of integrating the current T over whole the t-plane, we consider the integration along the unit circle in the t-plane. The reasons why we choose the unit circle, are that the approximation

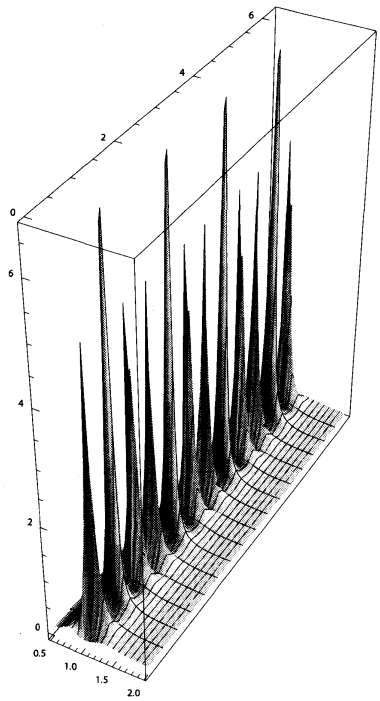
$$G_n(t) := \frac{1}{2^n} \log(\| f_c^n(x, y) \|^2 + 1)^{\frac{1}{2}} \qquad (3.1)$$

of the Green's function $G(ct, c/t)$ restricted to the t-plane is symmetric with respect to the unit circle and that the rate of escape to infinity has the minimum on the unit circle if $c > 1$. We consider the trunated current restricted to the $Crit(f_c)$. We see that

$$dd^cG_n(x,y)\,|_{\binom{x=ct}{y=c/t}} = dd^cG_n(t) = \frac{r}{4\pi} \triangle\, G_n(r,\theta)dr \wedge d\theta, \qquad (3.2)$$

$$\text{where} \quad \triangle = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \theta^2}.$$

The graph of $r \triangle G_n(r,\theta)$ in the case $n = 3$ and $c = 1$ is depicted in the following.



We see many spikes on the unit circle .

If the unit circle in the t-plane intersects $supp(T)$, the graph of the truncated current $r \triangle G_n$ has many spikes in the circle whose heights approach infinity as $n \to \infty$. We call the unit circle in the t-plane the critical circle.

We measures the average growth rate of the spikes of $\frac{r}{4\pi} \triangle G_n$ along the unit circle in the t-plane. Set

$$g_n(c) := \frac{1}{4\pi}\int_0^{2\pi} (r \triangle G_n)\,|_{r=1}\, d\theta.$$

$$\text{Recall that} \quad dd^cG_n(t) = \frac{r}{4\pi} \triangle\, G_n(r,\theta)dr \wedge d\theta.$$

Lemma 3.1

$$(1) \quad \frac{\partial}{\partial r} G_n(re^{i\theta}) \mid_{r=1} = 0,$$

$$(2) \quad [\frac{\partial}{\partial r} G_n(r,\theta)]_0^{2\pi} = 0.$$

The proof is left to the reader. Then we have

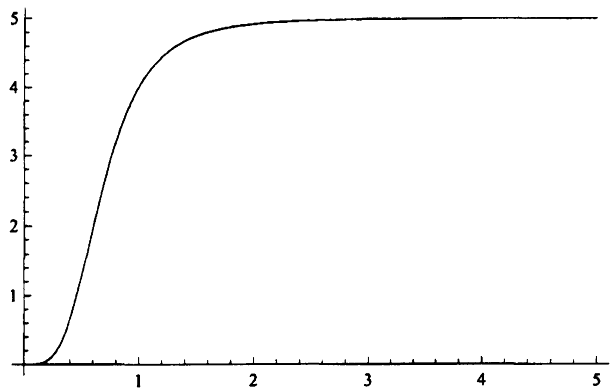
$$g_n(c) = \frac{1}{4\pi} \int_0^{2\pi} (\frac{\partial^2}{\partial r^2} G_n) \mid_{r=1} d\theta. \tag{3.3}$$

We consider the sequence $\{g_n(c)^{\frac{1}{n}}\}$ for $c > 0$.

We show some computer experiments of the graphs $g_1(c)$, $g_2(c)^{\frac{1}{2}}$ and $g_3(c)^{\frac{1}{3}}$.

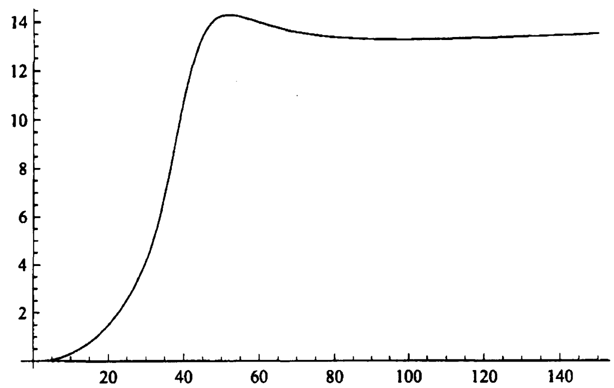
Case $n = 1$.

```
Plot[fsint, {c, 0, 5}, PlotRange → All]
```



Case $n = 2$.

```
g2 = ListLinePlot[T2]
```

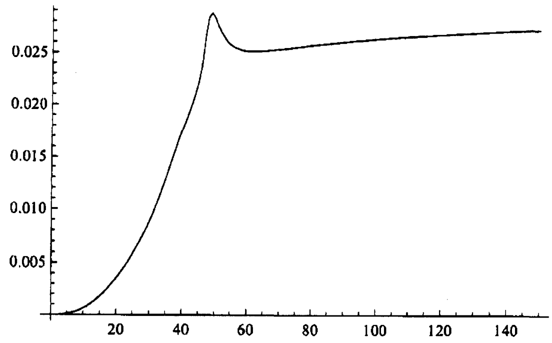


Case $n = 3$.

In[3] =

```
ListLinePlot[graph / (60 Pi)]
```

Out[3]=



4 A conjecture and some results

Based on the facts in the previous section we conjecture the following.

Conjecture 4.1. *If $c > 0$, there exists a limit*

$$\lim_{n\to\infty}(g_n(c)^{\frac{1}{n}}.$$

Concerning this conjecture we have some results.

Proposition 4.2. *If $c = 1$, then*

$$g_n(1) = 2^n \cdot \frac{57 + 53\sqrt{57}}{912},$$

$$\textit{and so} \qquad \lim_{n\to\infty} g_n(1)^{\frac{1}{n}} = 2.$$

Proof. Let

$$f_c^n(ct, c/t) =: (u_n(t), v_n(t)), \quad K_n(t) := 1 + \mid u_n(t) \mid^2 + \mid v_n(t) \mid^2.$$

$$\textit{Then} \qquad G_n(t) = \frac{1}{2^{n+1}} \log K_n(t).$$

We will calculate $g_n(1)$. By (3.3), it equals

$$\frac{1}{4\pi}\int_0^{2\pi} (\frac{\partial^2}{\partial r^2} G_n) \mid_{r=1} d\theta.$$

By Lemma 3.1.(1), we see that

$$\frac{\partial^2}{\partial r^2}\log K_n(re^{i\theta})\mid_{r=1}=(\frac{1}{K_n(re^{i\theta})}\frac{\partial^2}{\partial r^2}\log K_n(re^{i\theta}))\mid_{r=1}.$$

Set

$$k_m(r,\theta):=(\frac{1}{r})^{2^{m+1}}+4(\frac{1}{r})^{2^m}+1+4r^{2^m}+r^{2^{m+1}}+4((\frac{1}{r})^{2^{m-1}}+r^{2^{m-1}})\cos(3\cdot 2^{m-1}\theta).$$

Clearly

$$k_m(r,\theta)=k_1(r^{2^{m-1}},2^{m-1}\theta).$$

Claim. *Assume that $c=1$ and $n\geq 2$. Then*

$$K_n(re^{i\theta})=k_n(r,\theta).$$

The proof of this Claim is straightforward.
Hence

$$(\frac{1}{K_n(re^{i\theta})}\frac{\partial^2}{\partial r^2}\log K_n(re^{i\theta}))\mid_{r=1}=2^{2n+1}\frac{8+\cos(3\cdot 2^{n-1}\theta)}{11+8\cos(3\cdot 2^{n-1}\theta)}.$$

Clearly

$$\int_0^{2\pi}(\frac{8+\cos(3\cdot 2^{n-1}\theta)}{11+8\cos(3\cdot 2^{n-1}\theta)})d\theta=\int_0^{2\pi}(\frac{8+\cos(3\theta)}{11+8\cos(3\theta)})d\theta=\frac{57+53\sqrt{57}}{288}\pi.$$

Then we obtain the proposition. □

Next we consider a case where the parameter c approaches ∞.

Proposition 4.3.

$$\lim_{c\to\infty}g_n(c)=\frac{5}{8}\cdot 2^n,$$

$$\text{and so}\qquad \lim_{n\to\infty}(\lim_{c\to\infty}g_n(c))^{\frac{1}{n}}=2.$$

Proof. First we note that $K_n(c,r,\theta)$ and $\frac{\partial^2}{\partial r^2}K_n(c,r,\theta)$ are polynomials in c with the same degree. Then $\frac{1}{K_n}\frac{\partial^2}{\partial r^2}K_n$ converges uniformly as $c\to\infty$.

$$\lim_{c\to\infty}g_n(c)=\frac{1}{4\pi}\cdot\frac{1}{2^{n+1}}\int_0^{2\pi}\lim_{c\to\infty}(\frac{1}{K_n}\frac{\partial^2}{\partial r^2}K_n)\mid_{r=1}d\theta.$$

Both $u_n(t)$ and $v_n(t)$ are also polynomials in c. Let $uh_n(t)$ and $vh_n(t)$ be the highest degree terms with respect to c of $u_n(t)$ and $v_n(t)$. Then we can easily see that if $n\geq 2$, then

$$uh_n(t)=(c^2(t^2-\frac{2}{t}))^{2^{n-1}},$$

and

$$vh_n(t) = (c^2(\frac{1}{t^2} - 2t))^{2^{n-1}}.$$

$$\text{Set} \qquad Kh_n = | uh_n(re^{i\theta}) |^2 + | vh_n(re^{i\theta}) |^2 .$$

$$\text{Then} \qquad \lim_{c\to\infty} \frac{1}{K_n}\frac{\partial^2}{\partial r^2}K_n = \frac{1}{Kh_n}\frac{\partial^2}{\partial r^2}Kh_n.$$

By direct calculations, we have

$$(\frac{1}{Kh_n}\frac{\partial^2}{\partial r^2}Kh_n) |_{r=1}= \frac{1}{(5-4\cos 3\theta)^2}[4\cdot 2^{n-1}\{6(6+2^{n-1})+(-45+8\cdot 2^{n-1})\cos 3\theta+2\cdot 2^{n-1}\cos 6\theta\}].$$

$$\text{Therefore} \qquad \int(\frac{1}{Kh_n}\frac{\partial^2}{\partial r^2}Kh_n) |_{r=1} \, d\theta = I_n(\theta),$$

where $I_n(\theta) = 2^{n-2}\{2^n \cdot \theta + 6 \cdot 2^n \tan^{-1}(3\tan\frac{3\theta}{2}) + \frac{\sin 3\theta(12\cdot 2^n - 24)}{(5-4\cos 3\theta)}.$

Hence

$$\int_0^{2\pi}(\frac{1}{Kh_n}\frac{\partial^2}{\partial r^2}Kh_n) |_{r=1} \, d\theta = [I_n(\theta)]_{-\pi}^{\frac{\pi}{3}} + [I_n(\theta)]_{\frac{\pi}{3}}^{\pi} + [I_n(\theta)]_{\pi}^{\frac{5}{3}\pi} = 5\pi \cdot 4^n.$$

Hence

$$\lim_{c\to\infty}\frac{1}{4\pi}\int_0^{2\pi}\lim_{c\to\infty}(\frac{\partial^2}{\partial r^2}G_n) |_{r=1} \, d\theta = \frac{5}{8}\cdot 2^n. \qquad \square$$

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