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A quandle cocycle invariant with non-commutative flows for a handlebody-knot

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Abstract

This is a summary of the construction of the quandle cocycle invariant obtained in the joint work with Iwakiri, Jang and Oshiro [7]. Iwakiri and the author [6] introduced a notion of a flow, and defined a quandle cocycle invariant for handlebody-knots. The quandle cocycle invariant given in this article is defined by using “non-commutative” flows.

1 A G-family of quandles

A quandle [8, 9] is a non-empty set $X$ with a binary operation $*: X \times X \rightarrow X$ satisfying

- $x*x = x \ (x \in X)$,
- $*x: X \rightarrow X$ is bijective \ $(x \in X)$,
- $(x*y)*z = (x* z)*(y*z) \ (x, y, z \in X)$.

An Alexander quandle $(M,*)$ is a $\Lambda$-module $M$ with the binary operation defined by $x*y = tx + (1-t)y$, where $\Lambda := \mathbb{Z}[t, t^{-1}]$. A conjugation quandle $(G,*)$ is a group $G$ with the binary operation defined by $x*y = y^{-1}xy$.

A G-family of quandles is a non-empty set $X$ with a family of binary operations $*^g: X \times X \rightarrow X \ (g \in G)$ satisfying

- $x*^gx = x \ (x \in X, g \in G)$,
- $x*^ghy = (x*^g y)*^h y, x*^e y = x \ (x, y \in X, g, h \in G)$,
\[ (x *^g y) *^h z = (x *^h z) *^{h^{-1}gh} (y *^h z) \quad (x, y, z \in X, g, h \in G). \]

**Proposition 1.** Let \( G \) be a group, and \((X, \{ *^g \}_{g \in G})\) a \( G \)-family of quandles.

1. For any \( g \in G \), \((X, *^g)\) is a quandle.
2. We define \( * : (X \times G) \times (X \times G) \to X \times G \) by
   \[ (x, g) * (y, h) = (x *^h y, h^{-1}gh). \]

Then \((X \times G, *)\) is a quandle. We call the quandle \((X \times G, *)\) given in Proposition 1 the associated quandle of \( X \).

**Proposition 2.** Let \( R \) be a ring, \( G \) a group, and \( X \) a right \( R[G] \)-module. We define a binary operation \( *^g : X \times X \to X \) by \( x *^g y = xg + y(e - g) \). Then \( X \) is a \( G \)-family of quandles.

Let \( X \) be a \( G \)-family of quandles, and \( Q \) the associated quandle of \( X \). The associated group of \( X \), denoted by \( As(X) \), is defined by

\[
As(X) = \left\{ q \in Q \mid \begin{array}{l}
q_1 * q_2 = q_2^{-1} q_1 q_2 (q_1, q_2 \in Q), \\
(x, gh) = (x, g)(x, h) (x \in X, g, h \in G)
\end{array} \right\}.
\]

An \( X \)-set \( Y \) is a set equipped with a right action of the associated group \( As(X) \). We denote by \( y * q \) the image of an element \( y \in Y \) by the action \( q \in As(X) \). We also denote \( y * (x, g) \) by \( y *^g x \). Any singleton set \( \{ y \} \) is an \( X \)-set with the trivial action, which is a trivial \( X \)-set. The set \( X \) is also an \( X \)-set with the action defined by \( y * (x, g) = y *^g x \) for \( y \in X, (x, g) \in Q \).

## 2 A handlebody-link

A handlebody-link is a disjoint union of handlebodies embedded in the 3-sphere \( S^3 \). Two handlebody-links are equivalent if there is an orientation-preserving self-homeomorphism of \( S^3 \) which sends one to the other. A spatial graph is a finite graph embedded in \( S^3 \). Two spatial graphs are equivalent if there is an orientation-preserving self-homeomorphism of \( S^3 \) which sends one to the other. When a handlebody-link \( H \) is a regular neighborhood of a spatial graph \( K \), we say that \( H \) is represented by \( K \). In this article,
Figure 1:

Figure 2:

A trivalent graph may contain circle components. Then any handlebody-link can be represented by some spatial trivalent graph. A diagram of a handlebody-link is a diagram of a spatial trivalent graph which represents the handlebody-link. An IH-move is a local spatial move on spatial trivalent graphs as described in Figure 1.

**Theorem 3** ([5]). For spatial trivalent graphs $K_1$ and $K_2$, the following are equivalent.

- $K_1$ and $K_2$ represent an equivalent handlebody-link.
- $K_1$ and $K_2$ are related by a finite sequence of IH-moves.
- Diagrams of $K_1$ and $K_2$ are related by a finite sequence of the moves depicted in Figure 2.

### 3 A coloring with G-family of quandles

Let $D$ be a diagram of a handlebody-link $H$. Putting an orientation to each edge in $D$, we obtain a diagram $D$ of an oriented spatial trivalent graph. We may represent an orientation of an edge by a normal orientation, which is obtained by rotating a usual orientation $\pi/2$ counterclockwise on the diagram.
For an arc incident to a vertex $\omega$, we define $\epsilon(\alpha; \omega) \in \{1, -1\}$ by

$$
\epsilon(\alpha; \omega) = \begin{cases} 
1 & \text{the orientation of the arc } \alpha \text{ points to the vertex } \omega, \\
-1 & \text{otherwise.}
\end{cases}
$$

We denote by $A(D)$ (resp. $R(D)$) the set of arcs (resp. complementary regions) of $D$. Let $D$ be a diagram of an oriented spatial trivalent graph. Let $X$ be a $G$-family of quandles, $Y$ an $X$-set, and $Q$ be the associated quandle of $X$. Let $p_X$ and $p_G$ be the projections from $Q$ to $X$ and $G$, respectively.

An $X_Y$-coloring of $D$ is a map $C : A(D) \cup R(D) \to Q \cup Y$ satisfying the following conditions (see Figures 3, 4):

- $C(A(D)) \subset Q, C(R(D)) \subset Y$.
- Let $\chi_3$ be the over-arc at a crossing $\chi$. Let $\chi_1, \chi_2$ be the under-arc at the crossing $\chi$ such that the normal orientation of $\chi_3$ points from $\chi_1$ to $\chi_2$. Then
  $$
  C(\chi_2) = C(\chi_1) * C(\chi_3).
  $$
- Let $\omega_1, \omega_2, \omega_3$ be the arcs incident to a vertex $\omega$. Then
  $$
  (p_X \circ C)(\omega_1) = (p_X \circ C)(\omega_2) = (p_X \circ C)(\omega_3),
  $$(p_G \circ C)(\omega_1)^{\epsilon(\omega_1; \omega)}(p_G \circ C)(\omega_2)^{\epsilon(\omega_2; \omega)}(p_G \circ C)(\omega_3)^{\epsilon(\omega_3; \omega)} = e.
- For any arc $\alpha \in A(D)$, we have
  $$
  C(\alpha_1) * C(\alpha) = C(\alpha_2),
  $$
  where $\alpha_1, \alpha_2$ are the regions facing the arc $\alpha$ so that the normal orientation of $\alpha$ points from $\alpha_1$ to $\alpha_2$.

We denote by $\text{Col}_X(D)_Y$ the set of $X_Y$-colorings of $D$.

For two diagrams $D$ and $E$ which locally differ, we denote by $A(D, E)$ (resp. $R(D, E)$) the set of arcs (resp. regions) that $D$ and $E$ share.

**Lemma 4.** Let $X$ be a $G$-family of quandles, and $Y$ an $X$-set. Let $D$ be a diagram of an oriented spatial trivalent graph. Let $E$ be a diagram obtained by applying one of the R1-R6 moves to the diagram $D$ once, where we choose orientations for $E$ which agree with those for $D$ on $A(D, E)$. For $C \in \text{Col}_X(D)_Y$, there is a unique $X_Y$-coloring $C_{D,E} \in \text{Col}_X(E)_Y$ such that $C_{A(D,E)} = C_{D,E}|_{A(D,E)}$ and $C_{R(D,E)} = C_{D,E}|_{R(D,E)}$. 


Figure 3:

Figure 4:
4 A homology

Let $X$ be a $G$-family of quandles, $Y$ an $X$-set, and $Q$ the associated quandle of $X$. Let $B_n(X)_Y$ be the free abelian group generated by the elements of $Y \times Q^n$ if $n \geq 0$, and let $B_n(X)_Y = 0$ otherwise. We put

$((y, q_1, \ldots, q_i) * q, q_{i+1}, \ldots, q_n) := (y * q, q_1 * q, \ldots, q_i * q, q_{i+1}, \ldots, q_n)$

for $y \in Y$ and $q, q_1, \ldots, q_n \in Q$. We define a boundary homomorphism $\partial_n : B_n(X)_Y \to B_{n-1}(X)_Y$ by

$$\partial_n(y, q_1, \ldots, q_n) = \sum_{i=1}^{n} (-1)^i (y, q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_n) - \sum_{i=1}^{n} (-1)^i ((y, q_1, \ldots, q_{i-1}) * q_{i}, q_{i+1}, \ldots, q_n)$$

for $n > 0$, and $\partial_n = 0$ otherwise. Then $B_*(X)_Y = (B_n(X)_Y, \partial_n)$ is a chain complex (see [1, 2, 3, 4]).

Let $D_n(X)_Y$ be the subgroup of $B_n(X)_Y$ generated by the elements of

$$\bigcup_{i=1}^{n-1} \{(y, q_1, \ldots, q_{i-1}, (x, g), (x, h), q_{i+2}, \ldots, q_n) \mid y \in Y, x \in X, g, h \in G, q_1, \ldots, q_n \in Q\}$$

and

$$\bigcup_{i=1}^{n} \left\{ (y, q_1, \ldots, q_{i-1}, (x, gh), q_{i+1}, \ldots, q_n) \mid y \in Y, x \in X, g, h \in G, q_1, \ldots, q_n \in Q \right\}$$

$$\bigcup_{i=1}^{n} \left\{ -((y, q_1, \ldots, q_{i-1}) * (x, g), (x, h), q_{i+1}, \ldots, q_n) \mid y \in Y, x \in X, g, h \in G, q_1, \ldots, q_n \in Q \right\}$$

Lemma 5. For $n \in \mathbb{Z}$, we have $\partial_n(D_n(X)_Y) \subset D_{n-1}(X)_Y$. Thus $D_*(X)_Y = (D_n(X)_Y, \partial_n)$ is a subcomplex of $B_*(X)_Y$.

We put $C_n(X)_Y = B_n(X)_Y / D_n(X)_Y$. Then $C_*(X)_Y = (C_n(X)_Y, \partial_n)$ is a chain complex. For an abelian group $A$, we define the cochain complex $C^*(X; A)_Y = \text{Hom}(C_*(X)_Y, A)$. We denote by $H_n(X)_Y$ the $n$th homology group of $C_*(X)_Y$. 
5 A cocycle invariant

Let $D$ be a diagram of an oriented spatial trivalent graph. For an $X_Y$-coloring $C \in \text{Col}_X(D)_Y$, we define the weight $w(\chi;C) \in C_2(X)_Y$ at a crossing $\chi$ of $D$ as follows. Let $\chi_1, \chi_2$ and $\chi_3$ be respectively the under-arcs and the over-arc at a crossing $\chi$ such that the normal orientation of $\chi_3$ points from $\chi_1$ to $\chi_2$. Let $R_\chi$ be the region facing $\chi_1$ and $\chi_3$ such that the normal orientations $\chi_1$ and $\chi_3$ point from $R_\chi$ to the opposite regions with respect to $\chi_1$ and $\chi_3$, respectively. Then we define

$$w(\chi;C) = \epsilon(\chi)(C(R_\chi), C(\chi_1), C(\chi_3)),$$

where $\epsilon(\chi) \in \{1, -1\}$ is the sign of a crossing $\chi$. We define a chain $W(D;C) \in C_2(X)_Y$ by

$$W(D;C) = \sum_{\chi} w(\chi;C),$$

where $\chi$ runs over all crossings of $D$.

Lemma 6. The chain $W(D;C)$ is a 2-cycle of $C_*(X)_Y$. Further, for cohomologous 2-cocycles $\theta, \theta'$ of $C^*(X;A)_Y$, we have $\theta(W(D;C)) = \theta'(W(D;C))$.

Lemma 7. Let $D$ be a diagram of an oriented spatial trivalent graph. Let $E$ be a diagram obtained by applying one of the R1–R6 moves to the diagram $D$ once, where we choose orientations for $E$ which agree with those for $D$ on $A(D,E)$. For $C \in \text{Col}_X(D)_Y$ and $C_{D,E} \in \text{Col}_X(E)_Y$ such that $C|_{A(D,E)} = C_{D,E}|_{A(D,E)}$ and $C|_{R(D,E)} = C_{D,E}|_{R(D,E)}$, we have $[W(D;C)] = [W(E;C_{D,E})] \in H_2(X)_Y$.

We denote by $G_H$ (resp. $G_K$) the fundamental group of the exterior of a handlebody-link $H$ (resp. a spatial graph $K$). When $H$ is represented by $K$, $G_H$ and $G_K$ are identical. Let $D$ be a diagram of an oriented spatial trivalent graph $K$. By the definition of an $X_Y$-coloring $C$ of $D$, the map $p_G \circ C|_{A(D)}$ represents a homomorphism from $G_K$ to $G$, which we denote by $\rho_C \in \text{Hom}(G_K, G)$. For $\rho \in \text{Hom}(G_K, G)$, we define

$$\text{Col}_X(D;\rho)_Y = \{C \in \text{Col}_X(D)_Y | \rho_C = \rho\}.$$
For a 2-cocycle $\theta$ of $C^*(X; A)_Y$, we define
\[
\mathcal{H}(D) := \{[W(D; C)] \in H_2(X)_Y \mid C \in \text{Col}_X(D)_Y\},
\]
\[
\Phi_\theta(D) := \{\theta(W(D; C)) \in A \mid C \in \text{Col}_X(D)_Y\},
\]
\[
\mathcal{H}(D; \rho) := \{[W(D; C)] \in H_2(X)_Y \mid C \in \text{Col}_X(D; \rho)_Y\},
\]
\[
\Phi_\theta(D; \rho) := \{\theta(W(D; C)) \in A \mid C \in \text{Col}_X(D; \rho)_Y\}
\]
as multisets.

**Lemma 8.** Let $D$ be a diagram of an oriented spatial trivalent graph $K$. For $\rho, \rho' \in \text{Hom}(G_K, G)$ such that $\rho$ and $\rho'$ are conjugate, we have
\[
\mathcal{H}(D; \rho) = \mathcal{H}(D; \rho') \quad \Phi_\theta(D; \rho) = \Phi_\theta(D; \rho').
\]
We denote by $\text{Conj}(G_K, G)$ the set of conjugacy classes of homomorphisms from $G_K$ to $G$. By Lemma 8, $\mathcal{H}(D; \rho)$ and $\Phi_\theta(D; \rho)$ are well-defined for $\rho \in \text{Conj}(G_K, G)$.

**Lemma 9.** Let $D$ be a diagram of an oriented spatial trivalent graph $K$. Let $E$ be a diagram obtained from $D$ by reversing the orientation of an edge $e$. For $\rho \in \text{Hom}(G_K, G)$, we have
\[
\mathcal{H}(D) = \mathcal{H}(E), \quad \mathcal{H}(D; \rho) = \mathcal{H}(E; \rho),
\]
\[
\Phi_\theta(D) = \Phi_\theta(E), \quad \Phi_\theta(D; \rho) = \Phi_\theta(E; \rho).
\]
By Lemma 9, $\mathcal{H}(D)$, $\Phi_\theta(D)$, $\mathcal{H}(D; \rho)$ and $\Phi_\theta(D; \rho)$ are well-defined for a diagram $D$ of an unoriented spatial trivalent graph, which is a diagram of a handlebody-link. For a diagram $D$ of a handlebody-link $H$, we define
\[
\mathcal{H}^\text{hom}(D) := \{\mathcal{H}(D; \rho) \mid \rho \in \text{Hom}(G_H, G)\},
\]
\[
\Phi_\theta^\text{hom}(D) := \{\Phi_\theta(D; \rho) \mid \rho \in \text{Hom}(G_H, G)\},
\]
\[
\mathcal{H}^\text{conj}(D) := \{\mathcal{H}(D; \rho) \mid \rho \in \text{Conj}(G_H, G)\},
\]
\[
\Phi_\theta^\text{conj}(D) := \{\Phi_\theta(D; \rho) \mid \rho \in \text{Conj}(G_H, G)\}
\]
as "multisets of multisets." We remark that, for $X_Y$-colorings $C$ and $C_{D,E}$ in Lemma 7, we have $\rho_C = \rho_{C_{D,E}}$. Then, by Lemmas 6–9, we have the following theorem.

**Theorem 10.** Let $X$ be a $G$-family of quandles, $Y$ an $X$-set. Let $\theta$ be a 2-cocycle of $C^*(X; A)_Y$. Let $H$ be a handlebody-link represented by a diagram $D$. Then the followings are invariants of a handlebody-link $H$.
\[
\mathcal{D}(H), \quad \Phi_\theta(D), \quad \mathcal{H}^\text{hom}(D), \quad \Phi_\theta^\text{hom}(D), \quad \mathcal{H}^\text{conj}(D), \quad \Phi_\theta^\text{conj}(D).
\]
References


