QUANDLE COCYCLE INVARIANTS
OF ROLL-SPUN KNOTS

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ABSTRACT. We generalize the class of roll-spun knots in 2-knot theory and study the quandle colorings for such a 2-knot. We also explain how to calculate the quandle cocycle invariant and prove that the invariant of any roll-spun knot is trivial if the second homology group of the quandle vanishes.

1. INTRODUCTION

For an oriented surface-knot $F$ and a third cohomology class $\theta \in H^3(X; A)$ of a quandle $X$, the calculation of the quandle cocycle invariant of $F$ with respect to $\theta$ is given as follows:

$$C \in \text{Col}_X(F) \leadsto \gamma(C) \in H_3(X) \leadsto \Phi_X(F) \leadsto \Phi_\theta(F).$$

More precisely, each $X$-coloring $C$ for $F$ defines a third homology class $\gamma(C) \in H_3(X)$ by taking the sum of weights on triple points of a diagram, and such classes form the multi-set

$$\Phi_X(F) = \{\gamma(C) \in H_3(X) \mid C \in \text{Col}_X(F)\}.$$ 

Under the Kronecker product $\langle \ , \ angle : H_3(X) \otimes H^3(X; A) \to A$, the cocycle invariant $\Phi_\theta(F)$ is the evaluation of $\Phi_X(F)$ by $[\theta]$;

$$\Phi_\theta(F) = \{\langle \gamma(C), \theta \rangle \in A \mid C \in \text{Col}_X(F)\}.$$ 

The deform-spun knot [8] is a 2-knot obtained from a tangle of a 1-knot with its motion. The spinning process is originally introduced by Artin [1], and generalized to twist-spinning by Fox [5] and Zeeman [11]. The quandle cocycle invariant of a twist-spun knot is calculated in some cases (cf. [2, 3, 6, 7]).

In this note, we introduce a 2-knot $F(K, K')$ associated with a tangle diagram $K$ and a 1-knot diagram $K'$. In particular, $F(K, K')$ is a roll-spun knot in the special case. Under some condition for a quandle $X$, we prove that there is a one-to-one correspondence between $\text{Col}_X(F(K, K'))$ and $\text{Col}_X(K)$; that is, each $X$-coloring $C$ for $K$ can be extended to that for $F(K, K')$ naturally which is denoted by $\overline{C}$. 
On the other hand, we define a normal subgroup $G_0(X)$ of the adjoint group $G(X)$ and a shifting map $S^w_2 : H_2(X) \to H_3(X)$ for every element $w \in G_0(X)$ so that we have
\[ \gamma(C) = S^w_2(\gamma(C)) \]
for any $C \in \text{Col}_X(K)$, where $w(C) \in G_0(X)$ and $\gamma(C) \in H_2(X)$ are the element of $G_0(X)$ and the second homology class associated with $C$. This implies that $\Phi_X(F(K, K'))$ is calculated in terms of $\text{Col}_X(K)$, and so is $\Phi_\theta(F(K, K'))$. As an application, we give a sufficient condition for $\Phi_X(F(K, K'))$ to be trivial.

2. Definition of $F(K, K')$

Let $K$ be an oriented tangle diagram and $K'$ an oriented knot diagram. We assume that $K'$ is located on a 2-sphere $S^2$ embedded in $\mathbb{R}^3$. We replace a tubular neighborhood of $K'$ in $S^2$ with a product $K \times S^1$, where the modification near a crossing of $K'$ is illustrated in Figure 1. This modification is realized by the connected sum of two copies of $K$ as cross-sections such that one of $K$'s passes through the other $K$.

![Figure 1](image_url)

We denote by $F(K, K')$ the 2-knot presented by this diagram. Let $\omega(K)$ and $\omega(K')$ denote the writhes of $K$ and $K'$, respectively. Then we have the following.

Lemma 2.1. If $\omega(K_0) = \omega(K_1)$ and $\omega(K'_0) = \omega(K'_1)$, then $F(K_0, K'_0) \cong F(K_1, K'_1)$.
Proposition 2.2. $F(K, K')$ is a deform-spun knot.

We denote by $\tau^r\rho^s K$ the $r$-twist-$s$-roll-spinning of $K$ (cf. [8]).

Theorem 2.3. If $K'$ is a diagram of the trivial knot, then $F(K, K') \cong \tau^{-rs}\rho^{-s}K$, where $r = \omega(K)$ and $s = \omega(K')$. In particular, if $K$ is a tangle diagram with $\omega(K) = 0$, then $F(K, K') \cong \rho^{-s}K$.

3. Definition of $G_0(X)$

For a quandle $X$ and an element $x \in X$, we denote by $\varphi_x : X \to X$ the right action by $x$; that is, $\varphi_x(a) = a \cdot x$. The axiom of the right distribution induces the equality

$$\varphi_y \circ \varphi_x = \varphi_{x \cdot y} \circ \varphi_y$$

for any $x, y \in X$. Let $W(X)$ denote the set of words on $X$. For a word $w = x_1^{\epsilon_1} \cdots x_n^{\epsilon_n} \in W(X)$, we define a quandle isomorphism $\varphi_w : X \to X$ to be

$$\varphi_w(a) = \varphi_{x_n}^{\epsilon_n} \circ \cdots \circ \varphi_{x_1}^{\epsilon_1}(a).$$

We also use the notation $\varphi_w(a) = a \cdot w$. We remark that $w$ in the definition of $\varphi_w$ can be regarded as an element of the adjoint group

$$G(X) = \langle x \in X \mid a \cdot b = b^{-1}ab \ (a, b \in X) \rangle.$$ 

The index of an element $w = x_1^{\epsilon_1} \cdots x_n^{\epsilon_n} \in G(X)$ is defined by $\operatorname{ind}(w) = \epsilon_1 + \cdots + \epsilon_n$.

**Definition 3.1.** $G_0(X) = \{w \in G(X) \mid \varphi_w = \text{id}_X \text{ and } \operatorname{ind}(w) = 0\}$.

We remark that $G_0(X)$ is a normal subgroup of $G(X)$.

**Lemma 3.2.** (i) $G_0(R_p) = \{0\}$, where $R_p = \mathbb{Z}[t, t^{-1}]/(p, t + 1)$ for odd prime $p$.

(ii) $G_0(S_4) = \mathbb{Z}_2$, where $S_4 = \mathbb{Z}[t, t^{-1}]/(2, t^2 + t + 1)$.

**Example 3.3.** We consider the case $X = S_4 = \mathbb{Z}[t, t^{-1}]/(2, t^2 + t + 1)$. The element $w = 1 \cdot t^{-1} \cdot 0 \cdot (t + 1)^{-1}$ satisfies $\varphi_w = \text{id}_{S_4}$; in fact, we have

$$
\begin{array}{cccc}
0 & \mapsto & t + 1 & \mapsto & \varphi_{t+1}^{-1} \\
1 & \mapsto & 1 & \mapsto & 0 \\
t & \mapsto & 0 & \mapsto & 1 \\
t + 1 & \mapsto & t & \mapsto & t + 1
\end{array}
$$

Since $\operatorname{ind}(w) = 0$, it holds that $w \in G_0(S_4)$. Figure 2 shows that $w^2 = 1$ in $G_0(S_4)$. Moreover, we see that $w$ is the generator of $G_0(S_4) \cong \mathbb{Z}_2$. 

4. Definition of $S_n^w : H_n(X) \rightarrow H_{n+1}(X)$

Let $C_n(X)$ denote the quandle $n$-chain group which is the free abelian group generated by the $n$-tuples $(a_1, \ldots, a_n) \in X^n$ with $a_i \neq a_{i+1}$ for any $1 \leq i \leq n-1$.

**Definition 4.1.** For an $n$-chain $\gamma = \sum \pm (a_1, \ldots, a_n) \in C_n(X)$, a word $w \in W(X)$, and an element $x \in X$, we denote by

$$(\gamma * w, x) = \sum \pm (a_1 * w, \ldots, a_n * w, x).$$

**Definition 4.2.** For a word $w = x_1^{\epsilon_1} \ldots x_k^{\epsilon_k} \in W(X)$, we define words $w(i)$ ($1 \leq i \leq k$) by

$$w(i) = \begin{cases} x_1^{\epsilon_1} \ldots x_{i-1}^{\epsilon_{i-1}} & (\epsilon_i = +1) \\ x_1^{\epsilon_1} \ldots x_{i-1}^{\epsilon_{i-1}} x_i^{\epsilon_i} & (\epsilon_i = -1) \end{cases}$$

**Definition 4.3.** For a word $w = x_1^{\epsilon_1} \ldots x_k^{\epsilon_k} \in W(X)$, the shifting map $S_n^w : C_n(X) \rightarrow C_{n+1}(X)$ is defined by

$$S_n^w(\gamma) = \sum_{i=1}^k \epsilon_i (\gamma * w(i), x_i).$$

**Lemma 4.4.** If $w \in G_0(X)$, then $S_n^w$ induces a shifting map $H_n(X) \rightarrow H_{n+1}(X)$.

**Example 4.5.** For the element $w = 1 \cdot t^{-1} \cdot 0 \cdot (t+1)^{-1} \in G_0(S_4)$, the shifting map $S_n^w : H_n(X) \rightarrow H_{n+1}(X)$ is given by

$$S_n^w(\gamma) = +(\gamma, 1) - (\gamma * (1 \cdot t^{-1}), t) + (\gamma * (1 \cdot t^{-1}), 0) - (\gamma, 1 + t).$$
See Figure 3.

![Figure 3](image)

**FIGURE 3**

5. Col$_X(F(K, K'))$ AND Col$_X(K)$

Let $K$ be a tangle diagram with $k$ crossings and $\omega(K) = 0$, and $C \in \text{Col}_X(K)$ an $X$-coloring for $K$. Let $\epsilon_i$ and $x_i$ ($1 \leq i \leq k$) be the sign and the color of the upper arc at $i$th lower crossing along $K$, respectively. The element of $G(X)$ associated with $C$ is given by

$$w(C) = x_1^{\epsilon_1} x_2^{\epsilon_2} \ldots x_k^{\epsilon_k}.$$ 

**Example 5.1.** We consider the $S_4$-coloring $C$ for the tangle diagram of the figure-eight knot as shown in Figure 4. Then the element associated with $C$ is given by $w(C) = 1 \cdot t^{-1} \cdot 0 \cdot (t + 1)^{-1}$.

![Figure 4](image)

**FIGURE 4**

**Lemma 5.2.** Let $a$ and $a'$ be the colors assigned to the initial and terminal arcs of $K$, respectively. Then it holds that $\varphi_w(a) = a'$.

We consider the following condition (#) for a quandle $X$;

(#) For any tangle diagram $K$ with $\omega(K) = 0$ and any $X$-coloring $C$ for $K$, it holds that $w(C) \in G_0(X)$. 

We remark that since $\omega(K) = 0$, we have $\text{ind}(w(C)) = 0$. Therefore, the condition $(\#)$ is equivalent to $\varphi_{w(C)} = \text{id}_X$.

**Proposition 5.3.** Any Alexander quandle satisfies the condition $(\#)$.

**Theorem 5.4.** Suppose that a quandle $X$ satisfies the condition $(\#)$. If $K$ is a tangle diagram with $\omega(K) = 0$, then there is a one-to-one correspondence between $\text{Col}_X(F(K, K'))$ and $\text{Col}_X(K)$.

6. **Computation of $\Phi_X(F(K, K'))$**

We consider the connected sum of two copies of a tangle diagram $K$ colored by $C \in \text{Col}_X(K)$. Let $\gamma \in H_3(X)$ be the class associated with the motion where the small tangle passes through the big one as shown in Figure 5. We divide $\gamma$ into $\gamma_+$ and $\gamma_- \in H_3(X)$ corresponding to the motions where the small tangle passes over and under the transverse arc, respectively.

![Figure 5](image_url)

The third homology class $\gamma_+$ is the sum of weights on the triple points as shown in Figure 6 which is equivalent to the shadow cocycle invariant of $K$.

**Lemma 6.1.** $\gamma_+ = 0$.

On the other hand, the third homology class $\gamma_-$ is the sum of triple points as shown in Figure 7. Let $\gamma(C) \in H_2(X)$ denote the class associated with the $X$-coloring $C$ for $K$. Then we have the following.
Lemma 6.2. $\gamma_{-} = S_{2}^{w(C)}(\gamma(C))$.

Theorem 6.3. Suppose that a quandle $X$ satisfies the condition $(\#)$. If $K$ is a tangle diagram with $\omega(K) = 0$, then it holds that

$$\Phi_{X}(F(K, K')) = \{-\omega(K') : S_{2}^{w(C)}(\gamma(C)) \mid C \in \text{Col}_{X}(K)\}$$

Corollary 6.4. Suppose that a quandle $X$ satisfies the condition $(\#)$ with $G_{0}(X) = 0$ or $H_{2}(X) = 0$. If $K$ is a tangle diagram with $\omega(K) = 0$, then $\Phi_{X}(F(K, K'))$ is trivial.

After the conference, Nosaka pointed out that $G_{0}(X)$ and $H_{2}(X)$ are isomorphic for any Alexander quandle (cf. [4]). Therefore, the conditions $G_{0}(X) = 0$ and $H_{2}(X) = 0$ are equivalent if $X$ is an Alexander quandle.

REFERENCES


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