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<tr>
<td>Author(s)</td>
<td>Hirose, Susumu</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2011), 1766: 81-90</td>
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<tr>
<td>Issue Date</td>
<td>2011-09</td>
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<td>URL</td>
<td><a href="http://hdl.handle.net/2433/171424">http://hdl.handle.net/2433/171424</a></td>
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<tr>
<td>Type</td>
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On diffeomorphisms over non-orientable surfaces embedded in the 4-sphere

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1. INTRODUCTION

We put an annulus in \( \mathbb{R}^4 \), and deform this in \( \mathbb{R}^4 \) with fixing its boundary as shown in Figure 1. We can change crossing from (3) to (4) because this annulus is in \( \mathbb{R}^4 \). After this deformation, this annulus is twisted two times along the core. This means that this double twist can be extended to the ambient \( \mathbb{R}^4 \). In this note, we will discuss how many diffeomorphisms over the embedded surface are extendable to the ambient 4-space.

For some special embeddings of closed surfaces in 4-manifolds, we have answers to the above problem (for example, [9], [3], [4]). An embedding \( e \) of the orientable surface \( \Sigma_g \) into \( S^4 \) is called standard if there is an embedding of 3-dimensional handlebody into \( S^4 \) such that whose boundary is the image of \( e \). In [9] and [3], we showed:

\[ \text{Figure 1} \]

\[ (1), (2), (3), (4), (5), (6) \]

\[ \text{Figure 1} \]

1 This research was supported by Grant-in-Aid for Scientific Research (C) (No. 20540083), Japan Society for the Promotion of Science.
Theorem 1.1 ([9] (g = 1), [3] (g ≥ 2)). Let $\Sigma_g$ be standardly embedded in $S^4$. A diffeomorphism $\phi$ over the $\Sigma_g$ is extendable to $S^4$ if and only if $\phi$ preserves the Rokhlin quadratic form of the $\Sigma_g$.

In this note, we will introduce some approach to the same kind of problem for non-orientable surfaces embedded in $S^4$.

2. Setting

Let $N_g$ be a connected non-orientable surface constructed from $g$ projective planes by connected sum. We call $N_g$ the closed non-orientable surface of genus $g$. For a smooth embedding $e$ of $N_g$ into $S^4$, Guillou and Marin ([2] see also [8]) defined a quadratic form $q_e : H_1(N_g; \mathbb{Z}_2) \rightarrow \mathbb{Z}_4$ as follows: Let $C$ be an immersed circle on $N_g$, and $D$ be a connected orientable surface immersed in $S^4$ such that $\partial D = C$, and $D$ is not tangent to $N_g$. Let $\nu_D$ be the normal bundle of $D$, then $\nu_D|_C$ is a solid torus with the unique trivialization induced from any trivialization of $\nu_D$. Let $N_{N_g}(C)$ be the tubular neighborhood of $C$ in $N_g$, then $N_{N_g}(C)$ is an twisted annulus or Möbius band in $\nu_D|_C$. We denote by $n(D)$ the number of right hand half-twist of $N_{N_g}(C)$ with respect to the trivialization of $\nu_D|_C$. Let $D \cdot F$ be mod-2 intersection number of $D$ and $F$, $Self(C)$ be mod-2 double points number of $C$, and 2 be an injection $\mathbb{Z}_2 \rightarrow \mathbb{Z}_4$ defined by $2[n]_2 = 2[n]_4$. Then the number $n(D) + 2D \cdot F + 2Self(C) \pmod{4}$ depend only on the mod-2 homology class $[C]$ of $C$. Hence, we define

$$q_e([C]) := n(D) + 2D \cdot F + 2Self(C) \pmod{4}.$$ 

This map $q_e$ is called Guillou and Marin quadratic form, since $q_e$ satisfies

$$q_e(x + y) = q_e(x) + q_e(y) + 2 < x, y >_2,$$

where $< x, y >_2$ means mod-2 intersection number between $x$ and $y$. This quadratic form $q_e$ is a non-orientable analogy of Rokhlin quadratic form.
A diffeomorphism $\phi$ over $N_g$ is $e$-extendable if there is an orientation preserving diffeomorphism $\Phi$ of $S^4$ such that the following diagram is commutative,

\[
\begin{array}{ccc}
N_g & \xrightarrow{e} & S^4 \\
\phi \downarrow & & \downarrow \phi \\
N_g & \xrightarrow{e} & S^4.
\end{array}
\]

If the diffeomorphisms $\phi_1$ over $N_g$ is $e$-extendable, and $\phi_1$ and $\phi_2$ are isotopic, then $\phi_2$ is $e$-extendable. Therefore, $e$-extendability is a property about isotopy classes of diffeomorphisms over $N_g$. The group $\mathcal{M}(N_g)$ of isotopy classes of diffeomorphisms over $N_g$ is called the mapping class group of $N_g$. An element $\phi$ of $\mathcal{M}(N_g)$ is $e$-extendable if there is an $e$-extendable representative of $\phi$. By the definition of $q_e$, we can see that if $\phi \in \mathcal{M}(N_g)$ is $e$-extendable then $\phi$ preserves $q_e$, i.e. $q_e(\phi_*(x)) = q_e(x)$ for any $x \in H_1(N_g;\mathbb{Z}_2)$. What we would like to know is whether $\phi \in \mathcal{M}(N_g)$ is $e$-extendable when $\phi$ preserves $q_e$. But the answer for this problem would be depend on the embedding $e$. So, we will introduce an embedding which seems to be simplest.

Let $S^3 \times [-1, 1]$ be a closed tubular neighborhood of the equator $S^3$ in $S^4$. Then $S^4 - S^3 \times (-1, 1)$ consists of two 4-balls. Let $D_+^4$ be the northern component of
them, and $D^4_-$ be the southern component of them. An embedding $ps : N_g \hookrightarrow S^4$ is $p$-standard if $ps(N_g) \subset S^3 \times [-1,1]$ and as shown in Figure 2. For the basis $\{e_1, \ldots, e_g\}$ of $H_1(N_g; \mathbb{Z}_2)$ shown in Figure 2, $q_{ps}(e_i) = 1$. Since $<e_i, e_j>_{2} = \delta_{ij}$, $q_{ps}(e_{i_1} + e_{i_2} + \cdots + e_{i_t}) = t$. The problem which we consider is the following:

**Problem 2.1.** If $\phi$ preserves $q_{ps}$, is $\phi \in \mathcal{M}(N_g)$ $ps$-extendable?

In order to approach this problem, we review the generators for $\mathcal{M}(N_g)$.

### 3. Generators for $\mathcal{M}(N_g)$

![Figure 3](image)

**Figure 3.** $M$ with circle indicates a place where to attach a Möbius band

A simple closed curve $c$ on $N_g$ is A-circle (resp. M-circle), if the tubular neighborhood of $c$ is an annulus (resp. a Möbius band). We denote by $t_C$ the Dehn twist about an A-circle $c$ on $N_g$. Lickorish [6] showed that $\mathcal{M}(N_g)$ is not generated by Dehn twists, and that Dehn twists and $Y$-homeomorphisms generate $\mathcal{M}(N_g)$. We review the definition of $Y$-homeomorphism. Let $m$ be an M-circle and $a$ be an oriented A-circle in $N_g$ such that $m$ and $a$ transversely intersect in one point. Let $K \subset N_g$ be a regular neighborhood of $m \cup a$, which is homeomorphic to the Klein bottle with a hole, and let $M$ be a regular neighborhood of $m$, which is a Möbius band. We denote by $Y_{m,a}$ a homeomorphism over $N_g$ which may be described as the result of pushing $M$ once along $a$ keeping the boundary of $K$ fixed (see Figure 3). We call
$Y_{m,a}$ a $Y$-homeomorphism. Since $Y$-homeomorphisms act on $H_1(N_g;\mathbb{Z}_2)$ trivially, $Y$-homeomorphisms do not generate $\mathcal{M}(N_g)$. Szepietowski [11] showed an interesting results on the proper subgroup of $\mathcal{M}(N_g)$ generated by all $Y$-homeomorphisms.

**Theorem 3.1 ([11]).** $\Gamma_2(N_g) = \{ \phi \in \mathcal{M}(N_g) \mid \phi_* : H_1(N_g;\mathbb{Z}_2) \to H_1(N_g;\mathbb{Z}_2) = id \}$ is generated by $Y$-homeomorphisms.

In Appendix, we give a quick proof for this Theorem.

Chillingworth showed that $\mathcal{M}(N_g)$ is finitely generated.

**Theorem 3.2 ([1]).** $t_{a_1}, \ldots, t_{a_{g-1}}, t_{b_2}, \ldots, t_{b_{\lfloor \Sigma \rfloor}}, Y_{m_{g-1},a_{g-1}}$ generate $\mathcal{M}(N_g)$.

![Figure 4](image)

**Figure 4**

4. LOWER GENUS CASES

When genus $g$ is at most 3, Problem 2.1 has a trivial answer.

The case where genus $g = 1$: $\mathcal{M}(N_1)$ is trivial.

The case where genus $g = 2$: $\mathcal{M}(N_2)$ is generated by two elements $t_{a_1}$ and $Y_{m_1,a_1}$. Since the tubular neighborhood of $a_1$ in $N_2$ is a Hopf-band in $S^3 \times \{0\}$, $t_{a_1}$ is $ps$-extendable by [4, §2]. Since a sliding of a Möbius band along the tube illustrated in Figure 5 is an extension of $Y_{m_1,a_1}$, $Y_{m_1,a_1}$ is $ps$-extendable. Therefore, any element of $\mathcal{M}(N_2)$ is $ps$-extendable.
The case where genus $g = 3$: $\mathcal{M}(N_3)$ is generated by three elements $t_{a_1}$, $t_{a_2}$ and $Y_{m_2,a_2}$. By the same argument as in the above case, it is shown that any element of $\mathcal{M}(N_3)$ is ps-extendable.

5. Higher genus cases

In the case where genus $g = 4$, $t_{b_4}$ does not preserve $q_{ps}$ because $q_{ps}((t_{b_4})_*(x_1)) = q_{ps}(x_2 + x_3 + x_4) = 3 \neq 1 = q_{ps}(x_1)$. Therefore, $t_{b_4}$ is not ps-extendable. We should consider the following subgroup of $\mathcal{M}(N_g)$,

$$\mathcal{N}_g = \{ \phi \in \mathcal{M}(N_g) | q_{ps}(\phi_*(x)) = q_{ps}(x) \text{ for any } x \in H_1(N_g; Z_2) \} .$$

In order to find a finite system of generators of $\mathcal{N}_g$, we introduce a group

$$\mathcal{O}_g = \{ \phi_* \in Aut(H_1(N_g; Z_2), <, >_{2}) | \phi \in \mathcal{N}_g \} .$$

Then we have a natural short exact sequence

$$0 \rightarrow \Gamma_2(N_g) \rightarrow \mathcal{N}_g \rightarrow \mathcal{O}_g \rightarrow 0.$$

Since $\Gamma_2(N_g)$ is a finite index subgroup of $\mathcal{M}(N_g)$ and $\mathcal{O}_g$ is a finite group, theoretically, there is a finite system of generators for $\mathcal{N}_g$. But we would like to find an explicit system of generators. Nowik found a system of generators for $\mathcal{O}_g$ explicitly. For $a \in H_1(N_g; Z_2)$, define $T_a : H_1(N_g; Z_2) \rightarrow H_1(N_g; Z_2)$ (transvection) by $T_a(x) = x + <x, a >_2 a$, where $<, >_2$ means mod-2 intersection form. We remark that if $l$ is a simple closed curve on $N_g$ such that $[l] = a \in H_1(N_g; Z_2)$, then $(t_l)_* = T_a$. Nowik proved:

**Theorem 5.1.** $\mathcal{O}_g$ is generated by $T_a$ about $a$ with $q_{ps}(a) = 2$. 
If we can find a finite system of generators for $\Gamma_2(N_g)$ explicitly, we can get a finite system of generators for $N_g$. In the case where genus $g = 4$, we find that $\Gamma_2(N_4)$ is generated by the elements shown in Figure 6. Considering the action of Dehn twists corresponding to Nowik's generators of $\mathcal{O}_4$ on our system of generators for $\Gamma_2(N_4)$ by the conjugation, we see that $N_4$ is generated by the 7 elements shown in Figure 7.

We can get an affirmative answer to Problem 2.1 when genus $g = 4$, if we answer the following Problem positively.

*Problem 5.2.* Is $Y$-homeomorphism $Y_{m,a}$ indicated in the above figure $ps$-extendable?
Appendix. A quick proof of Theorem 3.1

Nowik showed,

**Theorem 5.3** ([10]). $\Gamma_2(N_g)$ is generated by the following three types of elements:
1. $(t_c)^2$ about non-separating A-circles $c$ (i.e., $N_g - c$ is connected) in $N_g$,
2. $t_c$ about separating A-circles $c$ (i.e., $N_g - c$ is not connected) in $N_g$,

This Theorem is proved by the same type of argument as in [5]. If we see that any elements of type (1) and (2) are products of Y-homeomorphisms, then we see Theorem 3.1. For (1), Szepietowski showed,

**Lemma 5.4** (Lemma 3.1 of [11]). For any non-separating A-circle $c$ in $N_g$, $(t_c)^2$ is a product of two Y-homeomorphisms.

**Proof.** There exists an $M$-circle $m$ which intersects $c$ in one point. Since $Y_{m,c}$ exchanges the two sides of $c$, we see $Y_{m,c} t_c Y_{m,c}^{-1} = t_c^{-1}$. Therefore, $(t_c)^2 = t_c (Y_{m,c} t_c Y_{m,c}^{-1})^{-1} = t_c Y_{m,c} t_c^{-1} Y_{m,c}^{-1} = Y_{t_c(m),c} Y_{m,c}^{-1}$. \[\square\]

Let $c$ be a separating A-circle, then at least one component $F$ of $N_g - c$ is non-orientable. Let $k$ be the genus of $F$.

![Figure 8](image)

If $k$ is an odd integer, we set $l = (k - 1)/2$. Then $F$ is as shown in Figure 8. By the chain relation, $t_a t_c = (t_{c_1} t_{c_2} \cdots t_{c_{2l+1}})^{2l+2}$. Let $G_F$ be a subgroup of $\mathcal{M}(N_g)$ generated by $t_{c_1}, t_{c_2}, \ldots, t_{c_{2l+1}}$, and $B_{2l+2}$ be the group of $2l + 2$ string braid group generated by $\sigma_1, \sigma_2, \ldots, \sigma_{2l+1}$ where $\sigma_i$ is a braid exchanging the $i$-th string and the $i + 1$-st string. Then there is a surjection $\pi : B_{2l+2} \to G_F$ defined by $\pi(\sigma_i) = t_{c_i}$. By
this surjection, \( \pi(\text{a full twist}) = (t_{c_1}t_{c_2} \cdots t_{2l+1})^{2l+2} \). Since a full twist is a pure braid and the subgroup of pure braids in \( B_{2l+2} \) is generated by \((\sigma_1 \cdots \sigma_l)\sigma_{l+1}^{-1}(\sigma_1 \cdots \sigma_l)^{-1}\), \((t_{c_1}t_{c_2} \cdots t_{2l+1})^{2l+2}\) is a product of \((t_{c_1} \cdots t_{c_l})t_{c_{l+1}}t_{c_1} \cdots t_{c_l})^{-1} = (t_{t_{c_1} \cdots t_{c_l}(c_{l+1})})^2\). By the above Lemma, \((t_{t_{c_1} \cdots t_{c_l}(c_{l+1})})^2\) is a product of \(Y\)-homeomorphisms. Since \(t_a\) is isotopic to the identity, \(t_c\) is a product of \(Y\)-homeomorphisms.

If \(k\) is an even integer, we set \(l = (k - 2)/2\). Then \(F\) is as shown in Figure 9. By the chain relation, \(t_at_c = (t_{c_1}t_{c_2} \cdots t_{2l+1})^{2l+2}\). By the same argument as above, we see that \(t_at_c\) is a product of \(Y\)-homeomorphisms. Let \(y\) be a \(Y\)-homeomorphism whose support is a Klein bottle with one boundary \(a\), then \(t_a = y^2\). Therefore, \(t_c\) is a product of \(Y\)-homeomorphisms.

Remark 5.5. Szepietowski showed that (2) is a product of \(Y\)-homeomorphisms in Lemma 3.2 of [11] by using the lantern relation.

References


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