A quandle cocycle invariant with non-commutative flows for a handlebody-knot

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Abstract

This is a summary of the construction of the quandle cocycle invariant obtained in the joint work with Iwakiri, Jang and Oshiro [7]. Iwakiri and the author [6] introduced a notion of a flow, and defined a quandle cocycle invariant for handlebody-knots. The quandle cocycle invariant given in this article is defined by using “non-commutative” flows.

1 A G-family of quandles

A quandle [8, 9] is a non-empty set $X$ with a binary operation $*: X \times X \to X$ satisfying

- $x * x = x$ ($x \in X$),
- $*x : X \to X$ is bijective ($x \in X$),
- $(x * y) * z = (x * z) * (y * z)$ ($x, y, z \in X$).

An Alexander quandle $(M, *)$ is a $\Lambda$-module $M$ with the binary operation defined by $x * y = tx + (1 - t)y$, where $\Lambda := \mathbb{Z}[t, t^{-1}]$. A conjugation quandle $(G, *)$ is a group $G$ with the binary operation defined by $x * y = y^{-1}xy$.

A $G$-family of quandles is a non-empty set $X$ with a family of binary operations $*_g : X \times X \to X$ ($g \in G$) satisfying

- $x *_g x = x$ ($x \in X, g \in G$),
- $x *_{gh} y = (x *_{g} y) *^{h} y$, $x *^{e} y = x$ ($x, y \in X, g, h \in G$),
\( (x \ast^g y)^h z = (x^h z)^{h^{-1}gh}(y \ast^h z) \) \( (x, y, z \in X, g, h \in G) \).

**Proposition 1.** Let \( G \) be a group, and \( (X, \{ \ast^g \}_{g \in G}) \) a \( G \)-family of quandles.

1. For any \( g \in G \), \( (X, \ast^g) \) is a quandle.
2. We define \( \ast : (X \times G) \times (X \times G) \rightarrow X \times G \) by
   \[
   (x, g) \ast (y, h) = (x^h y, h^{-1}gh).
   \]
   Then \( (X \times G, \ast) \) is a quandle

We call the quandle \( (X \times G, \ast) \) given in Proposition 1 the *associated quandle* of \( X \).

**Proposition 2.** Let \( R \) be a ring, \( G \) a group, and \( X \) a right \( R[G] \)-module. We define a binary operation \( \ast^g : X \times X \rightarrow X \) by \( x \ast^g y = xg + y(e - g) \). Then \( X \) is a \( G \)-family of quandles.

Let \( X \) be a \( G \)-family of quandles, and \( Q \) the associated quandle of \( X \). The *associated group* of \( X \), denoted by \( \text{As}(X) \), is defined by

\[
\text{As}(X) = \left\langle q \in Q \mid q_1 \ast q_2 = q_2^{-1}q_1q_2 \ (q_1, q_2 \in Q), \quad (x, gh) = (x, g)(x, h) \ (x \in X, g, h \in G) \right\rangle.
\]

An \( X \)-set \( Y \) is a set equipped with a right action of the associated group \( \text{As}(X) \). We denote by \( y \ast q \) the image of an element \( y \in Y \) by the action \( q \in \text{As}(X) \). We also denote \( y \ast (x, g) \) by \( y \ast^g x \). Any singleton set \( \{y\} \) is an \( X \)-set with the trivial action, which is a trivial \( X \)-set. The set \( X \) is also an \( X \)-set with the action defined by \( y \ast (x, g) = y \ast^g x \) for \( y \in X, (x, g) \in Q \).

### 2 A handlebody-link

A *handlebody-link* is a disjoint union of handlebodies embedded in the 3-sphere \( S^3 \). Two handlebody-links are *equivalent* if there is an orientation-preserving self-homeomorphism of \( S^3 \) which sends one to the other. A *spatial graph* is a finite graph embedded in \( S^3 \). Two spatial graphs are *equivalent* if there is an orientation-preserving self-homeomorphism of \( S^3 \) which sends one to the other. When a handlebody-link \( H \) is a regular neighborhood of a spatial graph \( K \), we say that \( H \) is represented by \( K \). In this article,
A trivalent graph may contain circle components. Then any handlebody-link can be represented by some spatial trivalent graph. A *diagram* of a handlebody-link is a diagram of a spatial trivalent graph which represents the handlebody-link. An *IH-move* is a local spatial move on spatial trivalent graphs as described in Figure 1.

**Theorem 3** ([5]). For spatial trivalent graphs $K_1$ and $K_2$, the following are equivalent.

- $K_1$ and $K_2$ represent an equivalent handlebody-link.
- $K_1$ and $K_2$ are related by a finite sequence of IH-moves.
- Diagrams of $K_1$ and $K_2$ are related by a finite sequence of the moves depicted in Figure 2.

### 3 A coloring with $G$-family of quandles

Let $D$ be a diagram of a handlebody-link $H$. Putting an orientation to each edge in $D$, we obtain a diagram $D$ of an oriented spatial trivalent graph. We may represent an orientation of an edge by a normal orientation, which is obtained by rotating a usual orientation $\pi/2$ counterclockwise on the diagram.
For an arc incident to a vertex $\omega$, we define $\epsilon(\alpha; \omega) \in \{1, -1\}$ by

$$
\epsilon(\alpha; \omega) = \begin{cases} 
1 & \text{the orientation of the arc } \alpha \text{ points to the vertex } \omega, \\
-1 & \text{otherwise.}
\end{cases}
$$

We denote by $A(D)$ (resp. $R(D)$) the set of arcs (resp. complementary regions) of $D$. Let $D$ be a diagram of an oriented spatial trivalent graph. Let $X$ be a $G$-family of quandles, $Y$ an $X$-set, and $Q$ be the associated quandle of $X$. Let $p_X$ and $p_G$ be the projections from $Q$ to $X$ and $G$, respectively. An $X_Y$-coloring of $D$ is a map $C : A(D) \cup R(D) \rightarrow Q \cup Y$ satisfying the following conditions (see Figures 3, 4).

- $C(A(D)) \subset Q$, $C(R(D)) \subset Y$.
- Let $\chi_3$ be the over-arc at a crossing $\chi$. Let $\chi_1, \chi_2$ be the under-arc at the crossing $\chi$ such that the normal orientation of $\chi_3$ points from $\chi_1$ to $\chi_2$. Then
  $$C(\chi_2) = C(\chi_1) \ast C(\chi_3).$$
- Let $\omega_1, \omega_2, \omega_3$ be the arcs incident to a vertex $\omega$. Then
  $$\begin{align*}
  (p_X \circ C)(\omega_1) &= (p_X \circ C)(\omega_2) = (p_X \circ C)(\omega_3), \\
  (p_G \circ C)(\omega_1)^{\epsilon(\omega_1; \omega)}(p_G \circ C)(\omega_2)^{\epsilon(\omega_2; \omega)}(p_G \circ C)(\omega_3)^{\epsilon(\omega_3; \omega)} &= e.
  \end{align*}$$
- For any arc $\alpha \in A(D)$, we have
  $$C(\alpha_1) \ast C(\alpha) = C(\alpha_2),$$
  where $\alpha_1, \alpha_2$ are the regions facing the arc $\alpha$ so that the normal orientation of $\alpha$ points from $\alpha_1$ to $\alpha_2$.

We denote by $\text{Col}_{X}(D)_Y$ the set of $X_Y$-colorings of $D$.

For two diagrams $D$ and $E$ which locally differ, we denote by $A(D,E)$ (resp. $R(D,E)$) the set of arcs (resp. regions) that $D$ and $E$ share.

**Lemma 4.** Let $X$ be a $G$-family of quandles, and $Y$ an $X$-set. Let $D$ be a diagram of an oriented spatial trivalent graph. Let $E$ be a diagram obtained by applying one of the R1–R6 moves to the diagram $D$ once, where we choose orientations for $E$ which agree with those for $D$ on $A(D,E)$. For $C \in \text{Col}_{X}(D)_Y$, there is a unique $X_Y$-coloring $C_{D,E} \in \text{Col}_{X}(E)_Y$ such that $C|_{A(D,E)} = C_{D,E}|_{A(D,E)}$ and $C|_{R(D,E)} = C_{D,E}|_{R(D,E)}$. 
Figure 3:

$(x, h^{-1}g^{-1}) \quad (x, hg) \quad (x, g^{-1}h^{-1})$

$(x, g) \quad (x, h) \quad (x, g) \quad (x, h)$

Figure 4:

$y_1 \quad y_1 * q$

$q$
4 A homology

Let $X$ be a $G$-family of quandles, $Y$ an $X$-set, and $Q$ the associated quandle of $X$. Let $B_n(X)_Y$ be the free abelian group generated by the elements of $Y \times Q^n$ if $n \geq 0$, and let $B_n(X)_Y = 0$ otherwise. We put

$$((y, q_1, \ldots, q_i) * q, q_{i+1}, \ldots, q_n) := (y * q, q_1 * q, \ldots, q_i * q, q_{i+1}, \ldots, q_n)$$

for $y \in Y$ and $q, q_1, \ldots, q_n \in Q$. We define a boundary homomorphism $\partial_n : B_n(X)_Y \rightarrow B_{n-1}(X)_Y$ by

$$\partial_n(y, q_1, \ldots, q_n) = \sum_{i=1}^{n}(-1)^i(y, q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_n)$$

$$-\sum_{i=1}^{n}(-1)^i((y, q_1, \ldots, q_{i-1}) * q_i, q_{i+1}, \ldots, q_n)$$

for $n > 0$, and $\partial_n = 0$ otherwise. Then $B_*(X)_Y = (B_n(X)_Y, \partial_n)$ is a chain complex (see [1, 2, 3, 4]).

Let $D_n(X)_Y$ be the subgroup of $B_n(X)_Y$ generated by the elements of

$$\bigcup_{i=1}^{n-1}\{(y, q_1, \ldots, q_i, (x, g), (x, h), q_{i+2}, \ldots, q_n) \mid y \in Y, x \in X, g, h \in G, q_1, \ldots, q_n \in Q\}$$

and

$$\bigcup_{i=1}^{n}\{(y, q_1, \ldots, q_{i-1}, (x, gh), q_{i+1}, \ldots, q_n) \mid y \in Y, x \in X, g, h \in G, q_1, \ldots, q_n \in Q\}.$$ 

Lemma 5. For $n \in \mathbb{Z}$, we have $\partial_n(D_n(X)_Y) \subset D_{n-1}(X)_Y$. Thus $D_*(X)_Y = (D_n(X)_Y, \partial_n)$ is a subcomplex of $B_*(X)_Y$.

We put $C_n(X)_Y = B_n(X)_Y / D_n(X)_Y$. Then $C_*(X)_Y = (C_n(X)_Y, \partial_n)$ is a chain complex. For an abelian group $A$, we define the cochain complex $C^*(X; A)_Y = \text{Hom}(C_*(X)_Y, A)$. We denote by $H_n(X)_Y$ the $n$th homology group of $C_*(X)_Y$. 

5 A cocycle invariant

Let $D$ be a diagram of an oriented spatial trivalent graph. For an $X_Y$-coloring $C \in \text{Col}_X(D)_Y$, we define the weight $w(\chi; C) \in C_2(X)_Y$ at a crossing $\chi$ of $D$ as follows. Let $\chi_1, \chi_2$ and $\chi_3$ be respectively the under-arcs and the over-arc at a crossing $\chi$ such that the normal orientation of $\chi_3$ points from $\chi_1$ to $\chi_2$. Let $R_\chi$ be the region facing $\chi_1$ and $\chi_3$ such that the normal orientations $\chi_1$ and $\chi_3$ point from $R_\chi$ to the opposite regions with respect to $\chi_1$ and $\chi_3$, respectively. Then we define

$$w(\chi; C) = \epsilon(\chi)(C(R_\chi), C(\chi_1), C(\chi_3)),$$

where $\epsilon(\chi) \in \{1, -1\}$ is the sign of a crossing $\chi$. We define a chain $W(D; C) \in C_2(X)_Y$ by

$$W(D; C) = \sum_\chi w(\chi; C),$$

where $\chi$ runs over all crossings of $D$.

**Lemma 6.** The chain $W(D; C)$ is a 2-cycle of $C_*(X)_Y$. Further, for cohomologous 2-cocycles $\theta, \theta'$ of $C^*(X; A)_Y$, we have $\theta(W(D; C)) = \theta'(W(D; C))$.

**Lemma 7.** Let $D$ be a diagram of an oriented spatial trivalent graph. Let $E$ be a diagram obtained by applying one of the R1–R6 moves to the diagram $D$ once, where we choose orientations for $E$ which agree with those for $D$ on $A(D, E)$. For $C \in \text{Col}_X(D)_Y$ and $C_{D,E} \in \text{Col}_X(E)_Y$ such that $C|_{A(D,E)} = C_{D,E}|_{A(D,E)}$ and $C|_{R(D,E)} = C_{D,E}|_{R(D,E)}$, we have $[W(D; C)] = [W(E; C_{D,E})] \in H_2(X)_Y$.

We denote by $G_H$ (resp. $G_K$) the fundamental group of the exterior of a handlebody-link $H$ (resp. a spatial graph $K$). When $H$ is represented by $K$, $G_H$ and $G_K$ are identical. Let $D$ be a diagram of an oriented spatial trivalent graph $K$. By the definition of an $X_Y$-coloring $C$ of $D$, the map $p_G \circ C|_{A(D)}$ represents a homomorphism from $G_K$ to $G$, which we denote by $\rho_C \in \text{Hom}(G_K, G)$. For $\rho \in \text{Hom}(G_K, G)$, we define

$$\text{Col}_X(D; \rho)_Y = \{C \in \text{Col}_X(D)_Y \mid \rho_C = \rho\}.$$
For a 2-cocycle $\theta$ of $C^*(X;A)_Y$, we define

\begin{align*}
\mathcal{H}(D) &:= \{[W(D;C)] \in H_2(X)_Y \mid C \in \text{Col}_X(D)_Y\}, \\
\Phi_\theta(D) &:= \{\theta(W(D;C)) \in A \mid C \in \text{Col}_X(D)_Y\}, \\
\mathcal{H}(D;\rho) &:= \{[W(D;C)] \in H_2(X)_Y \mid C \in \text{Col}_X(D;\rho)_Y\}, \\
\Phi_\theta(D;\rho) &:= \{\theta(W(D;C)) \in A \mid C \in \text{Col}_X(D;\rho)_Y\}
\end{align*}

as multisets.

**Lemma 8.** Let $D$ be a diagram of an oriented spatial trivalent graph $K$. For $\rho, \rho' \in \text{Hom}(G_K, G)$ such that $\rho$ and $\rho'$ are conjugate, we have

\[ \mathcal{H}(D;\rho) = \mathcal{H}(D;\rho') \quad \Phi_\theta(D;\rho) = \Phi_\theta(D;\rho'). \]

We denote by $\text{Conj}(G_K, G)$ the set of conjugacy classes of homomorphisms from $G_K$ to $G$. By Lemma 8, $\mathcal{H}(D;\rho)$ and $\Phi_\theta(D;\rho)$ are well-defined for $\rho \in \text{Conj}(G_K, G)$.

**Lemma 9.** Let $D$ be a diagram of an oriented spatial trivalent graph $K$. Let $E$ be a diagram obtained from $D$ by reversing the orientation of an edge $e$. For $\rho \in \text{Hom}(G_K, G)$, we have

\[ \mathcal{H}(D) = \mathcal{H}(E), \quad \mathcal{H}(D;\rho) = \mathcal{H}(E;\rho), \]
\[ \Phi_\theta(D) = \Phi_\theta(E), \quad \Phi_\theta(D;\rho) = \Phi_\theta(E;\rho). \]

By Lemma 9, $\mathcal{H}(D), \Phi_\theta(D), \mathcal{H}(D;\rho)$ and $\Phi_\theta(D;\rho)$ are well-defined for a diagram $D$ of an unoriented spatial trivalent graph, which is a diagram of a handlebody-link. For a diagram $D$ of a handlebody-link $H$, we define

\[ \mathcal{H}^\text{hom}(D) := \{\mathcal{H}(D;\rho) \mid \rho \in \text{Hom}(G_H, G)\}, \]
\[ \Phi_\theta^\text{hom}(D) := \{\Phi_\theta(D;\rho) \mid \rho \in \text{Hom}(G_H, G)\}, \]
\[ \mathcal{H}^\text{conj}(D) := \{\mathcal{H}(D;\rho) \mid \rho \in \text{Conj}(G_H, G)\}, \]
\[ \Phi_\theta^\text{conj}(D) := \{\Phi_\theta(D;\rho) \mid \rho \in \text{Conj}(G_H, G)\} \]

as "multisets of multisets." We remark that, for $X_Y$-colorings $C$ and $C_{D,E}$ in Lemma 7, we have $\rho_C = \rho_{C_{D,E}}$. Then, by Lemmas 6–9, we have the following theorem.

**Theorem 10.** Let $X$ be a $G$-family of quandles, $Y$ an $X$-set. Let $\theta$ be a 2-cocycle of $C^*(X;A)_Y$. Let $H$ be a handlebody-link represented by a diagram $D$. Then the followings are invariants of a handlebody-link $H$.

\[ D(H), \quad \Phi_\theta(D), \quad \mathcal{H}^\text{hom}(D), \quad \Phi_\theta^\text{hom}(D), \quad \mathcal{H}^\text{conj}(D), \quad \Phi_\theta^\text{conj}(D). \]
References


