A quandle cocycle invariant with non-commutative flows for a handlebody-knot (Intelligence of Low-dimensional Topology)

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Citation
数理解析研究所講究録 (2011), 1766: 63-71

Issue Date
2011-09

URL
http://hdl.handle.net/2433/171426

Type
Departmental Bulletin Paper
A quandle cocycle invariant with non-commutative flows for a handlebody-knot

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Abstract

This is a summary of the construction of the quandle cocycle invariant obtained in the joint work with Iwakiri, Jang and Oshiro [7]. Iwakiri and the author [6] introduced a notion of a flow, and defined a quandle cocycle invariant for handlebody-knots. The quandle cocycle invariant given in this article is defined by using “non-commutative” flows.

1 A $G$-family of quandles

A quandle [8, 9] is a non-empty set $X$ with a binary operation $*: X \times X \rightarrow X$ satisfying

- $x * x = x \ (x \in X),$
- $x * x : X \rightarrow X$ is bijective $(x \in X),$
- $(x * y) * z = (x * z) * (y * z) \ (x, y, z \in X).$

An Alexander quandle $(M, *)$ is a $\Lambda$-module $M$ with the binary operation defined by $x * y = tx + (1 - t)y$, where $\Lambda := \mathbb{Z}[t, t^{-1}]$. A conjugation quandle $(G, *)$ is a group $G$ with the binary operation defined by $x * y = y^{-1}xy$.

A $G$-family of quandles is a non-empty set $X$ with a family of binary operations $*^g : X \times X \rightarrow X \ (g \in G)$ satisfying

- $x *^g x = x \ (x \in X, g \in G),$
- $x *^g^h y = (x *^g y) *^h y, x *^e y = x \ (x, y \in X, g, h \in G),$
• $(x \ast^g y)^h z = (x^h z)^{h^{-1} g h} (y^h z)$ \quad (x, y, z \in X, g, h \in G)$.

Proposition 1. Let $G$ be a group, and $(X, \{\ast^g\}_{g \in G})$ a $G$-family of quandles.

(1) For any $g \in G$, $(X, \ast^g)$ is a quandle.

(2) We define $\ast : (X \times G) \times (X \times G) \to X \times G$ by

$$(x, g) \ast (y, h) = (x^h y, h^{-1} g h).$$

Then $(X \times G, \ast)$ is a quandle.

We call the quandle $(X \times G, \ast)$ given in Proposition 1 the associated quandle of $X$.

Proposition 2. Let $R$ be a ring, $G$ a group, and $X$ a right $R[G]$-module. We define a binary operation $\ast^g : X \times X \to X$ by $x \ast^g y = xg + ye - g$. Then $X$ is a $G$-family of quandles.

Let $X$ be a $G$-family of quandles, and $Q$ the associated quandle of $X$. The associated group of $X$, denoted by $\text{As}(X)$, is defined by

$$\text{As}(X) = \left\{ q \in Q \mid q_1 \ast q_2 = q_2^{-1} q_1 q_2 \ (q_1, q_2 \in Q), \quad (x, gh) = (x, g)(x, h) \ (x \in X, g, h \in G) \right\}.$$

An $X$-set $Y$ is a set equipped with a right action of the associated group $\text{As}(X)$. We denote by $y \ast q$ the image of an element $y \in Y$ by the action $q \in \text{As}(X)$. We also denote $y \ast (x, g)$ by $y \ast^g x$. Any singleton set $\{y\}$ is an $X$-set with the trivial action, which is a trivial $X$-set. The set $X$ is also an $X$-set with the action defined by $y \ast (x, g) = y \ast^g x$ for $y \in X$, $(x, g) \in Q$.

2 A handlebody-link

A handlebody-link is a disjoint union of handlebodies embedded in the 3-sphere $S^3$. Two handlebody-links are equivalent if there is an orientation-preserving self-homeomorphism of $S^3$ which sends one to the other. A spatial graph is a finite graph embedded in $S^3$. Two spatial graphs are equivalent if there is an orientation-preserving self-homeomorphism of $S^3$ which sends one to the other. When a handlebody-link $H$ is a regular neighborhood of a spatial graph $K$, we say that $H$ is represented by $K$. In this article,
Figure 1:

Figure 2:

A trivalent graph may contain circle components. Then any handlebody-link can be represented by some spatial trivalent graph. A diagram of a handlebody-link is a diagram of a spatial trivalent graph which represents the handlebody-link. An IH-move is a local spatial move on spatial trivalent graphs as described in Figure 1.

Theorem 3 ([5]). For spatial trivalent graphs $K_1$ and $K_2$, the following are equivalent.

- $K_1$ and $K_2$ represent an equivalent handlebody-link.
- $K_1$ and $K_2$ are related by a finite sequence of IH-moves.
- Diagrams of $K_1$ and $K_2$ are related by a finite sequence of the moves depicted in Figure 2.

3 A coloring with $G$-family of quandles

Let $D$ be a diagram of a handlebody-link $H$. Putting an orientation to each edge in $D$, we obtain a diagram $D$ of an oriented spatial trivalent graph. We may represent an orientation of an edge by a normal orientation, which is obtained by rotating a usual orientation $\pi/2$ counterclockwise on the diagram.
For an arc incident to a vertex \( \omega \), we define \( \epsilon(\alpha; \omega) \in \{1, -1\} \) by

\[
\epsilon(\alpha; \omega) = \begin{cases} 
1 & \text{the orientation of the arc } \alpha \text{ points to the vertex } \omega, \\
-1 & \text{otherwise.}
\end{cases}
\]

We denote by \( \mathcal{A}(D) \) (resp. \( \mathcal{R}(D) \)) the set of arcs (resp. complementary regions) of \( D \). Let \( D \) be a diagram of an oriented spatial trivalent graph. Let \( X \) be a \( G \)-family of quandles, \( Y \) an \( X \)-set, and \( Q \) be the associated quandle of \( X \). Let \( p_X \) and \( p_G \) be the projections from \( Q \) to \( X \) and \( G \), respectively.

An \( X_Y \)-coloring of \( D \) is a map \( C : \mathcal{A}(D) \cup \mathcal{R}(D) \to Q \cup Y \) satisfying the following conditions (see Figures 3, 4).

- \( C(\mathcal{A}(D)) \subset Q \), \( C(\mathcal{R}(D)) \subset Y \).
- Let \( \chi_3 \) be the over-arc at a crossing \( \chi \). Let \( \chi_1, \chi_2 \) be the under-arc at the crossing \( \chi \) such that the normal orientation of \( \chi_3 \) points from \( \chi_1 \) to \( \chi_2 \). Then
  \[ C(\chi_2) = C(\chi_1) \ast C(\chi_3). \]
- Let \( \omega_1, \omega_2, \omega_3 \) be the arcs incident to a vertex \( \omega \). Then
  \[
  (p_X \circ C)(\omega_1) = (p_X \circ C)(\omega_2) = (p_X \circ C)(\omega_3),
  \]
  \[
  (p_G \circ C)(\omega_1)^{\epsilon(\omega_1; \omega)}(p_G \circ C)(\omega_2)^{\epsilon(\omega_2; \omega)}(p_G \circ C)(\omega_3)^{\epsilon(\omega_3; \omega)} = e.
  \]
- For any arc \( \alpha \in \mathcal{A}(D) \), we have
  \[ C(\alpha_1) \ast C(\alpha) = C(\alpha_2), \]
  where \( \alpha_1, \alpha_2 \) are the regions facing the arc \( \alpha \) so that the normal orientation of \( \alpha \) points from \( \alpha_1 \) to \( \alpha_2 \).

We denote by \( \text{Col}_{X_Y}(D) \) the set of \( X_Y \)-colorings of \( D \).

For two diagrams \( D \) and \( E \) which locally differ, we denote by \( \mathcal{A}(D, E) \) (resp. \( \mathcal{R}(D, E) \)) the set of arcs (resp. regions) that \( D \) and \( E \) share.

**Lemma 4.** Let \( X \) be a \( G \)-family of quandles, and \( Y \) an \( X \)-set. Let \( D \) be a diagram of an oriented spatial trivalent graph. Let \( E \) be a diagram obtained by applying one of the R1–R6 moves to the diagram \( D \) once, where we choose orientations for \( E \) which agree with those for \( D \) on \( \mathcal{A}(D, E) \). For \( C \in \text{Col}_{X_Y}(D) \), there is a unique \( X_Y \)-coloring \( C_{D,E} \in \text{Col}_{X_Y}(E) \) such that \( C|_{\mathcal{A}(D,E)} = C_{D,E}|_{\mathcal{A}(D,E)} \) and \( C|_{\mathcal{R}(D,E)} = C_{D,E}|_{\mathcal{R}(D,E)} \).
Figure 3:

Figure 4:
4 A homology

Let $X$ be a $G$-family of quandles, $Y$ an $X$-set, and $Q$ the associated quandle of $X$. Let $B_n(X)_Y$ be the free abelian group generated by the elements of $Y \times Q^n$ if $n \geq 0$, and let $B_n(X)_Y = 0$ otherwise. We put

$$(y, q_1, \ldots, q_i \ast q, q_{i+1}, \ldots, q_n) := (y * q, q_1 \ast q, \ldots, q_i \ast q, q_{i+1}, \ldots, q_n)$$

for $y \in Y$ and $q, q_1, \ldots, q_n \in Q$. We define a boundary homomorphism $\partial_n : B_n(X)_Y \rightarrow B_{n-1}(X)_Y$ by

$$\partial_n(y, q_1, \ldots, q_n) = \sum_{i=1}^{n} (-1)^i (y, q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_n) - \sum_{i=1}^{n} (-1)^i ((y, q_1, \ldots, q_{i-1}) \ast q_i, q_{i+1}, \ldots, q_n)$$

for $n > 0$, and $\partial_n = 0$ otherwise. Then $B_\ast(X)_Y = (B_n(X)_Y, \partial_n)$ is a chain complex (see [1, 2, 3, 4]).

Let $D_n(X)_Y$ be the subgroup of $B_n(X)_Y$ generated by the elements of

$$\bigcup_{i=1}^{n-1} \left\{ (y, q_1, \ldots, q_{i-1}, (x, g), (x, h), q_{i+2}, \ldots, q_n) \mid y \in Y, x \in X, g, h \in G \right\} \bigcup_{i=1}^{n} \left\{ -(y, q_1, \ldots, q_{i-1}) \ast (x, g), (x, h), q_{i+1}, \ldots, q_n \mid y q_1 \in Y, x, q_n \in Q \right\}$$

Lemma 5. For $n \in \mathbb{Z}$, we have $\partial_n(D_n(X)_Y) \subset D_{n-1}(X)_Y$. Thus $D_\ast(X)_Y = (D_n(X)_Y, \partial_n)$ is a subcomplex of $B_\ast(X)_Y$.

We put $C_n(X)_Y = B_n(X)_Y/D_n(X)_Y$. Then $C_\ast(X)_Y = (C_n(X)_Y, \partial_n)$ is a chain complex. For an abelian group $A$, we define the cochain complex $C^\ast(X; A)_Y = \text{Hom}(C_\ast(X)_Y, A)$. We denote by $H_n(X)_Y$ the $n$th homology group of $C_\ast(X)_Y$. 
5 A cocycle invariant

Let $D$ be a diagram of an oriented spatial trivalent graph. For an $X_Y$-coloring $C \in \text{Col}_X(D)_Y$, we define the weight $w(\chi; C) \in C_2(X)_Y$ at a crossing $\chi$ of $D$ as follows. Let $\chi_1, \chi_2$ and $\chi_3$ be respectively the under-arcs and the over-arc at a crossing $\chi$ such that the normal orientation of $\chi_3$ points from $\chi_1$ to $\chi_2$. Let $R_\chi$ be the region facing $\chi_1$ and $\chi_3$ such that the normal orientations $\chi_1$ and $\chi_3$ point from $R_\chi$ to the opposite regions with respect to $\chi_1$ and $\chi_3$, respectively. Then we define

$$w(\chi; C) = \epsilon(\chi)(C(R_\chi), C(\chi_1), C(\chi_3)),$$

where $\epsilon(\chi) \in \{1, -1\}$ is the sign of a crossing $\chi$. We define a chain $W(D; C) \in C_2(X)_Y$ by

$$W(D; C) = \sum_{\chi} w(\chi; C),$$

where $\chi$ runs over all crossings of $D$.

Lemma 6. The chain $W(D; C)$ is a 2-cycle of $C_*(X)_Y$. Further, for cohomologous 2-cocycles $\theta, \theta'$ of $C^*(X; A)_Y$, we have $\theta(W(D; C)) = \theta'(W(D; C))$.

Lemma 7. Let $D$ be a diagram of an oriented spatial trivalent graph. Let $E$ be a diagram obtained by applying one of the R1–R6 moves to the diagram $D$ once, where we choose orientations for $E$ which agree with those for $D$ on $A(D, E)$. For $C \in \text{Col}_X(D)_Y$ and $C_{D,E} \in \text{Col}_X(E)_Y$ such that $C|_{A(D,E)} = C_{D,E}|_{A(D,E)}$ and $C|_{R(D,E)} = C_{D,E}|_{R(D,E)}$, we have $[W(D; C)] = [W(E; C_{D,E})] \in H_2(X)_Y$.

We denote by $G_H$ (resp. $G_K$) the fundamental group of the exterior of a handlebody-link $H$ (resp. a spatial graph $K$). When $H$ is represented by $K$, $G_H$ and $G_K$ are identical. Let $D$ be a diagram of an oriented spatial trivalent graph $K$. By the definition of an $X_Y$-coloring $C$ of $D$, the map $p_G \circ C|_{A(D)}$ represents a homomorphism from $G_K$ to $G$, which we denote by $\rho_C \in \text{Hom}(G_K, G)$. For $\rho \in \text{Hom}(G_K, G)$, we define

$$\text{Col}_X(D; \rho)_Y = \{C \in \text{Col}_X(D)_Y \mid \rho_C = \rho\}.$$
For a 2-cocycle $\theta$ of $C^*(X; A)_Y$, we define

\begin{align*}
\mathcal{H}(D) &:= \{[W(D; C)] \in H_2(X)_Y \mid C \in \text{Col}_X(D)_Y\}, \\
\Phi_\theta(D) &:= \{\theta(W(D; C)) \in A \mid C \in \text{Col}_X(D)_Y\}, \\
\mathcal{H}(D; \rho) &:= \{[W(D; C)] \in H_2(X)_Y \mid C \in \text{Col}_X(D; \rho)_Y\}, \\
\Phi_\theta(D; \rho) &:= \{\theta(W(D; C)) \in A \mid C \in \text{Col}_X(D; \rho)_Y\}
\end{align*}

as multiset.

**Lemma 8.** Let $D$ be a diagram of an oriented spatial trivalent graph $K$. For $\rho, \rho' \in \text{Hom}(G_K, G)$ such that $\rho$ and $\rho'$ are conjugate, we have

\[\mathcal{H}(D; \rho) = \mathcal{H}(D; \rho'), \quad \Phi_\theta(D; \rho) = \Phi_\theta(D; \rho').\]

We denote by $\text{Conj}(G_K, G)$ the set of conjugacy classes of homomorphisms from $G_K$ to $G$. By Lemma 8, $\mathcal{H}(D; \rho)$ and $\Phi_\theta(D; \rho)$ are well-defined for $\rho \in \text{Conj}(G_K, G)$.

**Lemma 9.** Let $D$ be a diagram of an oriented spatial trivalent graph $K$. Let $E$ be a diagram obtained from $D$ by reversing the orientation of an edge $e$. For $\rho \in \text{Hom}(G_K, G)$, we have

\[\mathcal{H}(D) = \mathcal{H}(E), \quad \mathcal{H}(D; \rho) = \mathcal{H}(E; \rho), \quad \Phi_\theta(D) = \Phi_\theta(E), \quad \Phi_\theta(D; \rho) = \Phi_\theta(E; \rho).\]

By Lemma 9, $\mathcal{H}(D)$, $\Phi_\theta(D)$, $\mathcal{H}(D; \rho)$ and $\Phi_\theta(D; \rho)$ are well-defined for a diagram $D$ of an unoriented spatial trivalent graph, which is a diagram of a handlebody-link. For a diagram $D$ of a handlebody-link $H$, we define

\[\mathcal{H}_{\text{hom}}(D) := \{\mathcal{H}(D; \rho) \mid \rho \in \text{Hom}(G_H, G)\}, \]

\[\Phi_{\theta, \text{hom}}(D) := \{\Phi_\theta(D; \rho) \mid \rho \in \text{Hom}(G_H, G)\}, \]

\[\mathcal{H}_{\text{conj}}(D) := \{\mathcal{H}(D; \rho) \mid \rho \in \text{Conj}(G_H, G)\}, \]

\[\Phi_{\theta, \text{conj}}(D) := \{\Phi_\theta(D; \rho) \mid \rho \in \text{Conj}(G_H, G)\}\]

as "multisets of multisets." We remark that, for $X_Y$-colorings $C$ and $C_{D,E}$ in Lemma 7, we have $\rho_C = \rho_{C_{D,E}}$. Then, by Lemmas 6–9, we have the following theorem.

**Theorem 10.** Let $X$ be a $G$-family of quandles, $Y$ an $X$-set. Let $\theta$ be a 2-cocycle of $C^*(X; A)_Y$. Let $H$ be a handlebody-link represented by a diagram $D$. Then the followings are invariants of a handlebody-link $H$.

\[D(H), \quad \Phi_\theta(D), \quad \mathcal{H}_{\text{hom}}(D), \quad \Phi_{\theta, \text{hom}}(D), \quad \mathcal{H}_{\text{conj}}(D), \quad \Phi_{\theta, \text{conj}}(D).\]
References


