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ON IMMERSED ORIENTED SURFACES AND THEIR
PLANE PROJECTIONS

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1. INTRODUCTION

Throughout the report, all manifolds and maps are differentiable of class $C^\infty$. Let $M$ be a closed connected surface, $f$ and $g : M \to \mathbb{R}^3$ immersions. Let $F : M \times [0, 1] \to \mathbb{R}^3$ be a homotopy between $f$ and $g$. That is, $F|M \times \{0\} = f$ and $F|M \times \{1\} = g$ hold. We call $F$ a regular homotopy between $f$ and $g$ if a level preserving map $(F \times id) : M \times [0, 1] \to \mathbb{R}^3 \times [0, 1]$ defined by $(F \times id)(x, t) = (F(x, t), t)$ is an immersion. James and Thomas [4] proved that the space of immersions $f : M \to \mathbb{R}^3$ has $2^{\dim H_1(M; \mathbb{Z}_2)}$ connected components. Pinkall [6] associated to any immersion $f : M \to \mathbb{R}^3$ a $\mathbb{Z}_4$-valued quadratic form $g_f : H_1(M; \mathbb{Z}_2) \to \mathbb{Z}_4$ and proved that two immersions $f$ and $g : M \to \mathbb{R}^3$ are regularly homotopic if and only if $g_f = g_g$ holds.

Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be an orthogonal projection and $\tilde{f} : M \to \mathbb{R}^2$ a stable map. If a stable map $\tilde{f} : M \to \mathbb{R}^2$ has an immersion $f : M \to \mathbb{R}^3$ such that $\pi \circ f = \tilde{f}$, we call that $f$ has an immersion lift $\tilde{f}$ and that $f$ is an immersion lift over $\tilde{f}$. Let $f$ and $g : M \to \mathbb{R}^3$ be immersion lifts over $\tilde{f} : M \to \mathbb{R}^2$. If there exists a regular homotopy $F : M \times [0, 1] \to \mathbb{R}^3$ between $f$ and $g$ which satisfies that $\pi \circ (F|M \times \{t\}) = \tilde{f}$ for any $t \in [0, 1]$, then we call that $f$ and $g$ are $\tilde{f}$-regularly homotopic.

In this report, when $M$ is a closed connected oriented surface, we study $\tilde{f}$-regular homotopy classes for a fixed stable map $\tilde{f} : M \to \mathbb{R}^2$ which has an immersion lift.

This report is organized as follows. In Section 2, we give the definition and see properties of a stable map $\tilde{f} : M \to \mathbb{R}^2$. In Section 3, we restate Haefliger’s theorem [3] by putting sings on the apparent contour of $\tilde{f}$. In Section 4, we determine $\tilde{f}$-regular homotopy classes for a fixed stable map $\tilde{f}$ which has an immersion lift. In Section 5, we give the definitions of generic homotopy and regular homotopy lift. By using the method obtained in the previous section, we introduce a necessary and sufficient condition for existence of regular homotopy lift over the given generic homotopy. As an application, we construct a generic homotopy whose regular homotopy lift corresponds to a sphere eversion.

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2. Stable map

In this section, we give the definition and see properties of a stable map. Let \( \tilde{f} : M \to \mathbb{R}^2 \) be a smooth map of a closed connected surface \( M \) into the plane. We denote the set of such maps by \( C^\infty(M, \mathbb{R}^2) \), which is equipped with the Whitney \( C^\infty \)-topology. A smooth map \( \tilde{f} \) is said to be a stable map if in \( C^\infty(M, \mathbb{R}^2) \), there exists an open neighborhood \( U \) of \( \tilde{f} \) such that for any \( \tilde{g} \in U, \tilde{g} \) is \( C^\infty \) right-left equivalent to \( \tilde{f} \), i.e., there exist two diffeomorphisms \( \Phi : M \to M \) and \( \varphi : \mathbb{R}^2 \to \mathbb{R}^2 \) such that the diagram

\[
\begin{array}{ccc}
M & \xrightarrow{\Phi} & M \\
\downarrow \tilde{f} & & \downarrow \tilde{g} \\
\mathbb{R}^2 & \xrightarrow{\varphi} & \mathbb{R}^2
\end{array}
\]

is commutative.

For a smooth map \( \tilde{f} : M \to \mathbb{R}^2 \), we denote by \( S(\tilde{f}) \) the set of the points in \( M \) where the rank of the differential of \( \tilde{f} \) is strictly less than two. We say that \( S(\tilde{f}) \subset M \) is a singular set of \( \tilde{f} \) and \( \tilde{f}(S(\tilde{f})) \subset \mathbb{R}^2 \) is an apparent contour of \( \tilde{f} \).

The following characterizations of stable maps are well-known (see [2, 8], for example).

**Proposition 2.1.** A smooth map \( \tilde{f} : M \to \mathbb{R}^2 \) of a closed surface \( M \) is a stable map if and only if the following conditions are satisfied.

(i) For every \( q \in M \), there exist local coordinates \((x, y)\) and \((X, Y)\) around \( q \in M \) and \( \tilde{f}(q) \in \mathbb{R}^2 \) respectively such that one of the following holds:

(a) \((X \circ \tilde{f}, Y \circ \tilde{f}) = (x, y) \) (\( q \) : regular point),

(b) \((X \circ \tilde{f}, Y \circ \tilde{f}) = (x, y^2) \) (\( q \) : fold point),

(c) \((X \circ \tilde{f}, Y \circ \tilde{f}) = (x, y^3 - xy) \) (\( q \) : cusp point).

(ii) If \( q \in M \) is a cusp point, then \( \tilde{f}^{-1}(\tilde{f}(q)) \cap S(\tilde{f}) = \{q\} \).

(iii) The map \( \tilde{f}((S(\tilde{f}) \setminus \{cusp points\})) \) is an immersion with normal crossings.

Note that \( S(\tilde{f}) \) is a compact 1-dimensional submanifold of \( M \) and the number of cusp points is finite. Let \( U \subset M \) be a tubular neighborhood of \( S(\tilde{f}) \). Then the restriction of \( \tilde{f} \) on the closure \( cl(M \setminus U) \) is an immersion. By the apparent contour of \( \tilde{f} \), \( \mathbb{R}^2 \) is naturally stratified into 2-, 1- and 0-dimensional strata. The union of 1- and 0-dimensional strata forms \( \tilde{f}(S(\tilde{f})) \). On each 1-dimensional stratum, we can define an orientation as follows. We fix the canonical orientation on \( \mathbb{R}^2 \). Let \( \Omega \) be a connected component of \( \mathbb{R}^2 \setminus \tilde{f}(S(\tilde{f})) \). We associate to \( \Omega \) a non-negative integer \( n_{\tilde{f}}(\Omega) \), which is the number of points in the fiber of \( \tilde{f} \) over any point of \( \Omega \). Every 1-dimensional stratum is adjacent to exactly two connected components of \( \mathbb{R}^2 \setminus \tilde{f}(S(\tilde{f})) \). Since these two components have distinct \( n_{\tilde{f}}(\Omega) \)-values, we can orient each 1-dimensional stratum in \( \tilde{f}(S(\tilde{f})) \) so that the region with the larger \( n_{\tilde{f}}(\Omega) \)-value is on its left. Since \( \tilde{f}((S(\tilde{f}) \setminus \{cusp points\})) \) is an immersion, \( S(\tilde{f}) \setminus \{cusp points\} \) is also oriented.
Suppose that $M$ is an oriented closed surface and $\mathbb{R}^2$ is oriented plane. Let $q$ be a cusp point of a stable map $\tilde{f} : M \to \mathbb{R}^2$. For a sufficiently small neighborhood $U$ of $f(q)$, the map $\tilde{f}|V : V \to U$ has degree $\pm 1$, where $V$ is the component of $\tilde{f}^{-1}(U)$ containing $q$. We call $q$ is a positive (resp. negative) cusp if the local degree of $\tilde{f}$ at $q$ equals $+1$ (resp. $-1$).

3. IMMERSION LIFT

In the following, we assume that $M$, $\mathbb{R}^3$ and $\mathbb{R}^2$ are oriented. In this case, Haefliger's theorem is restated as follows.

**Theorem 3.1** (Haefliger [3]). A stable map $\tilde{f} : M \to \mathbb{R}^2$ has an immersion lift if and only if each connected component of $S(\tilde{f})$ has even number of cusp points.

Let $\tilde{f} : M \to \mathbb{R}^2$ be a stable map which has an immersion lift. On each connected component of fold points $S(\tilde{f}) \setminus \{\text{cusp points}\}$, we can put a sign $+1$ or $-1$ which satisfies the following rule.

- Let $C$ and $C'$ be two connected components which adjacent to the same cusp point. Then $C$ and $C'$ have the opposite signs.

If a sign of $C$ is $+1$ (resp. $-1$), we call $C$ a positive (resp. negative) fold and a sign can be put on each image of fold component. Such a stable map $\tilde{f}$ is called a signed stable map.

Let $\tilde{f} : M \to \mathbb{R}^2$ be a signed stable map and $U$ a tubular neighborhood of $S(\tilde{f})$. Since $M$ is oriented, $U \setminus S(\tilde{f})$ is divided into two regions $U_+$ and $U_-$ where $\tilde{f}|U_+$ (resp. $\tilde{f}|U_-$) is an orientation preserving (resp. reversing) immersion. We construct an immersion lift $f_U : U \to \mathbb{R}^3$ over $\tilde{f}|U$ which satisfies the following.

1. If $C$ is a positive fold, $f$ is defined as Figure 1(a).
2. If $C$ is a negative fold, $f$ is defined as Figure 1(b).
3. If $q$ is a positive cusp and negative fold comes in $q$ for the orientation of $S(\tilde{f})$, $f$ is defined as Figure 2(a).
4. If $q$ is a positive cusp and positive fold comes in $q$ for the orientation of $S(\tilde{f})$, $f$ is defined as Figure 2(b).
5. If $q$ is a negative cusp and negative fold comes in $q$ for the orientation of $S(\tilde{f})$, $f$ is defined as Figure 2(c).
6. If $q$ is a negative cusp and positive fold comes in $q$ for the orientation of $S(\tilde{f})$, $f$ is defined as Figure 2(d).

**Definition 3.2.** Let $\tilde{f} : M \to \mathbb{R}^2$ be a signed stable map and $U$ a tubular neighborhood of $S(\tilde{f})$. If an immersion $f : M \to \mathbb{R}^3$ satisfies the above rules (1)-(6) on $f|U$, we call $f$ an immersion lift over the signed stable map $\tilde{f}$.

4. $\tilde{f}$-REGULAR HOMOTOPY

In this section, we state that $\tilde{f}$-regular homotopy classes can be determined.

**Theorem 4.1.** If $f$ and $g : M \to \mathbb{R}^3$ are immersion lifts over the signed stable map $\tilde{f} : M \to \mathbb{R}^2$, then $f$ and $g$ are $\tilde{f}$-regularly homotopic.
Corollary 4.2. Let $\tilde{f}: M \to \mathbb{R}^2$ be a stable map which has an immersion lift. (Note that $\tilde{f}$ is not signed.) The number of $\tilde{f}$-regular homotopy classes is $2^{|S(\tilde{f})|}$, where $|S(\tilde{f})|$ is the number of connected components of $S(\tilde{f})$.

We have a following example which is related to Theorem 4.1.

Example 4.3. Let $T^2$ be an oriented torus and $l$ and $m$ longitude and meridian of $T^2$, respectively. Let $\tilde{f}$ and $\tilde{g}: T^2 \to \mathbb{R}^2$ be signed stable maps which satisfy the following properties. They do not have cusp points, $\tilde{f}(S(\tilde{f})) = \tilde{g}(S(\tilde{g}))$, both signs are the same, $\tilde{f}|l = \tilde{g}|l$ and $\tilde{g}|m$ are plane curves whose rotation numbers equal 2 (or -2), $\tilde{f}|m$ is a simple closed plane curve. See Figure 3. By the theorem of Pinkall [6], immersion lifts $\tilde{f}$ and $\tilde{g}: T^2 \to \mathbb{R}^3$ over $\tilde{f}$ and $\tilde{g}$ respectively are not regularly homotopic.

Theorem 3.1 and Example 4.3 mean that if $M \neq S^2$, an apparent contour with sign does not determine a regular homotopy class. We need information of immersion $\tilde{f}|(M \backslash S(\tilde{f}))$.

5. REGULAR HOMOTOPY LIFT OVER A GENERIC HOMOTOPY

Let $\tilde{f}$ and $\tilde{g}: M \to \mathbb{R}^2$ be stable maps and $\tilde{F}: M \times [0, 1] \to \mathbb{R}^2$ a homotopy between $\tilde{f}$ and $\tilde{g}$. If $\tilde{F}$ satisfies the following conditions, we call $\tilde{F}$ a generic homotopy between $\tilde{f}$ and $\tilde{g}$ (see [5]).

(1) There is a finite set of parameter values $0 < t_1 < \cdots < t_n < 1$ (possibly empty) in $(0, 1)$.

(2) For any $t \in (0, 1) \backslash \{t_1, \ldots, t_n\}$, $\tilde{F}|M \times \{t\}: M \times \{t\} \to \mathbb{R}^2$ is a stable map.

(3) For each $t_i$ and a sufficiently small positive value $\varepsilon$, the moves of apparent contours of $\tilde{F}|M \times \{t\}$ ($t \in (t_i - \varepsilon, t_i + \varepsilon)$) are classified into lips (type $L$), beaks (type $B$), swallowtail (type $S$), cusp-fold (type $C$), self-tangency (type $K$) or triple point (type $T$).
FIGURE 2. Immersion lifts if (a) $q$ is a positive cusp and negative fold comes in, (b) $q$ is a positive cusp and positive fold comes in, (c) $q$ is a negative cusp and negative fold comes in, (b) $q$ is a negative cusp and positive fold comes in.

We call each $t_i$ a bifurcation point on a generic homotopy $\tilde{F}$.

Let $\tilde{F} : M \times [0, 1] \to \mathbb{R}^2$ be a generic homotopy between signed stable maps $\tilde{f}$ and $\tilde{g}$ and let $f$ and $g$ immersion lifts over $\tilde{f}$ and $\tilde{g}$, respectively. If there exists a regular homotopy $F : M \times [0, 1] \to \mathbb{R}^3$ between $f$ and $g$ such that $\pi \circ F = \tilde{F}$, we call $F$ a regular homotopy lift over $\tilde{F}$.

Theorem 5.1. Let $\tilde{f}$ and $\tilde{g} : M \to \mathbb{R}^2$ be signed stable maps. If there exists a generic homotopy $\tilde{F} : M \times [0, 1] \to \mathbb{R}^2$ between $\tilde{f}$ and $\tilde{g}$ which preserves sign
FIGURE 3. Two stable maps \( \tilde{f} \) and \( \tilde{g} : T^2 \to \mathbb{R}^2 \) which satisfy that \( \tilde{f}(S(\tilde{f})) = \tilde{g}(S(\tilde{g})) \) and both apparent contours have positive signs. But their immersion lifts \( f \) and \( g : T^2 \to \mathbb{R}^3 \) are not regularly homotopic.

convention as depicted in Figures 4 and 5, then \( \tilde{F} \) has a regular homotopy lift \( F : M \times [0, 1] \to \mathbb{R}^3 \).

As an application of Theorem 5.1, we have the following example.

**Example 5.2.** If \( \tilde{f} \) and \( \tilde{g} : S^2 \to \mathbb{R}^2 \) are signed stable maps such that \( \tilde{f}(S(\tilde{f})) = \tilde{g}(S(\tilde{g})) = D^2 \), \( \tilde{f}(S(\tilde{f})) = \tilde{g}(S(\tilde{g})) \) is a simple closed curve and the sign of \( S(\tilde{f}) \) (resp. \( S(\tilde{g}) \)) is +1 (resp. -1). Then there is a generic homotopy \( \tilde{F} : S^2 \times [0, 1] \to \mathbb{R}^2 \) between \( \tilde{f} \) and \( \tilde{g} \) which has a regular homotopy lift \( F : S^2 \times [0, 1] \to \mathbb{R}^3 \). See Figure 6. By the definitions of \( \tilde{f}, \tilde{g} \), the regular homotopy lift \( F \) over \( \tilde{F} \) corresponds to an eversion of the embedded sphere.

Our eversion in Example 5.2 is almost same as the eversion given by Francis [1]. But in his picture, self intersections of immersed spheres were not drawn. Professor Mikami Hirasawa and the author draw a regular homotopy over the generic homotopy of Figure 6, precisely. So, we can follow how self intersections move during our sphere eversion. Our eversion will appear in their preparing paper.

**REFERENCES**

FIGURE 4. Bifurcations of type $L, B, S$ and $C$ which have regular homotopy lifts. Here, $\alpha = \pm 1$ and $\beta = \pm 1$ and $\alpha$ and $\beta$ vary independently.

FIGURE 5. Bifurcations of type $K$ and $T$ which have regular homotopy lifts. Here, $\alpha = \pm 1$, $\beta = \pm 1$ and $\gamma = \pm 1$ and $\alpha, \beta$ and $\gamma$ vary independently.

Figure 6. A sequence of apparent contours of a generic homotopy between $f$ and $\tilde{g} : S^2 \to \mathbb{R}^2$ which has a regular homotopy lift. This regular homotopy corresponds to a sphere eversion.