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Kyoto University
On the Kashaev invariant of twist knots

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Let $K$ denote the knot represented by the diagram with $n + 3$ crossings below.
In this talk, we compute its Kashaev invariant, which is a special value of the
colored Jones polynomial, and study its asymptotic behavior.

1. Kashaev's invariant

Let $N$ be a positive integer and

$$\mathcal{N} = \{0, 1, \ldots, N - 1\}.$$ 

Then, for $i, j, k, l \in \mathcal{N}$, we define $\theta_{kl}^{ij}$ by

$$\theta_{kl}^{ij} = \begin{cases} 1 & \text{if } [i - j] + [j - l] + [l - k - 1] + [k - i] = N - 1, \\ 0 & \text{otherwise}, \end{cases}$$

where $[m] \in \mathcal{N}$ denotes the residue of $m$ modulo $N$. Furthermore, for $x \in \mathbb{C}$, we define $(x)_m$ by

$$(x)_m = (1 - x)(1 - x^2) \cdots (1 - x^{[m]}).$$

In what follows, we put

$$q = \exp \frac{2\pi \sqrt{-1}}{N}.$$ 

Then, the Kashaev invariant $\langle K \rangle_N$ of $K$ is obtained by contracting the tensors associated to the following critical points, where

$$R_{kl}^{ij} = \frac{Nq^{-\frac{1}{2} + i - k}}{(q)_{[i-j]}(\overline{q})_{[j-l]}(q)_{[l-k-1]}(\overline{q})_{[k-i]}} \cdot \theta_{kl}^{ij},$$

$$\overline{R}_{kl}^{ij} = \frac{Nq^{\frac{1}{2} + j - l}}{(\overline{q})_{[i-j]}(q)_{[j-l]}(\overline{q})_{[l-k-1]}(q)_{[k-i]}} \cdot \theta_{kl}^{ij}.$$
Proposition. The Kashaev invariant $\langle K \rangle_N$ of $K$ is given by

$$N^{n+1} \sum_{0 \leq i_1 \leq \cdots \leq i_n < N} \frac{1}{(q)_{i_1} (\overline{q})_{i_n}} \prod_{\nu=1}^{n-1} \frac{1}{(\overline{q})_{i_{\nu}} (q)_{i_{\nu+1}-i_{\nu}} (\overline{q})_{N-1-i_{\nu+1}}}.$$ 

Example. Suppose $n = 3$. Then,

$$\langle K \rangle_N = \sum_{c \leq d \leq b} \frac{Nq^{-\frac{1}{2}+b+1}}{(q)_{b-d} (\overline{q})_d (\overline{q})_{N-1-b}} \cdot \frac{Nq^{-\frac{1}{2}-c}}{(\overline{q})_{N-1-d} (q)_{d-c} (\overline{q})_c} \cdot \sum_{a=c}^{N-1} \frac{Nq^c}{(\overline{q})_{N-1-a} (q)_{a}} \cdot \frac{Nq^{-\frac{1}{2}+c}}{(q)_{a-c} (\overline{q})_{c} (\overline{q})_{N-1-a}} \times \sum_{e=b}^{N-1} \frac{Nq^{-c}}{(q)_{e} (\overline{q})_{N-1-e}} \cdot \frac{Nq^{-\frac{1}{2}-b}}{(q)_{N-1-e} (q)_{e-b} (\overline{q})_b},$$

from the picture below. This is further equal to

$$\sum_{c \leq d \leq b} \frac{Nq^{-\frac{1}{2}+b+1}}{(q)_{b-d} (\overline{q})_d (\overline{q})_{N-1-b}} \cdot \frac{Nq^{-\frac{1}{2}-c}}{(\overline{q})_{N-1-d} (q)_{d-c} (\overline{q})_c} \cdot \frac{N^2 q^{c-b}}{(q)_c (\overline{q})_b},$$

where we used the following lemma.

Lemma. $(q)_m (\overline{q})_{N-1-m} = N$ and

$$\sum_{i \leq m \leq j} \frac{1}{(q)_{j-m} (\overline{q})_{m-i}} = 1.$$
2. QUANTUM DILOGARITHMS

Let

$$\psi_N(z) = \exp \int_{-\infty}^{\infty} \frac{e^{\sqrt{N}(2z+1)t}dt}{4t \sinh(t/\sqrt{N}) \sinh(\sqrt{N}t)},$$

and

$$p_k = \frac{2k+1}{2N}.$$

Then, the sets of poles and zeros of $\psi_N$ are given by $\{p_k \mid k \geq N\}$ and $\{p_k \mid k < 0\}$ respectively, and

$$\frac{1}{(q)_k} = \frac{\psi_N(p_k)}{\psi_N(p_0)}, \quad \frac{1}{(\overline{q})_k} = \frac{\psi_N(1-p_0)}{\psi_N(1-p_k)},$$

$$\frac{\psi_N(p_0)}{\sqrt{N}} = \exp \frac{N}{2\pi \sqrt{-1}} \left( \frac{\pi^2}{6} - \frac{\pi^2}{2N} + \frac{\pi^2}{6N^2} \right) = \sqrt{N} \psi_N(1-p_0).$$

By using these quantum dilogarithms, we can write

$$\langle K \rangle_N = N^{n+1} \sum_{k_1 \leq \cdots \leq k_n} \frac{\psi_N(p_{k_1})}{\psi_N(p_0)} \frac{\psi_N(1-p_0)}{\psi_N(1-p_{k_n})} \prod_{\nu=1}^{n-1} \frac{\psi_N(1-p_{k_{\nu+1}})}{\psi_N(1-p_{k_{\nu}})} \frac{\psi_N(p_{k_{\nu+1}})}{\psi_N(p_0)} \frac{\psi_N(1-p_0)}{\psi_N(1-p_{k_{\nu+1}})}.$$

If we put

$$\Psi_N(z_1, \ldots, z_n) = e^{\frac{N(n-1)}{2\pi \sqrt{-1}} \left( \frac{\pi}{6} \nabla \right)} \frac{\prod_{\nu=1}^{n-1} \psi_N(z_{\nu+1} - z_{\nu} + p_0)}{\psi_N(1-z_n) \psi_N(1-z_{\nu+1})},$$

we have

$$\langle K \rangle_N = N^{\frac{n+1}{2}} \sum_{k_1=0}^{N-1} \cdots \sum_{k_n=0}^{N-1} \Psi_N(p_{k_1}, \ldots, p_{k_n})$$

because $\psi_N(p_{k_{\nu+1}} - p_{k_{\nu}}) = 0$ if $k_{\nu+1} < k_{\nu}$.

3. INTEGRALS

Let $Q_\nu = e^{2\pi \sqrt{-1}z_\nu}$ and

$$C = \{ x + y \sqrt{-1} \mid (x - \frac{1}{2})^2 + y^2 = (\frac{1}{2})^2 \}.$$

Then, by the residue theorem,

$$\langle K \rangle_N = (-1)^n N^{\frac{n+3}{2}} \int_C \frac{dz_1}{1+Q_1^N} \cdots \int_C \frac{dz_n}{1+Q_n^N} \Psi_N(z_1, \ldots, z_n).$$
Let
\[ A = \{ z \in C | \text{Im} \, z \geq 0 \}, \quad B = \{ z \in C | \text{Im} \, z \leq 0 \}. \]

Then, we have
\[
\int_C \frac{dz_\nu}{1 + Q_\nu^N} = \int_0^1 (Q_\nu^N - 1 + Q_\nu^{-N})dz_\nu + \int_A \frac{Q_\nu^{2N}}{1 + Q_\nu^N}dz_\nu - \int_B \frac{Q_\nu^{-2N}}{1 + Q_\nu^{-N}}dz_\nu
\]
because
\[
\int_A \frac{dz_\nu}{1 + Q_\nu^N} = \int_1^0 (1 - Q_\nu^N)dz_\nu + \int_A \frac{Q_\nu^{2N}}{1 + Q_\nu^N}dz_\nu,
\]
\[
\int_B \frac{dz_\nu}{1 + Q_\nu^N} = \int_B \frac{Q_\nu^{-N}dz_\nu}{1 + Q_{\overline{\nu}}^N} = \int_0^1 Q_\nu^{-N}dz_\nu - \int_B \frac{Q_\nu^{-2N}}{1 + Q_\nu^{-N}}dz_\nu.
\]

In what follows, for \( z \in \mathbb{C} \), we put
\[ x_z = \text{Re} \, 2\pi z, \quad y_z = \text{Im} \, 2\pi z, \quad \omega_z = -\arg(1 - e^{2\pi\sqrt{-1}z}). \]

**Lemma.**
\[ \lim_{y_{z_\nu} \to \pm \infty} \Psi_N(z_1, \ldots, z_n) Q_{\nu}^{\pm N} = 0. \]

**Proposition.**
\[ \langle K \rangle_N \sim (-1)^n N^{\frac{n+3}{2}} \int_0^1 dz_1 \cdots \int_0^1 dz_n \Psi_N(z_1, \ldots, z_n). \]

4. **ASYMPTOTICS**

Define
\[ \mathcal{L}(z) = \text{Li}_2(e^{2\pi\sqrt{-1}z}) + \beta_z(2\pi z - \frac{1}{2}\beta_z - \pi), \]
where \( \text{Li}_2 \) denotes Euler’s dilogarithm function and
\[ \beta_z = \begin{cases} 2\pi |\text{Re} \, z| & \text{if } \text{Im} \, z < 0, \\ 0 & \text{otherwise.} \end{cases} \]

Then, we have
\[ \psi_N(z) \sim e^{\frac{N}{2\pi\sqrt{-1}} \{ \mathcal{L}(z) + O(N^{-2}) \}}. \]

Furthermore, we put
\[ V(z) = \Lambda(x_z) + \Lambda(\omega_z) - \Lambda(x_z + \omega_z), \]
where \( \Lambda \) denotes Lobachevsky’s function. Then, we have
\[ \text{Im} \, \mathcal{L}(z) = y_z(\beta_z - \omega_z) + V(z), \quad \frac{\partial}{\partial y_z} \text{Im} \, \mathcal{L}(z) = \beta_z - \omega_z. \]
Define $H(z_1, \ldots, z_n)$ by

$$L(z_1) - L(z_n) + \frac{(n-1)\pi^2}{6} + \sum_{\nu=1}^{n-1} \{L(z_{\nu+1} - z_{\nu}) - L(1 - z_{\nu}) - L(z_{\nu+1})\}.$$  

For simplicity, we put $f(z_1, \ldots, z_n) = \text{Im} H(z_1, \ldots, z_n)$.

**Proposition.**

$$\Psi_N(z_1, \ldots, z_n) \sim e^{\frac{N}{2\pi}} \{H(z_1, \ldots, z_n) + O(N^{-1})\}.$$  

**Example.** Suppose $n = 3$. Then, $f(z_1, z_2, z_3)$ is equal to

$$V(z_1, z_2, z_3) + y_{z_1}(-\omega_{z_1} - \omega_{1-z_1} - \beta_{z_2-z_1} + \omega_{z_2-z_1})$$
$$+ y_{z_2}(\beta_{z_2-z_1} - \omega_{z_2-z_1} + \pi - x_{z_2})$$
$$+ y_{z_3}(\beta_{z_3-z_2} - \omega_{z_3-z_2} + \pi - x_{z_3}),$$

where $V(z_1, \ldots, z_n)$ is defined by

$$V(z_1) - V(z_n) + \sum_{\nu=1}^{n-1} \{V(z_{\nu+1} - z_{\nu}) - V(1 - z_{\nu}) - V(z_{\nu+1})\}.$$  

How does $f(z_1, z_2, z_3)$ behave when $y_{z_1}^2 + y_{z_2}^2 + y_{z_3}^2 \to \infty$? Since

$$\lim_{y_{z} \to \infty} (\beta_{z} - \omega_{z}) = 0, \quad \lim_{y_{z} \to -\infty} (\beta_{z} - \omega_{z}) = x_{z} - \pi,$$

we check its behavior along the following 3 lines;

$y_{z_2-z_1} = y_{z_3-z_2} = 0, \quad y_{z_1} = y_{z_3-z_2} = 0, \quad y_{z_1} = y_{z_2-z_1} = 0$

with $x_{z_1}, x_{z_2}, x_{z_3}$ fixed. For simplicity, we put

$$\lambda(y) = \frac{1}{2}(y - |y|).$$

Then, $f(z_1, z_2, z_3)$ is approximated by

$$\lambda(y_{z_1})(x_{z_1} - x_{z_2} - x_{z_3} + \pi) + \lambda(-y_{z_1})(x_{z_1} + x_{z_2} + x_{z_3} - 3\pi)$$

when $y_{z_2-z_1} = y_{z_3-z_2} = 0$, by

$$\lambda(y_{z_2})(-x_{z_1} - x_{z_3} + \pi) + \lambda(-y_{z_2})(x_{z_2} + x_{z_3} - 2\pi)$$

when $y_{z_1} = y_{z_3-z_2} = 0$, and by

$$\lambda(y_{z_3})(-x_{z_2}) + \lambda(-y_{z_3})(x_{z_3} - \pi)$$

when $y_{z_1} = y_{z_2-z_1} = 0$. Therefore, we can observe

$$\lim_{y_{z_1}^2 + y_{z_2}^2 + y_{z_3}^2 \to \infty} f(z_1, z_2, z_3) = \infty$$

if $x_{z_1}, x_{z_2}, x_{z_3}$ satisfy the following conditions.

$$\pi < -x_{z_1} + x_{z_2} + x_{z_3} < x_{z_1} + x_{z_2} + x_{z_3} < 3\pi,$$

$$\pi < x_{z_1} + x_{z_3} < x_{z_2} + x_{z_3} < 2\pi, \quad x_{z_3} < \pi.$$  

This region will play an important role in the following argument.
Let $\Delta$ be the set of $(z_1, \ldots, z_n) \in [0,1]^n$ satisfying
\[
\frac{1}{2}(n - \nu) < z_{\nu-1} + \sum_{k=\nu+1}^{n}z_k < \sum_{k=\nu+1}^{n}k < \frac{1}{2}(n - \nu + 1)
\]
for $1 < \nu \leq n$ and
\[
\frac{1}{2}(n - 2) < -z_1 + \sum_{k=2}^{n}z_k < \sum_{k=1}^{n}z_k < \frac{1}{2}n.
\]
The main purpose of this note is to show

**Proposition.** Let $(\zeta_1, \ldots, \zeta_n)$ be the solution to
\[
\frac{\partial H}{\partial z_{\nu}} \equiv 0 \mod 2\pi\sqrt{-1}
\]
satisfying $(\text{Re } \zeta_1, \ldots, \text{Re } \zeta_n) \in \Delta$. Then,
\[
\int_{\Delta} \Psi_N(z_1, \ldots, z_n) \, dz_1 \wedge \cdots \wedge dz_n \sim N^{-\frac{9}{8}N}e^{\overline{2\pi}}T - 5^{H(\zeta_1, \ldots, \zeta_n)}.
\]

Note that $f(\zeta_1, \ldots, \zeta_n)$ is equal to the complex volume of $K$.

**Proof.** Define $p: \mathbb{C}^n \to \mathbb{R}^n$ by
\[
p(z_1, \ldots, z_n) = (\text{Re } z_1, \ldots, \text{Re } z_n).
\]

Let $\Sigma$ be the set of $(z_1, \ldots, z_n) \in p^{-1}(\Delta)$ satisfying
\[
y_{z_1} = \log \frac{\sin \frac{1}{2}(n\pi - x_{z_1} + x_{z_2} + \cdots + x_{z_n})}{\sin \frac{1}{2}(n\pi + x_{z_1} + x_{z_2} + \cdots + x_{z_n})},
\]
\[
y_{z_2} = \log \frac{\sin(x_{z_1} + x_{z_2} + \cdots + x_{z_n})}{\sin(x_{z_2} + x_{z_3} + \cdots + x_{z_n})} + y_{z_1},
\]
\[
\vdots
\]
\[
y_{z_n} = \log \frac{\sin x_{z_{n-1}}}{\sin x_{z_n}} + y_{z_{n-1}},
\]
which is the unique solution to
\[
\frac{\partial f}{\partial y_{z_{\nu}}} = 0.
\]

Then, $f|_{\Sigma}$ takes its unique maximum at $(\zeta_1, \ldots, \zeta_n) \in \Sigma$. Let
\[
E_{\pm} = f^{-1}((-\infty, f(\zeta_1, \ldots, \zeta_n) \pm \epsilon]) \cap p^{-1}(\Delta)
\]
and $I$ the set of $(z_1(t), \ldots, z_n(t)) \in E_+ - E_-$ satisfying
\[
\frac{dz_{\nu}}{dt} = \frac{\partial f}{\partial z_{\nu}}, \quad \lim_{t \to -\infty} (z_1(t), \ldots, z_n(t)) = (\zeta_1, \ldots, \zeta_n).
Then, $\text{Re} H(z_1, \ldots, z_n)$ is constant on $I$ because

$$
\frac{d(\text{Re} H)}{dt} = \sum_{\nu=1}^{n} \text{Re} \left\{ \frac{\partial H}{\partial z_{\nu}} \cdot \frac{dz_{\nu}}{dt} + \frac{\partial H}{\partial \overline{z}_{\nu}} \cdot \frac{d\overline{z}_{\nu}}{dt} \right\} = \sum_{\nu=1}^{n} \text{Re} \left\{ \frac{\partial(\text{Re} H + \sqrt{-1} f)}{\partial z_{\nu}} \cdot \frac{dz_{\nu}}{dt} \right\} = \sum_{\nu=1}^{n} \text{Re} \left\{ 2\sqrt{-1} \cdot \frac{\partial f}{\partial z_{\nu}} \cdot \frac{\partial f}{\partial \overline{z}_{\nu}} \right\} = 0.
$$

Since $E_+ \simeq E_- \cup I \simeq \Sigma$, by the saddle point method, we have

$$
\int_{\Delta} \Omega_N = \int_{\Sigma} \Omega_N \sim \int_{I} \Omega_N \sim N^{-\frac{n}{2}} e^{\frac{N}{2\pi} \int_{\Sigma} H(z_1, \ldots, z_n)},
$$

where $\Omega_N = \Psi_N(z_1, \ldots, z_n) \, dz_1 \wedge \cdots \wedge dz_n$. \qed