<table>
<thead>
<tr>
<th>Title</th>
<th>Basic Topics on Tropical Geometry and Singularities (Geometry on Real Closed Field and its Application to Singularity Theory)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Ishikawa, Goo</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2011) 1764: 149-164</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2011-09</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/171393">http://hdl.handle.net/2433/171393</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
Basic Topics on Tropical Geometry and Singularities

(トロピカル幾何と特異性に関する基本的トピックス)

Goo Ishikawa
Department of Mathematics, Hokkaido University, Japan
(石川剛郎, 北海道大学数学教室, 日本)
e-mail: ishikawa@math.sci.hokudai.ac.jp

Motivated from real algebraic geometry, Viro, Mikhalkin, Shustin, Itenberg, and other mathematicians have developed the “tropical (algebraic) geometry” [8]. Algebraic curves are tropicalized to piecewise-linear curves. The method was used to construct topological types of real algebraic curves in Hilbert’s 16th problem [24].

In this rough sketch, we present several basic topics of tropical geometry, in particular, the notion of hyperfields introduced by Viro recently [25].

【Tropical Limits of Operations】

Let $\mathbb{R}_+$ denote the set of non-negative real numbers.

We fix $h > 0$ and consider the bijection

\[ h \log : \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\} \]

defined by

\[ u \mapsto h \log u, \quad e^{h} \leftarrow x. \]
On $\mathbb{R} \cup \{-\infty\}$, two operations
\[
\begin{align*}
  x +_h y & := h \log \left( e^{\frac{x}{h}} + e^{\frac{y}{h}} \right) \\
  x \times_h y & := h \log \left( e^{\frac{x}{h}} \cdot e^{\frac{y}{h}} \right) = x + y
\end{align*}
\]
are induced from the summension and the multiplication on $\mathbb{R}_+$. Set $m = \max\{x, y\}$. Then we have
\[h \log \left( e^{\frac{m}{h}} \right) \leq x +_h y \leq h \log \left( e^{\frac{m}{h}} + e^{\frac{m}{h}} \right),\]
namely,
\[m \leq x +_h y \leq m + h \log 2.\]
Therefore we have that
\[\lim_{h \downarrow 0} (x +_h y) = \max\{x, y\}.

**[Tropical Semi-Ring]**

$\mathbb{R}_{trop} = \mathbb{R} \cup \{-\infty\}$ with the two operations
\["x + y" := \max\{x, y\}, \quad "x \cdot y" := x + y,\]
is called the *tropical semi-ring* (or the *max-plus algebra*).

Moreover we set "$x/y" = x - y" if $y \neq -\infty$. Note that there is no tropical subtraction. The tropical sum is idempotent:
\["x + x" = x,\]
$-\infty$ being the tropical zero.

**[Tropical Polynomials]**

For a finite subset $A \subset \mathbb{Z}^n$, consider a "tropical" (Laurent) polynomial
\[F(x) = " \sum_{j \in A} c_j x^j" = \max\{c_j + j \cdot x \mid j \in A\},\]
$(c_j \in \mathbb{R})$, which is a PL-function on $\mathbb{R}^n$. Then the tropical hypersurface $Y_F \subset \mathbb{R}^n$ is defined by $F$ as the *corner locus* of $F$. 
Example 1. (tropical line). We consider
\[ F(x_1, x_2) = "ax_1 + bx_2 + c" = \max\{x_1 + a, x_2 + b, c\}. \]
Then \(Y_F\) consists of three half-lines meeting at one point.

\[ F(x_1, x_2) = "ax_1 + bx_2 + c" = \max\{x_1 + a, x_2 + b, c\}. \]

\[ Y_F = \{x_1 + a, x_2 + b, c\}. \]

【Tropical Hyperfields】

We define a \textit{multi-valued} addition \(\gamma\) on \(\mathbb{R} \cup \{-\infty\}: \) For \(a, b \in \mathbb{R} \cup \{-\infty\},\) we set
\[ a \gamma b := \begin{cases} \max\{a, b\}, & (a \neq b) \\ \{y \in \mathbb{R} \cup \{-\infty\} \mid y \leq a\}, & (a = b) \end{cases} \]

The multiplication is defined by the ordinary addition.
We set \(\mathbb{Y} = (\mathbb{R} \cup \{-\infty\}, \gamma, +)\) and we call it the \textit{tropical hyperfield}.
This implies the natural definition of "tropical zero".

Example 2. For \(a \in \mathbb{R},\) we define the function \(x \gamma a : \mathbb{Y} \to \mathbb{Y}.\) Then we have
\[ \text{graph}(x \gamma a) = \{y = a, x < a\} \cup \{y = x, a < x\} \cup \{y \leq a, x = a\}. \]
【Definition of Hyperfields】

Suppose there are given, on a set $X$, a multi-valued binary operation $\tau$ and a single-valued binary operation $\cdot$.

Then $(X, \tau, \cdot)$ is called a hyperfield if

- $a \tau b = b \tau a$, $a \tau (b \tau c) = (a \tau b) \tau c$
- $\exists 0 \in X$, $0 \tau a = a$, for any $a \in X$,
- $\forall a \in X, \exists \iota - a \in X$ (minus $a$) such that $0 \in a \tau (-a)$.
- $c \in a \tau b \iff (-c) \in (-a) \tau (-b)$
- The operation $\cdot$ is commutative, associative and $0 \cdot a = 0$ holds for any $a \in X$,
- $(X \setminus \{0\}, \cdot)$ is a commutative group, which will be denoted by $X^\times$,
- the "distributive law" holds: $a \cdot (b \tau c) = (a \cdot b) \tau (a \cdot c)$, $(b \tau c) \cdot a = (b \cdot a) \tau (c \cdot a)$.

**Lemma 3.** The tropical hyperfield $\mathcal{Y} = (\mathbb{R} \cup \{-\infty\}, \gamma, +)$ is a hyperfield.

In fact we have

- The zero-element is $-\infty$.
- For $a \in \mathcal{Y}$, $-a$ equals $a$, since $-\infty \in a \gamma b \iff b = a$.
- The commutative group $\mathcal{Y}^\times = (\mathbb{R}, +)$, the unit being $0 \in \mathbb{R}$.
For a tropical Laurent polynomial \( F(x) = \sum_{j \in A} c_j x^j \), we define \( v = -c : A \to \mathbb{R} \) by \( v(j) = -c_j, (j \in A) \). Then we set \( \bigcup(v) := \text{convex hull} \{ (j, y) \in \mathbb{R}^n \times \mathbb{R} \mid j \in A, y \geq v(j) \} \subset \mathbb{R}^{n+1} \).

We set \( \Delta = \Delta_F = \text{convex hull} (A) \subset \mathbb{R}^n \), and \( \tilde{\Delta} \) the union of compact faces of \( \bigcup(v) \). We call \( \Delta = \Delta_F \) the Newton polyhedra of \( F \).

Then \( \tilde{\Delta} \) projects to \( \Delta \) in bijection by \( \pi : \tilde{\Delta} \to \mathbb{R}^n, \pi(j, y) := j \). An integral subdivision of \( \Delta \) is induced from \( \tilde{\Delta} \). We obtain the convex function \( \overline{v} : \Delta \to \mathbb{R} \) having \( \tilde{\Delta} \) as its graph.

The tropical hypersurface \( Y_F \) is an \((n - 1)\)-dimensional regular polyhedral complex. (Regularity condition: the boundary of each \( i \)-cell is a union of \((i - 1)\)-cells.)

Along each \((n - 1)\)-cell \( I \), two functions \( c_j + j \cdot x, c_k + k \cdot x \) have the same value. From \( c_j + j \cdot x = c_k + k \cdot x \), we have the equation

\[
(k - j)x + (c_k - c_j) = 0
\]

of the hyperplane containing \( I \). Then the integer vector \( k - j \) is orthogonal to \( I \). Then there exist the unique positive integer \( w_I \) and the primitive integer vector \( n_I \) such that \( k - j = w_I n_I \).

For each \((n - 2)\)-cell \( C \), and \((n - 1)\)-cells \( I_1, I_2, \ldots, I_m \) adjacent to \( C \), if we fix a co-orientation of \( C \) and take primitive orthogonal vectors \( n_{I_j} \), then we have the \textit{balanced condition}

\[
w_{I_1} n_{I_1} + w_{I_2} n_{I_2} + \cdots + w_{I_m} n_{I_m} = 0.
\]
Thus the tropical hypersurface $Y$ is an $(n - 1)$-dimensional weighted rational polyhedral complex satisfying the regularity condition and the balanced condition.

Tropical hypersurface $Y_F$ is invariant under the deformations, called the fundamental deformations, of the tropical Laurent polynomial $F$.

1. Replace $c$ by $c' : A \to \mathbb{R}, c'(j) = c(j) + \text{const.}$
2. Replace $A$ by $A' = A + j_0, j_0 \in \mathbb{Z}^n$ and $c$ by $c' : A' \to \mathbb{R}, c'(j + j_0) = c(j)$.
3. Replace $c : A \to \mathbb{R}$ by $c' : A' \to \mathbb{R}$ such that convex hull $A' = \Delta$ and the convex function $-c' = -c$.

【Legendre Transformations】

Consider the contact manifold $M = \mathbb{R}^{2n+1}$ with coordinates

$$(x, y, p) = (x_1, \ldots, x_n, y, p_1, \ldots, p_n)$$

and with the contact form $\theta = dy - \sum_{i=1}^{n} p_i dx_i$.

Note that $-\theta = d(\sum_{i=1}^{n} p_i x_i - y) - \sum_{i=1}^{n} x_i dp_i$. Then we have the double Legendrian fibration:

$$\mathbb{R}^{n+1} \xleftarrow{\pi_1} \mathbb{R}^{2n+1} \xrightarrow{\pi_2} \mathbb{R}^{n+1},$$

$\pi_1(x, y, p) = (x, y), \pi_2(x, y, p) = (\tilde{y}, p), \tilde{y} = \sum_{i=1}^{n} p_i x_i - y$.

For a function $h : \Delta \to \mathbb{R}$ on a convex set $\Delta \subset \mathbb{R}^n$, the Legendre transformation of $h$ is defined as the set of supporting hyperplanes of the epi-graph of $h$. 
Lemma 4. The graph of tropical polynomial function

\[ F(x) = \sum_{j \in A} c_{j}x^{j} \]

and the graph of the convex function \( \overline{-c} : \mathbb{R}^{n} \to \mathbb{R} \) are the Legendre transformations to each other.

We consider the topological classification problem of tropical polynomial functions preserving corner loci.

Definition 5. Two tropical polynomials \( F(x) \) and \( G(x) \) are called topologically equivalent if there exist homeomorphisms \( \Phi : \mathbb{R}^{n} \to \mathbb{R}^{n} \) and \( \Psi : \mathbb{R} \to \mathbb{R} \) such that

\[ \Psi(F(x)) = G(\Phi(x)), \ \Phi(Y_{F}) = Y_{G}. \]

Proposition 6. There exists a semialgebraic set \( \Sigma \subset \mathbb{R}^{A} \) of codim > 0 such that, for any \( c \in \mathbb{R}^{A} \setminus \Sigma \), the decomposition of \( \Delta \) is simplicial.

For each connected component \( U \) of \( \mathbb{R}^{A} \setminus \Sigma \), the family \( F_{c}(x), c \in U \) of tropical polynomial functions is topologically trivial.

【 Topological Bifurcations of Singularities 】

The topology of a tropical polynomial with a non-simplicial decomposition bifurcates into a generic tropical polynomial.

Example 7. Let us consider the tropical polynomial

\[ F = \max\{0 + 0x_{1} + 0x_{2} + 0x_{1}x_{2} \} = \max\{0, x_{1}, x_{2}, x_{1} + x_{2}\}. \]
Then $F$ has the deformation:

$$F_{\lambda} = "\lambda + 0x_1 + 0x_2 + 0x_1x_2 " = \max\{\lambda, x_1, x_2, x_1 + x_2\}, (\lambda \in \mathbb{R}, \lambda \neq 0).$$

The tropical curve $Y_F$ bifurcates into $Y_{F_\lambda}(\lambda > 0, \lambda < 0)$. The decomposition of Newton polyhedron $\Delta_F$ bifurcates into $\Delta_{F_\lambda}(\lambda > 0, \lambda < 0)$.

**Amoeba and Pachworking**

For a complex Laurent polynomial

$$f(z) = \sum_{j \in A} b_j z^j \in \mathbb{C}[z_1^{\pm}, \ldots, z_n^{\pm}], \quad b_j \in \mathbb{C}^\times,$$

we have a hypersurface

$$Z_f = \{z \in (\mathbb{C}^\times)^n \mid f(z) = 0\} \subset (\mathbb{C}^\times)^n$$

in the complex torus $(\mathbb{C}^\times)^n$.

For a given function $v : A \to \mathbb{R}$, consider the family of polynomials,

$$f_t = f^v_t(z) := \sum_{j \in A} b_j t^{-v(j)} z^j, \quad (t > 0).$$

We call it the *patchworking polynomial* induced by $f$ and $v$. Note that $f_1 = f$. 

Let us define \( \Log_t : \mathbb{C}^n \to (\mathbb{R} \cup \{-\infty\})^n \) by
\[
\Log_t(z_1, \ldots, z_n) = (\log_t |z_1|, \ldots, \log_t |z_n|).
\]
We set \( \mathcal{A}_f = \Log(Z_f) \subset \mathbb{R}^n \) and we call it the amoeba of \( Z_f \).

**Proposition 8.** (Viro, Kapranov)
\[
\lim_{t \to \infty} \text{Hausdorff-dist}(\Log_t(Z_{f_t}), Y_{f^{t_{\text{rop}}}}) = 0
\]
where
\[
f_{\text{trop}}(x) := \max_{j \in A}(j \cdot x - v(j))
\]
(Legendre transformation of \( v \)).

**Example 9.** Amoeba of \( f(z_1, z_2) = z_1 + z_2 + 1 \).

---

**[Puiseux Series and Non-Archimedean Amoeba]**

Let us denote by \( \mathbb{C}[\mathbb{R}] \) the group algebra of the additive group \( \mathbb{R} \) over \( \mathbb{C} \). We consider its formal version:

A Puiseux-Laurent series of real power (Hahn series\([4]\)) is given by
\[
a := a(s) = \sum_{p \in I} \alpha_p s^p
\]
where \( \alpha_p \in \mathbb{C}^\times \) and the support \( I = I_a \subset \mathbb{R} \) of \( a \) is a well-ordered subset.

We set
\[
\mathbb{C}((\mathbb{R})) := \{ a(s) \mid a(s) : \text{Puiseux-Laurent series of real power} \} \cup \{0\}.
\]
Lemma 10. $K = \mathbb{C}((\mathbb{R}))$ is an algebraically closed field.

Define the valuation $\text{val} : \mathbb{C}((\mathbb{R})) \to \mathbb{R} \cup \{\infty\}$ on $\mathbb{C}((\mathbb{R}))$ by

$$\text{val}(a) := \min I_a \in \mathbb{R}, \quad (a \in \mathbb{C}((\mathbb{R})) \setminus \{0\}), \quad \text{val}(0) = \infty,$$

Then we have that $\text{val}(a) = \infty$ if and only if $a = 0$, and that

$$\text{val}(ab) = \text{val}(a) + \text{val}(b), \quad \text{val}(a + b) \geq \min\{\text{val}(a), \text{val}(b)\}.$$

We define the non-Archimedean norm on $\mathbb{C}((\mathbb{R}))$ by

$$\|a\| := e^{-\text{val}(a)} \quad (a \in \mathbb{C}((\mathbb{R}))^\times), \quad \|0\| = 0.$$

Then we have the tropical triangular inequality

$$\|a + b\| \leq \max\{\|a\|, \|b\|\} = "\|a\| + \|b\|"$$

Define $\text{Log} : \mathbb{C}((\mathbb{R}))^n \to (\mathbb{R} \cup \{-\infty\})^n$ by

$$\text{Log}(a_1, \ldots, a_n) := (\log \|a_1\|, \ldots, \log \|a_n\|) = (-\text{val}(a_1), \ldots, -\text{val}(a_n)).$$

Given a Laurent polynomial $f(z) = \sum a_j z^j \in \mathbb{K}[z, z^{-1}]$, we define

$$Z_f := \{z \in (\mathbb{K}^\times)^n \mid f(z) = 0\} \subset (\mathbb{K}^\times)^n.$$

Its Log-image $A_f := \text{Log}(Z_f) \subset (\mathbb{R} \cup \{-\infty\})^n$ is called the non-Archimedean amoeba of $Z_f$.

Define a tropical Laurent polynomial

$$f_{\text{trop}}(x) := "\sum_{j \in A} \log \|a_j\| x^j" = "\sum_{j \in A} (-\text{val}(a_j)) x^j" = \max_{j \in A}(j \cdot x - \text{val}(a_j)).$$

We call $f_{\text{trop}}(x)$ the tropicalization of $f(z)$.

Proposition 11. (Kapranov) Non-Archimedean amoeba is a tropical hypersurface: We have $A_f = Y_{f_{\text{trop}}}.$
The [Triangle hyperfield]

On $\mathbb{R}^+$, define the multi-valued addition
$$a \nabla b := \{ c \in \mathbb{R}^+ \mid |a - b| \leq c \leq a + b \} \quad = \quad \{|z + w| \mid |z| = a, |w| = b\}.$$

This reminds us the superposition of waves.

Then $\mathbb{R}_{+}^{tri} = (\mathbb{R}_{+}, \nabla, \cdot)$ is a hyperfield.

The [Amoeba hyperfield]

By the bijection $\log : \mathbb{R}_{+} \to \mathbb{R} \cup \{-\infty\}$, we have the hyperfield
$$\log(\mathbb{R}_{+}^{tri}) := (\mathbb{R} \cup \{-\infty\}, \ Y , +),$$

which is called the amoeba hyperfield:
$$a \ Y b := \{ c \in \mathbb{R} \cup \{-\infty\} \mid \log(|e^{0} - e^{b}|) \leq c \leq \log(e^{a} + e^{b}) \}.$$  

The [Tropical Limits of Amoeba Hyperfield]  

Define, on $\mathbb{R} \cup \{-\infty\}$,
$$a \ Y_{h} b := h(\frac{a}{h} \ Y_{\frac{b}{h}})$$
$$= \{ c \in \mathbb{R} \cup \{-\infty\} \mid h \log(|e^{\frac{a}{h}} - e^{\frac{b}{h}}|) \leq c \leq h \log(e^{\frac{a}{h}} + e^{\frac{b}{h}}), \}$$
$$a \ Y_{h} b = \{ c \in \mathbb{R} \cup \{-\infty\} \mid -\infty \leq c \leq a + h \log 2 \}$$
$$= [-\infty, a] =: a \ Y a.$$  

If $a \neq b$, then $a \ Y_{h} b \to \{ \text{max}\{a, b\} \}.$
$$\lim_{h \to 0} a \ Y_{h} b = a \ Y b,$$
$$\lim_{h \to 0} \log(\mathbb{R}_{+}^{tri})_{h} \to \mathcal{Y} (: \text{tropical hyperfield}).$$
We define a multi-valued addition $\sim$ on $C$: Let $a, b \in C$. If $|a| \neq |b|$, then we set $a \sim b := a$ if $|a| > |b|$, and $a \sim b := b$ if $|a| < |b|$.

Suppose $|a| = |b|$. If $b \neq -a$, then

$$a \sim b := \begin{cases} \text{the shortest arc connecting } a \text{ and } b \text{ on the circle } \{z \in C \mid |z| = |a|\} & \text{if } |a| > |b|, \\ z \in C \mid |z| = |a| & \text{if } |a| < |b|. \end{cases}$$

If $b = -a$, then set

$$a \sim b := \{z \in C \mid |z| \leq |a|\}.$$

We define the complex tropical hyperfield by

$$\mathcal{T}C := (C, \sim, \text{the usual multiplication}).$$

On $C$, we consider the bijection $S_h : C \to C$ defined by

$$S_h(z) := \begin{cases} |z|^h \frac{z}{|z|} & (z \neq 0), \\ 0 & (z = 0). \end{cases}$$

and we define

$$z +_h w := S_h^{-1}(S_h(z) + S_h(w)).$$

Then we have a family of fields $(C, +_h, \times)$, $h > 0$.

**Theorem 12.** (Viro [25]) Let

$$\Gamma = \{(z, w, z +_h w, h) \in C^3 \times R_+ \mid (z, w, h) \in C^2 \times R_+\}.$$

Then

$$\overline{\Gamma} \cap (C^3 \times \{0\}) := \{(a, b, a \sim b, 0) \mid (a, b) \in C^2\}.$$
Thus we have the diagram:

<table>
<thead>
<tr>
<th>Complex Algebraic Geometry</th>
<th>Complex Tropical Geometry</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C \cong C_h )</td>
<td>( C_0 = \mathcal{T}C )</td>
</tr>
<tr>
<td>( z \rightarrow</td>
<td>z</td>
</tr>
<tr>
<td>( R^\text{tri}<em>+ \cong R^\text{tri}</em>+ )</td>
<td>( R^\text{tri}_+ = \mathcal{Y}_x )</td>
</tr>
<tr>
<td>( x \rightarrow \log x )</td>
<td>( x \rightarrow \log x )</td>
</tr>
<tr>
<td>( \log(R^\text{tri}<em>+) \cong \log(R^\text{tri}</em>+) )</td>
<td>( \mathcal{Y} )</td>
</tr>
<tr>
<td>Amoeba Geometry</td>
<td>Tropical Geometry</td>
</tr>
</tbody>
</table>

**[Real Tropical Hyperfield]**

Question: What is the real counterpart of the complex tropical hyperfield?

We are naturally led to define the multi-valued addition \( \sim_R \) on \( \mathbb{R} \) induced from \( \sim \) on \( \mathbb{C} \): For \( a, b \in \mathbb{R} \), we set

\[
\begin{array}{l}
    a \sim_R b := a \quad \text{if } |a| > |b|, \\
    a \sim_R b := b \quad \text{if } |a| < |b|, \\
    a \sim_R a := a, \\
    a \sim_R (-a) := [-a, a].
\end{array}
\]

Theorem 13. \((\mathbb{R}, \sim_R, \times)\) is a hyperfield. Moreover let

\[
\Gamma_R = \{(a, b, a +_h b, h) \in \mathbb{R}^3 \times \mathbb{R}_+ \mid (a, b, h) \in \mathbb{R}^2 \times \mathbb{R}_+ \}.
\]

Then we have

\[
\overline{\Gamma_R} \cap (\mathbb{R}^3 \times \{0\}) = \{(a, b, a \sim_R b, 0) \mid (a, b) \in \mathbb{R}^2 \}.
\]
The real tropical hyperfield is, in some sense, a "double covering" of the tropical hyperfield via $x \mapsto \log |x|$. Therefore, "real tropical geometry" can be constructed as a "double covering" of tropical geometry.

**[Several Questions]**

**Question:** Is the complex tropical hyperfield $\mathcal{T}C$ is algebraically closed, in an appropriate sense?

**Question:** Are the real tropical hyperfield and the tropical hyperfield $\mathcal{Y}$ real closed?

**Question:** What is the real tropical algebraic geometry?

In Amoeba geometry, it is known the *Ronkin function*

$$
N_f(x) := \frac{1}{(2\pi i)^n} \cdot \int_{\text{Log}^{-1}(x)} \log |f(z)| \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n}
$$

is linear on each connected component of $\mathbb{R}^n \setminus \mathcal{A}_f$. We have $\text{grad} N_f : \mathbb{R}^n \setminus \mathcal{A}_f \rightarrow \Delta \cap \mathbb{Z}^n$ and $\text{grad} N_f$ separates every connected components of $\mathbb{R}^n \setminus \mathcal{A}_f$.

**Question:** Can the Ronkin function be described in terms of the amoeba hyperfield?

**References**


