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Hypersurfaces and Polar curves

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Introduction

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a non-constant germ of complex analytic function. From the book of J. Milnor [11] it is known that:

Theorem 0.1 *There is $\varepsilon_0 > 0$, such that, for any ε, $0 < \varepsilon \leq \varepsilon_0$, the map $\varphi_\epsilon : \mathbb{S}_\varepsilon \setminus \{f = 0\} \to \mathbb{S}^1$ defined by $\varphi_\varepsilon(z) = f(z)/|f(z)|$ on the sphere centered at 0 with radius ε, is a locally trivial C^∞ fibration.*

In this theorem, J. Milnor does not assume that the function f has isolated singularities.

The following theorem is consequence of [5] (Theorem 1.2.1):

Theorem 0.2 *There is $\varepsilon_0 > 0$, such that, for any ε, $0 < \varepsilon \leq \varepsilon_0$, there is $\eta_0 > 0$, such that, for any η, $0 < \eta \leq \eta_0$, the map $\varphi_{\epsilon,\eta} : (\mathbb{B}_\varepsilon \setminus \{f = 0\}) \cap f^{-1}(D_\eta^*) \to D_\eta^*$ induced by f on the intersection of the open ball centered at 0 with radius ε with the inverse image of the punctured open disc $D_\eta^* := D_\eta \setminus \{0\}$, is a locally trivial C^∞ fibration.*

The proof of 0.2 uses that, locally, the special fiber $f^{-1}(0)$ can be stratified with Thom condition. Namely, for $1 \gg \varepsilon > 0$, there is a stratification of $\mathbb{B}_\varepsilon$, such that $f^{-1}(0) \cap \mathbb{B}_\varepsilon$ is a union of strata and for any sequence $x_n \in \mathbb{B}_\varepsilon \setminus f^{-1}(0)$ which converges to $x \in f^{-1}(0) \cap \mathbb{B}_\varepsilon$ such that the sequence of tangent spaces $T_{x_n}(f^{-1}(f(x_n)))$ has a limit T, this limit contains the tangent space of the stratum which contains the point x.

The existence of such a stratification with Thom condition was proved by using Łojasiewicz inequality in [5], Theorem 1.2.1.

Theorem 5.11 of [11] proves that, for $1 \gg \varepsilon \gg |t| > 0$ a fiber of φ_ε and $f^{-1}(t) \cap \mathbb{B}_\varepsilon$ are diffeomorphic, and $\mathbb{S}_\varepsilon \setminus \{f = 0\}$ is diffeomorphic to $(\mathbb{B}_\varepsilon \setminus \{f = 0\}) \cap f^{-1}(S_\eta)$. Having Theorem 0.2, we know that $\partial\varphi_{\epsilon,\eta} : (\mathbb{B}_\varepsilon \setminus \{f = 0\}) \cap f^{-1}(S_\eta) \to S_\eta$ induced by f is a locally trivial C^∞ fibration. The fibrations $\partial\varphi_{\epsilon,\eta}$ and φ_ϵ have diffeomorphic total spaces and diffeomorphic fibers, so they must be isomorphic. We shall call Milnor fiber of f at 0 a fiber of any of these two fibrations.

1 Basic results

In [7] the following theorem is proved:

Theorem 1.1 *Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a non-constant germ of complex analytic function. Let $1 \gg \varepsilon \gg |t| > 0$, and let H be a general hyperplane of $\mathbb{C}^{n+1}$ through 0, then the manifold $f^{-1}(t) \cap \mathbb{B}_\varepsilon$ is obtained from a tubular neighbourhood of $f^{-1}(t) \cap \mathbb{B}_\varepsilon \cap H$ in $f^{-1}(t) \cap \mathbb{B}_\varepsilon$ by adding handles of index n.*

The proof of Theorem 1.1 uses Smale's interpretation of Morse theory by adding handles.

The key construction consists in introducing the Polar curve of f relatively to a general linear function ℓ. Namely, let ℓ be a linear function. We have a germ of map:

$$\Phi = (\ell, f) : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^2, 0).$$

The critical locus C_ℓ of Φ depends linearly on the linear form ℓ. A theorem of Bertini tells us that the singular points of the general element C_ℓ of this linear system lie in the fixed points of this system. Locally near 0, the fixed points are precisely the critical points of the function f. For a general C_ℓ, the points of $C_\ell \setminus \{f = 0\}$ are non-singular. Therefore, for a general ℓ, either $C_\ell \setminus \{f = 0\}$ is empty or it is non-singular of dimension 1. We denote by Γ_ℓ the closure of $C_\ell \setminus \{f = 0\}$. We call Γ_ℓ a relative polar curve of f at 0.

We can also define the relative polar curve of f at 0 in the following way. Let γ_f be the map from $\mathbb{B}_\varepsilon \setminus C(f)$ into $\mathbb{P}^n$ for $1 \gg \varepsilon > 0$, such that:

$$\gamma_f(x_0, \ldots, x_n) = (\partial f/\partial z_0(x_0, \ldots, x_n) : \ldots : \partial f/\partial z_n(x_0, \ldots, x_n)).$$

Let $\nu_f : \tilde{\mathbb{B}}_\varepsilon \to \mathbb{B}_\varepsilon$ be the blowing-up in $\mathbb{B}_\varepsilon$ of the Jacobian ideal of f generated by the partial derivatives of f. The map γ_f extends as $\tilde{\gamma}_f : \tilde{\mathbb{B}}_\varepsilon \to \mathbb{P}^n$. Let ℓ be a general line in $\mathbb{P}^n$, then $\nu_f(\tilde{\gamma}_f^{-1}(\ell)) = \Gamma_\ell$ is a polar curve of f at 0.

The relative polar curve of f at 0 can be empty. For instance, it is so, when f is an analytic product. When the critical locus of f is locally at 0 a non-singular curve Σ, one can show that the relative polar curve of f at 0 is empty if and only if the hypersurface $f^{-1}(0)$ has Milnor number constant along Σ.

Notice that the critical points of the restriction of the function $|\ell|$ to the difference of spaces

$$f^{-1}(t) \cap \mathbb{B}_\varepsilon \setminus f^{-1}(t) \cap \mathbb{B}_\varepsilon \cap H,$$

where $H = \{\ell = 0\}$, are the points of the intersection $\Gamma_f \cap f^{-1}(t) \cap \mathbb{B}_\varepsilon$.

By taking the cone on a projective hypersurface Δ in $\mathbb{P}^n$, Theorem 1.1 implies a Theorem of Zariski stated in [12], which has correct proofs in [2] or in [5]:

Corollary 1.2 *The fundamental group of the complement of the projective hypersurface Δ in $\mathbb{P}^n$ is isomorphic to the fundamental group of the complement in a general plane P of the intersection with P of the projective hypersurface Δ.*

Proof. First, we observe that the Milnor fiber F of the reduced cone $\tilde{\Delta}$ over the hypersurface Δ, defined by a homogeneous polynomial f of degree d, is diffeomorphic to $\mathbb{F} = f^{-1}(1)$, which is an abelian covering of degree d of the complement $\mathbb{P}^n \setminus \Delta$.

We have an exact sequence:

$$1 \to \pi_1(\mathbb{F}, *) \to \pi_1(\mathbb{P}^n \setminus \Delta, *) \to \mathbb{Z}/d\mathbb{Z} \to 0.$$

Theorem 1.1 gives that, if H is a general projective hyperplane, the inclusion $F \cap \tilde{H}$ into F, where $\tilde{H}$ is the hyperplane above H, induces an isomorphism $\pi_k(F \cap \tilde{H}, *) \simeq \pi_k(F, *)$, for $0 \leq k \leq n-2$. Therefore, if P is a general projective plane, the cone $\tilde{P}$ above P is a 3-dimensional affine space and $\pi_1(\mathbb{F} \cap \tilde{P}, *) \simeq \pi_1(F \cap \tilde{P}, *) \simeq \pi_1(F, *) \simeq \pi_1(\mathbb{F}, *)$.

The space $\mathbb{F} \cap \tilde{P}$ is an abelian covering of degree d of $P \setminus \Delta$. We have the exact sequence:

$$1 \to \pi_1(\mathbb{F} \cap \tilde{P}, *) \to \pi_1(P \setminus \Delta, *) \to \mathbb{Z}/d\mathbb{Z} \to 0.$$

Comparing this last exact sequence with the preceding one by using the inclusion maps, Theorem 1.1 gives a proof of our corollary.

Theorem 1.1 provides another proof of the Theorem of Zariski of Lefschetz type [12], stated in corollary 1.2. Using our results in [4], we shall extend this Theorem of Zariski to a more general setting.

2 Stratified Polar curve

Let $f : (X, 0) \to (\mathbb{C}, 0)$ be a germ of non-constant complex analytic function on a reduced space $X \subset \mathbb{C}^N$.

A theorem of H. Hironaka in [6] shows that, for $1 \gg \varepsilon > 0$, one can stratify $X \cap \mathbb{B}_\varepsilon$ and $f^{-1}(0) \cap \mathbb{B}\varepsilon$ in such a way that, for $1 \gg \varepsilon \gg \eta > 0$, the map $\phi_{\varepsilon,\eta} : X \cap \mathbb{B}_\varepsilon \cap f^{-1}(D_\eta) \to D_\eta$ induced by f can be stratified with Thom condition.

As it is done in [8], a fibration theorem analogous to Theorem 0.2 above can be proved:

Theorem 2.1 *There is $\varepsilon_0 > 0$, such that, for any ε, $0 < \varepsilon \leq \varepsilon_0$, there is $\eta_0 > 0$, such that, for any η, $0 < \eta \leq \eta_0$, the map $\varphi_{\epsilon,\eta} : \mathbb{B}_\varepsilon \cap X \cap f^{-1}(D_\eta^*) \to D_\eta^*$ induced by f on the intersection of the open ball centered at 0 with radius ε with the inverse image in X of the punctured open disc $D_\eta^* := D_\eta \setminus \{0\}$, is a locally trivial C^0 fibration.*

One notices that the statement is somehow the same as the one of Theorem 0.2, but now the fibration is C^0, because X might be singular.

Let ℓ be a linear form of the ambient space $\mathbb{C}^N$. We have a germ of map $\Phi = (\ell, f) : (X, 0) \to (\mathbb{C}^2, 0)$. First one observes:

Lemma 2.2 *There is a Zariski dense open space Ω of linear forms, such that, for any $\ell \in \Omega$, there are an open neighbourhood U of 0 in X and an open neighbourhood V of 0 in $\mathbb{C}^2$, such that Φ induces a stratified map from U onto V which satisfies Thom condition.*

Proof. To prove this lemma, consider $1 \gg \varepsilon \gg \eta > 0$, such that Theorem 2.1 holds. Let us stratify the map:

$$\varphi_{\epsilon,\eta} : \mathbb{B}_\varepsilon \cap X \cap f^{-1}(D_\eta) \to D_\eta$$

such that $\mathbb{B}_\varepsilon \cap X \cap f^{-1}(0)$ is a union of strata, the stratification of $\mathbb{B}_\varepsilon \cap X \cap f^{-1}(D_\eta)$ satisfies Whitney condition and Thom condition for f. Consider a linear form ℓ of $\mathbb{C}^N$ such that $\{\ell = 0\}$ is transverse in $\mathbb{C}^N$ to all the strata of $\mathbb{B}_\varepsilon \cap X \cap f^{-1}(D_\eta)$ except maybe at the point $\{0\}$.

We can see that by choosing the linear form in this way, the map Φ induces a Thom map for some neighbourhoods U and V. Since there are only a finite number of strata having 0 in its closure, it is enough to choose a linear form ℓ such that $\{\ell = 0\}$ is transverse except at 0 to a finite number of strata, i.e. a linear form lying in the intersection of Zariski dense open sets of linear forms.

Now, we can prove:

Proposition 2.3 *Let $1 \gg \varepsilon \gg \eta > 0$. Let us stratify $f : \mathbb{B}_\varepsilon \cap X \cap f^{-1}(D_\eta) \to D_\eta$ with Whitney stratifications and Thom condition. Let S_α, $\alpha \in A$, be the strata in finite number of $\mathbb{B}_\varepsilon \cap X \cap f^{-1}(D_\eta)$ which contain 0 in their closure. There is a Zariski open dense set Ω in the space of linear forms of $\mathbb{C}^N$, such that, for $\ell \in \Omega$, the restriction of (ℓ, f) to $S_\alpha \setminus f^{-1}(0)$ have an empty or non-singular critical locus C_α of dimension 1.*

Proof. Consider a stratum S_α, such that the closure $\bar{S}_\alpha$ contains 0. Let ℓ be a linear form of $\mathbb{C}^N$. The critical locus $C_\alpha(\ell)$ of the restriction of (ℓ, f) to the closure $\bar{S}_\alpha$ is an analytic subset of $\bar{S}_\alpha$ which depends linearly on ℓ, so, using the same theorem of Bertini as above, for ℓ general, i.e. for ℓ in a Zariski dense open set Ω_α of the space of linear forms of $\mathbb{C}^N$, the singular points of $C_\alpha(\ell)$ lie in the union of $f^{-1}(0) \cap \bar{S}_\alpha$ and the singular point of $\bar{S}_\alpha$. Therefore, for ℓ general, $C_\alpha(\ell) \cap (S_\alpha \setminus f^{-1}(0))$ is either empty or non-singular of dimension 1.

Since f is stratified, either $S_\alpha \subset f^{-1}(0)$ or $S_\alpha \cap f^{-1}(0) = \emptyset$, otherwise f has maximal rank on all the strata (S_α), in which case, for all α except at most one, $\dim S_\alpha \geq 2$ and, for a general ℓ, the critical locus of (ℓ, f) restricted to S_α is empty, except for the stratum of dimension 1 which contains 0, and in this last case the critical locus of the restriction of (ℓ, f) to this 1-dimensional stratum is the whole stratum itself.

We may suppose that, either $S_\alpha \subset f^{-1}(0)$ or $S_\alpha \cap f^{-1}(0) = \emptyset$. We have just seen that, for ℓ general, the critical space C_α of the restriction of (ℓ, f) to $S_\alpha \setminus f^{-1}(0)$ is empty or non-singular of dimension 1. The Zariski open dense set Ω of the proposition is the finite intersection $\cap_{\alpha \in A} \Omega_\alpha$.

Definition 2.4 *The relative polar curve of f relatively to a general linear form ℓ of $\mathbb{C}^N$ is the finite union of the closures $\cup_{\alpha \in A} \overline{C}_\alpha$. We shall call $\Gamma_\alpha := \overline{C}_\alpha$ a relative polar curve associated to the stratum S_α.*

Recall that, as above, C_α and also Γ_α can be empty.

In [8], we have introduced this notion in the particular case f was itself the restriction of a general linear form of $\mathbb{C}^N$ to the germ $(X, 0)$.

Notice that, since f is stratified, for t small enough, the intersections $\{f = t\} \cap S_\alpha$ give a Whitney stratification of the Milnor fiber $\{f = t\}$. The points of $\Gamma_\alpha \cap \{f = t\}$ are precisely the points of S_α where the restriction of ℓ to $\{f = t\} \cap S_\alpha$ has ordinary quadratic points.

3 Singular complement of projective hypersurfaces

Let X be a projective variety embedded in the complex projective space $\mathbb{P}^{N-1}$ and let $Y \subset X$ a projective hypersurface defined in X by one homogeneous equation $f = 0$. To these data correspond the cones $\tilde{X}$, $\tilde{Y}$ embedded in $\mathbb{C}^N$.

On the punctured cone $\tilde{X} \setminus \{0\}$, we have a natural action of the punctured group of non-zero complex numbers $\mathbb{C}^*$. Consider the fiber $\mathbb{F} = f^{-1}(1) \cap \tilde{X}$. Let σ be the natural map from $\tilde{X} \setminus \{0\}$ onto X. Let $y \in X \setminus Y$. The inverse image $\sigma^{-1}(y)$ is the punctured line $\tilde{y}^*$.

Assume that f is the restriction to X of a homogeneous polynomial of $\mathbb{P}^{N-1}$ of degree d. The intersection $\sigma^{-1}(y) \cap \mathbb{F}$ consists of d points $\{\xi^k y_0\}_{0 \leq k \leq d-1}$, where ξ is a primitive d-th root of unity and y_0 is one point of $\sigma^{-1}(y) \cap \mathbb{F}$.

Suppose that the fiber $\mathbb{F}$ of $f : \tilde{X} \to \mathbb{C}$ is connected, it is a topological covering of $X \setminus Y$ of degree d. Let H be a projective hyperplane of $\mathbb{P}^{N-1}$. If we want to compare $X \setminus Y$ and $(X \setminus Y) \cap H$, we can do it by comparing $\mathbb{F}$ and $\mathbb{F} \cap \tilde{H}$, where $\tilde{H}$ is the hyperplane lying above H in $\mathbb{C}^N$.

In this situation $\mathbb{F}$ is a singular space.

Let us define the notion of rectified homotopical depth introduced by Grothendieck and understood as in [4] (Theorem 4.1.2).

Definition 3.1 *Let X be a reduced complex analytic space. Let $(S_\alpha)_{\alpha \in A}$ be an analytic Whitney stratification of X. For each $\alpha \in A$, let $\mathcal{L}_\alpha$ be a complex link of S_α. We say that the rectified homotopical depth $rhd(X)$ of X is $\geq r$ if, for each $\alpha \in A$, the complex link $\mathcal{L}_\alpha$ is $(r - \dim S_\alpha - 2)$-connected.*

Remind that the complex link of a stratum S_α of a Whitney stratification is obtained in the following way. Let $x \in S_\alpha$. Assume that (X, x) is locally embedded in $\mathbb{C}^N$. Let $\mathcal{N}_\alpha$ be a slice of S_α at x, i.e. the intersection of X with a complex submanifold of $\mathbb{C}^N$ of codimension $\dim S_\alpha$ and transverse in $\mathbb{C}^N$ to S_α at x. A complex link of S_α is the Milnor fiber at x of a general linear form of $\mathbb{C}^N$ restricted to $\mathcal{N}_\alpha$.

One can prove that all complex links of a given stratum of a Whitney stratification are homeomorphic (see e.g. §2.3 Part II Chapter 2).

In [4] we indicate that, if X is equidimensional, the dimension of X bounds the rectified homotopical depth.

We shall show:

Theorem 3.2 *Let assume that the rectified homotopical depth of the projective variety X is $\dim X$ and let $Y \subset X$ a projective subset defined in X by one reduced homogeneous equation $f = 0$. For a general hyperplane H the inclusion of $(X \setminus Y) \cap H$ in $X \setminus Y$ gives an isomorphism of the fundamental groups $\pi_1((X \setminus Y) \cap H, *) \simeq \pi_1(X \setminus Y, *)$ for $\dim X \geq 3$.*

The proof of this theorem proceeds like what we have done for the Theorem 1.2 of Zariski of Lefschetz type ([12]). We first prove a local theorem:

Theorem 3.3 *Let (X, x) a germ of reduced equidimensional germ of complex analytic space embedded in $(\mathbb{C}^N, 0)$. Let*

$$f : (X, x) \to (\mathbb{C}, 0)$$

be a germ of non-constant complex analytic function on (X, x). Assume that $X \setminus f^{-1}(0)$ has rectified homotopical depth $\geq r$. Let $\mathbb{F}$ be a Milnor fiber of f at 0. For a general hyperplane H containing x, the relative homotopy $\pi_k(\mathbb{F}, \mathbb{F} \cap H, *) = 0$ if $k \leq r - 2$.

Proof of Theorem 3.3. As we have mentioned above, we are going to use our results in [4]. Let ℓ be a general linear form of $\mathbb{C}^N$, $H := \{\ell = 0\}$ the general hyperplane that it defines, and $Y := H \cap X$. Call $Z = f^{-1}(0) \cap X$. Denote by $\mathbb{S}_\varepsilon(x)$ the sphere centered at x with radius ε. Theorem 2.2 of [5] asserts that $\mathbb{S}_\varepsilon(x) \cap (X \setminus Z)$ is obtained from $\mathbb{S}_\varepsilon(x) \cap (X \setminus Z) \cap V_\alpha(Y)$ by adding cells of dimension $\geq r-1$, where:

$$V_\alpha(Y) = \{z \in X \mid |\ell(z)| \leq \alpha\}$$

with $\alpha > 0$ small enough.

Since H is a general hyperplane, one can show that the space $\mathbb{S}_\varepsilon(x) \cap (X \setminus Z) \cap V_\alpha(Y)$ retracts on the space $\mathbb{S}_\varepsilon(x) \cap (X \setminus Z) \cap Y$ and has the same homotopy type.

The local conic theorem (see [1] Lemma 3.2) implies that $\mathbb{S}_\varepsilon(x) \cap (X \setminus Z) \cap Y$ has the same homotopy type as $\mathbb{B}_\varepsilon(x) \cap (X \setminus Z) \cap Y$ and $\mathbb{S}_\varepsilon(x) \cap (X \setminus Z)$ has the same homotopy type as $\mathbb{B}_\varepsilon(x) \cap (X \setminus Z)$, where $\mathbb{B}_\varepsilon(x)$ is the ball of $\mathbb{C}^N$ having $\mathbb{S}_\varepsilon(x)$ as boundary.

Adapted to the stratified situation, the Lemma 5.9 of [11] constructs a vector field which gives a stratified homeomorphism of $\mathbb{B}_\varepsilon(x) \cap f^{-1}(D_\eta) \cap (X \setminus Z)$ onto $\mathbb{B}_\varepsilon(x) \cap (X \setminus Z)$, for $1 \gg \varepsilon \gg \eta > 0$.

Now, using the fibration theorem 2.1, one can show that $\mathbb{B}_\varepsilon(x) \cap f^{-1}(D_\eta) \cap (X \setminus Z) \simeq \mathbb{B}_\varepsilon(x) \cap (X \setminus Z)$ and the Milnor fiber $\mathbb{F}$ have the same homotopy type for dimensions $\neq 1$. For dimension 1, one has:

$$1 \to \pi_1(\mathbb{F}, *) \to \pi_1(\mathbb{B}_\varepsilon(x) \cap (X \setminus Z), *) \to \mathbb{Z} \to 0.$$

In the hyperplane section by H we have:

$$1 \to \pi_1(\mathbb{F} \cap H, *) \to \pi_1(\mathbb{B}_\varepsilon(x) \cap (X \setminus Z) \cap Y, *) \to \mathbb{Z} \to 0.$$

One finds that this implies, if $r \geq 3$, the relative homotopy $\pi_k(\mathbb{F}, \mathbb{F} \cap H, *) = 0$, for $k \leq r-2$.

Proof of Theorem 3.2. We can apply the Theorem 3.3 to obtain Theorem 3.2. We have supposed that the rectified homotopy depth of X equals $\dim X$. Let us show that the cone $\tilde{X}$ over X has rectified homotopical depth equal to $\dim \tilde{X} = \dim X + 1$.

Let us stratify X by $(S_\alpha)_{\alpha \in A}$, so that Y is a union of strata. First, notice that the punctured cones $\tilde{S}_\alpha \setminus \{0\}$ give a stratification of the punctured cone $\tilde{X} \setminus \{0\}$. By adding the trivial stratum $\{0\}$, this defines a Whitney stratification of the cone $\tilde{X}$.

The complex link of the punctured cone $\tilde{S}_\alpha \setminus \{0\}$ in $\tilde{X}$ is the Milnor fiber of the restriction of a general linear function to a slice of $\tilde{S}_\alpha \setminus \{0\}$ in $\tilde{X}$. It is clear that complex link of the punctured cone $\tilde{S}_\alpha \setminus \{0\}$ in $\tilde{X}$ is homeomorphic to the complex link of S_α in X. Since it is $(\dim X - \dim S_\alpha - 2)$-connected, it is equivalently $(\dim X + 1 - (\dim S_\alpha + 1) - 2)$-connected. So, the difference $\tilde{X} \setminus \tilde{Y}$ has rectified homotopical depth $\dim X + 1$.

Let F be the Milnor fiber of the restriction of f to $\tilde{X}$. The Theorem 3.3 implies that, for a general hyperplane $\tilde{H}$ of $\mathbb{C}^N$:

$$\pi_k(F \cap \tilde{H}, *) \simeq \pi_k(F, *)$$

when $k \leq \dim \tilde{X} - 3$. Therefore:

$$\pi_1(F \cap \tilde{H}, *) \simeq \pi_1(F, *)$$

when $1 \leq \dim X - 2$ or $\dim X \geq 3$.

Since the cone $\tilde{X}$ is homogeneous, the Milnor fiber F is homeomorphic to the fiber $\mathbb{F} = f^{-1}(1)$, and $F \cap H$ is homeomorphic to $\mathbb{F} \cap H$. Therefore:

$$\pi_1(\mathbb{F} \cap \tilde{H}, *) \simeq \pi_1(\mathbb{F}, *)$$

when $\dim X \geq 3$.

Using the covering $\mathbb{F} \to X \setminus Y$, we obtain:

$$\pi_1((X \setminus Y) \cap H, *) \simeq \pi_1((X \setminus Y), *)$$

when $\dim X \geq 3$ and where H is the general hyperplane of $\mathbb{P}^{N-1}$ defined by $\tilde{H}$.

Following the type of arguments given above, we can prove a stronger theorem than Theorem 3.2 with a general rectified homotopical depth, but dealing with homotopy groups. We leave it to the reader.

4 Comments

Notice that, in the proof of Theorem 3.3, we did not use Stratified Morse theory, but Theorem 2.2 of [4] instead.

Using Stratified Morse theory, we may see that we add cells at the points of $\mathbb{F} \cap (\cup_{\alpha \in A} \Gamma_\alpha)$, where Γ_α is the part from the stratum S_α of a relative polar curve of $f : X \to \mathbb{C}$ at x. However we cannot prove that at these points we are adding cells with the right connectivity. We conjecture that, in this particular setting where the rectified homotopical depth is maximum, this should be true.

There are many spaces X for which the rectified homotopical depth is maximum and equal to $\dim X$. In [9], we prove that local complex analytic hypersurfaces have maximum rectified homotopical depth. The same proof shows that local complete intersections have also maximum rectified homotopical depth.

In general, one can prove that a reduced equidimensional complex analytic space has maximum rectified homotopical depth if and only if it is a Milnor space in the sense of [10].

For instance, let $\mathcal{A}$ be an affine hyperplane arrangement. It is a hypersurface. So it has maximal depth.

One can consider $\mathcal{A}$ as a the complement of the hyperplane at ∞ of a projective hypersurface $\tilde{\mathcal{A}}$ in the projective space $\mathbb{P}^N$. Since the hyperplane at ∞ is defined by one equation in $\tilde{\mathcal{A}}$, we can apply Theorem 3.2.

We get that the fundamental group of $\mathcal{A}$ equals the fundamental group of the intersection with a general 3-plane E. We get:

Proposition 4.1 *The fundamental group of a plane arrangement in $\mathbb{C}^3$ whose planes have at least 3 distinct plane directions, such that none of them is parallel to the intersection of the two other plane directions, is trivial.*

Proof.[1] We assume that the planes are in general position. In this case we give a proof by induction on the number of planes. If the number of hyperplanes is 1, there is nothing to prove.

[1] We are indebted to Mutsuo Oka for this proof.

Let us suppose that an arrangement $\mathcal{A}'$ with $k \geq 1$ planes in $\mathbb{C}^3$ has a trivial fundamental group. The new arrangement $\mathcal{A}$ is the union of $\mathcal{A}'$ and of the plane P.

The intersection of P and $\mathcal{A}'$ is connected since it is the union of k lines in general position, where $k \geq 1$. We apply Van Kampen theorem. The fundamental group of $\mathcal{A}$ is the amalgamated product of the fundamental group of P and the fundamental group of $\mathcal{A}'$. Both these fundamental groups are trivial, so the fundamental group of $\mathcal{A}$ is also trivial.

Now, let us assume that the planes in $\mathcal{A}$ are not in general position. Let us order the planes

$$P_1^1, \dots, P_{n_1}^1, P_1^2, \dots, P_{n_2}^2, \dots, P_1^k, \dots, P_{n_k}^k,$$

in such a way that, for $1 \leq i \leq k$, all the planes P_ℓ^i are parallel between themselves, none of P_m^i is parallel to P_m^j for $i \neq j$.

We make another induction on k. For $k = 1$, the arrangement $\mathcal{A}$ consists of parallel planes. We may not speak of a fundamental group, but the arrangement is the disjoint union of planes having all trivial fundamental group. For $k = 2$, the arrangement $\mathcal{A}_1$ with the planes $P_1^1, \dots, P_{n_1}^1, P_1^2$ retracts by deformation on P_1^2. However, for $n_2 \geq 2$, the arrangement $\mathcal{A}_2$ given by $P_1^1, \dots, P_{n_1}^1, P_1^2, \dots, P_{n_2}^2$ has a non-trivial fundamental group.

Let us suppose that the third plane direction is not parallel to the intersection of the two first ones. All the generators of the fundamental group of $\mathcal{A}_2$ vanish in P_1^3. The arrangement given by $\mathcal{A}_2$ and the plane P_1^3 has a trivial fundamental group. By adding successively the planes P_j^3, for $2 \leq j \leq n_3$, using Van Kampen theorem, we prove by induction that the fundamental group of the arrangement given by $\mathcal{A}_2$ and $P_1^3, \dots, P_j^3$ is trivial.

Now it it is easy to prove by induction that arrangements with more plane directions have a trivial fundamental group.

It remains to consider the case the third plane direction is parallel to the intersection of the two first ones. It is easy to find that the fundamental group is not trivial. It remains like that, if all the plane directions are parallel to the intersection of the two first ones.

With Theorem 3.2 we obtain:

Corollary 4.2 *The fundamental group of a hyperplane arrangement in $\mathbb{C}^n$, for $n \geq 3$, which has at least 3 hyperplane directions, such that none of them is parallel to the intersection of the two other hyperplane directions, is trivial.*

Beware not to confuse the fundamental group of the complement of the arrangement $\mathcal{A}$ in $\mathbb{C}^n$ and the fundamental group of the arrangement itself. To compute the fundamental group of the complement of the arrangement $\mathcal{A}$ in $\mathbb{C}^n$, one has to use Corollary 1.2.

References

[1] D. Burghelea, A. Verona, *Local homological properties of analytic sets*, Manuscripta Math. **7** (1972), 55 - 66.

[2] D. Cheniot, *Une démonstration du théorème de Zariski sur les sections hyperplanes d'une hypersurface projective et du théorème de Van Kampen sur le groupe fondamental du complémentaire d'une courbe projective plane*, Compositio Math. **27** (1973), 141 - 158.

[3] M. Goresky - R. MacPherson, Stratified Morse theory, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) **14** Springer-Verlag, Berlin, (1988) 272 pp.

[4] H. Hamm - Lê Dũng Tráng, *Rectified homotopical depth and Grothendieck conjectures*, The Grothendieck Festschrift, Vol. II, 311 - 351, Progr. Math., 87 (1990), Birkhäuser Boston, Boston, MA.

[5] H. Hamm - Lê Dũng Tráng, *Un théorème de Zariski du type de Lefschetz*, Ann. Sci. École Norm. Sup. 4 (1973), 317 - 355.

[6] H. Hironaka, *Stratification and flatness. Real and complex singularities*, Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo (1976), pp. 199265, Sijthoff and Noordhoff, Alphen aan den Rijn, 1977.

[7] Lê Dũng Tráng, *Calcul du nombre de cycles évanouissants d'une hypersurface complexe*, Ann. Inst. Fourier **23** (1973), 261 - 270.

[8] Lê Dũng Tráng, *Some remarks on relative monodromy.* in Real and complex singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976), pp. 397 - 403. Sijthoff and Noordhoff, Alphen aan den Rijn, 1977

[9] Lê Dũng Tráng, *Sur les cycles vanouissants des espaces analytiques*, C. R. Acad. Sci. Paris Sr. A-B **288** (1979), A283 - A285.

[10] Lê Dũng Tráng, *Complex analytic functions with isolated singularities*, J. Algebraic Geom. **1** (1992), no. 1, 83 - 99.

[11] J. Milnor, Singular points of complex hypersurfaces, Ann. Math. Studies **61**, Princeton U. P., 1968.

[12] O. Zariski, A theorem on the Poincaré group of an algebraic hypersurface. Ann. of Math. **38** (1937), 131 - 141.

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