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Kyoto University
A quandle cocycle invariant with non-commutative flows for a handlebody-knot

Atsushi Ishii (University of Tsukuba)

Abstract

This is a summary of the construction of the quandle cocycle invariant obtained in the joint work with Iwakiri, Jang and Oshiro [7]. Iwakiri and the author [6] introduced a notion of a flow, and defined a quandle cocycle invariant for handlebody-knots. The quandle cocycle invariant given in this article is defined by using "non-commutative" flows.

1 A G-family of quandles

A quandle [8, 9] is a non-empty set $X$ with a binary operation $*: X \times X \to X$ satisfying

- $x * x = x \ (x \in X)$,
- $*x: X \to X$ is bijective $(x \in X)$,
- $(x * y) * z = (x * z) * (y * z) \ (x, y, z \in X)$.

An Alexander quandle $(M, *)$ is a $\Lambda$-module $M$ with the binary operation defined by $x * y = tx + (1 - t)y$, where $\Lambda := \mathbb{Z}[t, t^{-1}]$. A conjugation quandle $(G, \ast)$ is a group $G$ with the binary operation defined by $x \ast y = y^{-1}xy$.

A G-family of quandles is a non-empty set $X$ with a family of binary operations $*: X \times X \to X \ (g \in G)$ satisfying

- $x \ast^g x = x \ (x \in X, g \in G)$,
- $x \ast^{gh} y = (x \ast^g y) \ast^h y, x \ast^e y = x \ (x, y \in X, g, h \in G)$,
\[(x *^g y) *^h z = (x *^h z) *^{h^{-1} gh} (y *^h z) \quad (x, y, z \in X, \ g, h \in G).\]

**Proposition 1.** Let \(G\) be a group, and \((X, \{*_g\}_{g \in G})\) a \(G\)-family of quandles.

1. For any \(g \in G\), \((X, *_g)\) is a quandle.
2. We define \(* : (X \times G) \times (X \times G) \rightarrow X \times G\) by
   \[(x, g) * (y, h) = (x *^h y, h^{-1} gh).\]

Then \((X \times G, *)\) is a quandle.

We call the quandle \((X \times G, *)\) given in Proposition 1 the *associated quandle* of \(X\).

**Proposition 2.** Let \(R\) be a ring, \(G\) a group, and \(X\) a right \(R[G]\)-module. We define a binary operation \(*^g : X \times X \rightarrow X\) by \(x *^g y = xg + y(e - g)\). Then \(X\) is a \(G\)-family of quandles.

Let \(X\) be a \(G\)-family of quandles, and \(Q\) the associated quandle of \(X\). The *associated group* of \(X\), denoted by \(\text{As}(X)\), is defined by

\[\text{As}(X) = \left\langle q \in Q \left| \begin{array}{l} q_1 * q_2 = q_2^{-1} q_1 q_2 \ (q_1, q_2 \in Q), \\ (x, gh) = (x, g)(x, h) \ (x \in X, \ g, h \in G) \end{array} \right. \right\}.\]

An \(X\)-set \(Y\) is a set equipped with a right action of the associated group \(\text{As}(X)\). We denote by \(y * q\) the image of an element \(y \in Y\) by the action \(q \in \text{As}(X)\). We also denote \(y *^g x\) by \(y *^g x\). Any singleton set \(\{y\}\) is an \(X\)-set with the trivial action, which is a trivial \(X\)-set. The set \(X\) is also an \(X\)-set with the action defined by \(y *^g (x, g) = y *^g x\) for \(y \in X, \ (x, g) \in Q\).

## 2 A handlebody-link

A *handlebody-link* is a disjoint union of handlebodies embedded in the 3-sphere \(S^3\). Two handlebody-links are *equivalent* if there is an orientation-preserving self-homeomorphism of \(S^3\) which sends one to the other. A *spatial graph* is a finite graph embedded in \(S^3\). Two spatial graphs are *equivalent* if there is an orientation-preserving self-homeomorphism of \(S^3\) which sends one to the other. When a handlebody-link \(H\) is a regular neighborhood of a spatial graph \(K\), we say that \(H\) is represented by \(K\). In this article,
a trivalent graph may contain circle components. Then any handlebody-link can be represented by some spatial trivalent graph. A diagram of a handlebody-link is a diagram of a spatial trivalent graph which represents the handlebody-link. An IH-move is a local spatial move on spatial trivalent graphs as described in Figure 1.

**Theorem 3** ([5]). For spatial trivalent graphs $K_1$ and $K_2$, the following are equivalent.

- $K_1$ and $K_2$ represent an equivalent handlebody-link.
- $K_1$ and $K_2$ are related by a finite sequence of IH-moves.
- Diagrams of $K_1$ and $K_2$ are related by a finite sequence of the moves depicted in Figure 2.

### 3 A coloring with $G$-family of quandles

Let $D$ be a diagram of a handlebody-link $H$. Putting an orientation to each edge in $D$, we obtain a diagram $D$ of an oriented spatial trivalent graph. We may represent an orientation of an edge by a normal orientation, which is obtained by rotating a usual orientation $\pi/2$ counterclockwise on the diagram.
For an arc incident to a vertex $\omega$, we define $\epsilon(\alpha; \omega) \in \{1, -1\}$ by

$$
\epsilon(\alpha; \omega) = \begin{cases} 
1 & \text{the orientation of the arc } \alpha \text{ points to the vertex } \omega, \\
-1 & \text{otherwise.}
\end{cases}
$$

We denote by $\mathcal{A}(D)$ (resp. $\mathcal{R}(D)$) the set of arcs (resp. complementary regions) of $D$. Let $D$ be a diagram of an oriented spatial trivalent graph. Let $X$ be a $G$-family of quandles, $Y$ an $X$-set, and $Q$ be the associated quandle of $X$. Let $p_X$ and $p_G$ be the projections from $Q$ to $X$ and $G$, respectively. An $X_Y$-coloring of $D$ is a map $C : \mathcal{A}(D) \cup \mathcal{R}(D) \to Q \cup Y$ satisfying the following conditions (see Figures 3, 4).

- $C(\mathcal{A}(D)) \subset Q$, $C(\mathcal{R}(D)) \subset Y$.

- Let $\chi_3$ be the over-arc at a crossing $\chi$. Let $\chi_1, \chi_2$ be the under-arc at the crossing $\chi$ such that the normal orientation of $\chi_3$ points from $\chi_1$ to $\chi_2$. Then

$$
C(\chi_2) = C(\chi_1) \ast C(\chi_3).
$$

- Let $\omega_1, \omega_2, \omega_3$ be the arcs incident to a vertex $\omega$. Then

$$
(p_X \circ C)(\omega_1) = (p_X \circ C)(\omega_2) = (p_X \circ C)(\omega_3),
$$

$$
(p_G \circ C)(\omega_1)^{\epsilon(\omega_1; \omega)}(p_G \circ C)(\omega_2)^{\epsilon(\omega_2; \omega)}(p_G \circ C)(\omega_3)^{\epsilon(\omega_3; \omega)} = e.
$$

- For any arc $\alpha \in \mathcal{A}(D)$, we have

$$
C(\alpha_1) \ast C(\alpha) = C(\alpha_2),
$$

where $\alpha_1, \alpha_2$ are the regions facing the arc $\alpha$ so that the normal orientation of $\alpha$ points from $\alpha_1$ to $\alpha_2$.

We denote by $\text{Col}_{X}(D)_Y$ the set of $X_Y$-colorings of $D$.

For two diagrams $D$ and $E$ which locally differ, we denote by $\mathcal{A}(D, E)$ (resp. $\mathcal{R}(D, E)$) the set of arcs (resp. regions) that $D$ and $E$ share.

**Lemma 4.** Let $X$ be a $G$-family of quandles, and $Y$ an $X$-set. Let $D$ be a diagram of an oriented spatial trivalent graph. Let $E$ be a diagram obtained by applying one of the R1–R6 moves to the diagram $D$ once, where we choose orientations for $E$ which agree with those for $D$ on $\mathcal{A}(D, E)$. For $C \in \text{Col}_{X}(D)_Y$, there is a unique $X_Y$-coloring $C_{D,E} \in \text{Col}_{X}(E)_Y$ such that $C|_{\mathcal{A}(D,E)} = C_{D,E}|_{\mathcal{A}(D,E)}$ and $C|_{\mathcal{R}(D,E)} = C_{D,E}|_{\mathcal{R}(D,E)}$. 
Figure 3:

Figure 4:
4 A homology

Let $X$ be a $G$-family of quandles, $Y$ an $X$-set, and $Q$ the associated quandle of $X$. Let $B_n(X)_Y$ be the free abelian group generated by the elements of $Y \times Q^n$ if $n \geq 0$, and let $B_n(X)_Y = 0$ otherwise. We put

$$((y, q_1, \ldots, q_i) * q, q_{i+1}, \ldots, q_n) := (y * q, q_1 * q, \ldots, q_i * q, q_{i+1}, \ldots, q_n)$$

for $y \in Y$ and $q, q_1, \ldots, q_n \in Q$. We define a boundary homomorphism $\partial_n : B_n(X)_Y \to B_{n-1}(X)_Y$ by

$$\partial_n(y, q_1, \ldots, q_n) = \sum_{i=1}^{n}(-1)^i(y, q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_n) - \sum_{i=1}^{n}(-1)^i((y, q_1, \ldots, q_{i-1}) * q_i, q_{i+1}, \ldots, q_n)$$

for $n > 0$, and $\partial_n = 0$ otherwise. Then $B_*(X)_Y = (B_n(X)_Y, \partial_n)$ is a chain complex (see [1, 2, 3, 4]).

Let $D_n(X)_Y$ be the subgroup of $B_n(X)_Y$ generated by the elements of

$$\bigcup_{i=1}^{n-1}\left\{ (y, q_1, \ldots, q_{i-1}, (x, g), (x, h), q_{i+2}, \ldots, q_n) \mid y \in Y, x \in X, g, h \in G \right\}$$

and

$$\bigcup_{i=1}^{n}\left\{ (y, q_1, \ldots, q_{i-1}, (x, gh), q_{i+1}, \ldots, q_n) - (y, q_1, \ldots, q_{i-1}, (x, g), q_{i+1}, \ldots, q_n) \mid y \in Y, x \in X, g, h \in G \right\}.$$

**Lemma 5.** For $n \in \mathbb{Z}$, we have $\partial_n(D_n(X)_Y) \subset D_{n-1}(X)_Y$. Thus $D_*(X)_Y = (D_n(X)_Y, \partial_n)$ is a subcomplex of $B_*(X)_Y$.

We put $C_n(X)_Y = B_n(X)_Y/D_n(X)_Y$. Then $C_*(X)_Y = (C_n(X)_Y, \partial_n)$ is a chain complex. For an abelian group $A$, we define the cochain complex $C^*(X; A)_Y = \text{Hom}(C_*(X)_Y, A)$. We denote by $H_n(X)_Y$ the $n$th homology group of $C_*(X)_Y$. 

5 A cocycle invariant

Let $D$ be a diagram of an oriented spatial trivalent graph. For an $X_Y$-coloring $C \in \text{Col}_X(D)_Y$, we define the weight $w(\chi; C) \in C_2(X)_Y$ at a crossing $\chi$ of $D$ as follows. Let $\chi_1, \chi_2$ and $\chi_3$ be respectively the under-arcs and the over-arc at a crossing $\chi$ such that the normal orientation of $\chi_3$ points from $\chi_1$ to $\chi_2$. Let $R_\chi$ be the region facing $\chi_1$ and $\chi_3$ such that the normal orientations $\chi_1$ and $\chi_3$ point from $R_\chi$ to the opposite regions with respect to $\chi_1$ and $\chi_3$, respectively. Then we define

$$w(\chi; C) = \epsilon(\chi)(C(R_\chi), C(\chi_1), C(\chi_3)),$$

where $\epsilon(\chi) \in \{1, -1\}$ is the sign of a crossing $\chi$. We define a chain $W(D; C) \in C_2(X)_Y$ by

$$W(D; C) = \sum_\chi w(\chi; C),$$

where $\chi$ runs over all crossings of $D$.

Lemma 6. The chain $W(D; C)$ is a 2-cycle of $C_*(X)_Y$. Further, for cohomologous 2-cocycles $\theta, \theta'$ of $C^*(X; A)_Y$, we have $\theta(W(D; C)) = \theta'(W(D; C))$.

Lemma 7. Let $D$ be a diagram of an oriented spatial trivalent graph. Let $E$ be a diagram obtained by applying one of the R1–R6 moves to the diagram $D$ once, where we choose orientations for $E$ which agree with those for $D$ on $A(D, E)$. For $C \in \text{Col}_X(D)_Y$ and $C_{D,E} \in \text{Col}_X(E)_Y$ such that $C|_{A(D,E)} = C_{D,E}|_{A(D,E)}$ and $C|_{R(D,E)} = C_{D,E}|_{R(D,E)}$, we have $[W(D; C)] = [W(E; C_{D,E})] \in H_2(X)_Y$.

We denote by $G_H$ (resp. $G_K$) the fundamental group of the exterior of a handlebody-link $H$ (resp. a spatial graph $K$). When $H$ is represented by $K$, $G_H$ and $G_K$ are identical. Let $D$ be a diagram of an oriented spatial trivalent graph $K$. By the definition of an $X_Y$-coloring $C$ of $D$, the map $p_G \circ C|_{A(D)}$ represents a homomorphism from $G_K$ to $G$, which we denote by $\rho_C \in \text{Hom}(G_K, G)$. For $\rho \in \text{Hom}(G_K, G)$, we define

$$\text{Col}_X(D; \rho)_Y = \{C \in \text{Col}_X(D)_Y \mid \rho_C = \rho\}.$$
For a 2-cocycle $\theta$ of $C^*(X;A)_Y$, we define

$$\mathcal{H}(D) := \{[W(D;C)] \in H_2(X)_Y | C \in \text{Col}_X(D)_Y\},$$
$$\Phi_\theta(D) := \{\theta(W(D;C)) \in A | C \in \text{Col}_X(D)_Y\},$$
$$\mathcal{H}(D;\rho) := \{[W(D;C)] \in H_2(X)_Y | C \in \text{Col}_X(D;\rho)_Y\},$$
$$\Phi_\theta(D;\rho) := \{\theta(W(D;C)) \in A | C \in \text{Col}_X(D;\rho)_Y\}$$

as multisets.

**Lemma 8.** Let $D$ be a diagram of an oriented spatial trivalent graph $K$. For $\rho, \rho' \in \text{Hom}(G_K, G)$ such that $\rho$ and $\rho'$ are conjugate, we have

$$\mathcal{H}(D;\rho) = \mathcal{H}(D;\rho') \quad \Phi_\theta(D;\rho) = \Phi_\theta(D;\rho').$$

We denote by $\text{Conj}(G_K, G)$ the set of conjugacy classes of homomorphisms from $G_K$ to $G$. By Lemma 8, $\mathcal{H}(D;\rho)$ and $\Phi_\theta(D;\rho)$ are well-defined for $\rho \in \text{Conj}(G_K, G)$.

**Lemma 9.** Let $D$ be a diagram of an oriented spatial trivalent graph $K$. Let $E$ be a diagram obtained from $D$ by reversing the orientation of an edge $e$. For $\rho \in \text{Hom}(G_K, G)$, we have

$$\mathcal{H}(D) = \mathcal{H}(E), \quad \mathcal{H}(D;\rho) = \mathcal{H}(E;\rho),$$
$$\Phi_\theta(D) = \Phi_\theta(E), \quad \Phi_\theta(D;\rho) = \Phi_\theta(E;\rho).$$

By Lemma 9, $\mathcal{H}(D)$, $\Phi_\theta(D)$, $\mathcal{H}(D;\rho)$ and $\Phi_\theta(D;\rho)$ are well-defined for a diagram $D$ of an unoriented spatial trivalent graph, which is a diagram of a handlebody-link. For a diagram $D$ of a handlebody-link $H$, we define

$$\mathcal{H}_\text{hom}(D) := \{\mathcal{H}(D;\rho) | \rho \in \text{Hom}(G_H, G)\},$$
$$\Phi_\theta_\text{hom}(D) := \{\Phi_\theta(D;\rho) | \rho \in \text{Hom}(G_H, G)\},$$
$$\mathcal{H}_\text{conj}(D) := \{\mathcal{H}(D;\rho) | \rho \in \text{Conj}(G_H, G)\},$$
$$\Phi_\theta_\text{conj}(D) := \{\Phi_\theta(D;\rho) | \rho \in \text{Conj}(G_H, G)\}$$

as "multisets of multisets." We remark that, for $X_Y$-colorings $C$ and $C_{D,E}$ in Lemma 7, we have $\rho_C = \rho_{C_{D,E}}$. Then, by Lemmas 6-9, we have the following theorem.

**Theorem 10.** Let $X$ be a $G$-family of quandles, $Y$ an $X$-set. Let $\theta$ be a 2-cocycle of $C^*(X;A)_Y$. Let $H$ be a handlebody-link represented by a diagram $D$. Then the followings are invariants of a handlebody-link $H$.

$$\mathcal{D}(H), \quad \Phi_\theta(D), \quad \mathcal{H}_\text{hom}(D), \quad \Phi_\theta_\text{hom}(D), \quad \mathcal{H}_\text{conj}(D), \quad \Phi_\theta_\text{conj}(D).$$
References


