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Kyoto University
\theta\text{-correspondence for PGSp}(4) \text{ and } \text{PGU}(2,2)

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Introduction

Let $\mathcal{H}_2 = \{ Z = ^tZ \in M_2(\mathbb{C}) \mid \Im(Z) > 0 \}$ be the Siegel upper half space of degree 2. Let

$$\theta_m(Z) = \sum_{x \in \mathcal{H}_2} \exp \left( 2\pi i \left( \frac{1}{2}(x + \frac{m'}{2})Z^t(x + \frac{m'}{2}) + (x + \frac{m'}{2})^t(\frac{m''}{2}) \right) \right)$$

be the Igusa theta constant with $m = (m', m'') \in \mathbb{Q}^2 \times \mathbb{Q}^2$. For a congruence subgroup $\Gamma$ of $Sp_2(\mathbb{Z})$ (cf. $SL_4(\mathbb{Z})$), let $S_3(\Gamma)$ denote the space of Siegel modular cusp forms of weight 3 with respect to $\Gamma$, and let $S_{\Gamma}$ the Siegel modular 3-fold associated to $\Gamma$. van Geemen and van Straten showed that $S_3(\Gamma_2(4,8))$ is spanned by certain 6-tuple products $\prod_{j=1}^{6}\theta_{m_j}(Z)$ with $m_j \in \{0,1\}^4$ using the theta embedding of $S_{\Gamma(4,8)}$ into $\mathbb{P}^{13}$ (cf. [3]), where

$$\Gamma(4,8) = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma(4) \mid \text{diag}(B) \equiv \text{diag}(C) \equiv 0 \pmod{8} \right\}.$$

(0.1)

Through Igusa's transformation formula, $Sp_4(\mathbb{Z})$ acts on these 6-tuple products. They showed that $S_3(\Gamma(4,8))$ is decomposed into seven irreducible $Sp_4(\mathbb{Z})$-modules, and each module is generated by acting $Sp_4(\mathbb{Z})$ a 6-tuple product of Igusa theta constants:

$$S_3(\Gamma(4,8)) = \sum_{i=1}^{7} Sp_2(\mathbb{Z}) \cdot f_i$$

(0.2)

where $\cdot$ indicates the standard action of the elements of $Sp_2(\mathbb{R})$ to the Siegel modular forms of weight 3. Further, they showed that each 6-tuple products $f_i$ lie in irreducible cuspidal automorphic representations $\pi_{f_i}$ of PGSp$_4(\mathbb{A})$. Calculating some eigenvalues for Hecke operators on

$$f_\tau(Z) := \theta_{(0,0,0,0)}(Z)^2\theta_{(1,0,0,0)}(Z)\theta_{(0,1,0,0)}(Z)\theta_{(0,0,1,1)}(Z)\theta_{(0,0,0,1)}(Z),$$

they gave the following conjecture:

Conjecture (van Geemen and van Straten [2]). Let $\rho$ be the unique elliptic cusp form of weight 3 of level 32 with central character $\chi_{-4}$. Let $\mu$ be the größencharacter of $\mathbb{Q}(i)$ associated to the CM-elliptic curve $y^2 = x^3 - x$, and $\pi(\mu)$ be the CM-elliptic cusp form of weight 2 of level 32. Then, the irreducible cuspidal automorphic representation $\pi_{f_7}$ has the partial spinor L-function (of degree 4) $L(s, \rho \otimes \pi(\mu))$ outside of 2.

Here $\chi_{-4}$ indicates the quadratic character related to the extension $\mathbb{Q}(i)/\mathbb{Q}$. We will give a sketch of a proof of this conjecture.
Let $K = \mathbb{Q}(i)$. Let $\text{Gal}(K/\mathbb{Q}) = \{1, c\}$. Let $J = \begin{bmatrix} 0 & -1_2 \\ 1_2 & 0 \end{bmatrix}$ and

$$\text{GU}_{2,2}(K) = \{ g \in \text{GL}_4(K) \mid {}^t g^c J g = \nu(g) J, \nu(g) \in \mathbb{Q}^x \}.$$

We define the 6-dimensional quadratic space over $\mathbb{Q}$

$$X(\mathbb{Q}) = \left\{ x = \begin{bmatrix} 0 & u & a & d \\ -u & 0 & b & -a^c \\ -a & -b & 0 & -v \\ -d & a^c & v & 0 \end{bmatrix} \mid b, d, u, v \in \mathbb{Q}, a \in K \right\}$$

with norm form $(x, x) = (bd + uv + a\overline{a})$. Let

$$e_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, e_{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$e_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, e_{-2} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, e = iJ.$$

We define a right action $\rho$ of $\text{GU}_{2,2}(K)$ on $X(\mathbb{Q})$ by

$$\rho(h)x = h^{-1} \cdot x \cdot {}^t h^{-1}.$$

Via $\rho$, we have the isomorphism

$$\text{PGU}_{2,2}(K) \simeq \text{PGSO}_X(\mathbb{Q}).$$

We will denote by $F$ a $v$-adic completion of $\mathbb{Q}$. Let $Y(F)$ be the 4-dimensional symplectic space with symplectic form $\langle, \rangle$. Let $\{\epsilon_{+1}, \epsilon_{-1}, \epsilon_{+2}, \epsilon_{-2}\}$ be the standard basis of $Y(F)$ ($\langle \epsilon_{+i}, \epsilon_{-j}\rangle = \delta_{ij}$, $\langle \epsilon_{+i}, \epsilon_{+j}\rangle = 0$). Let $\text{Sp}_2(F)$ act from the right on $Y(F)$. We will use the two polarizations

$$Z = Y \otimes X = Z^+ + Z^-$$

$$= Z'^+ + Z'^-$$

with

$$Z^\pm = Y \otimes (Fe_{\pm 1} + Fe_{\pm 2}) + (Fe_{\pm 1} + Fe_{\pm 2}) \otimes (Fe + Fe')$$

$$Z'^\pm = (Fe_{\pm 1} + Fe_{\pm 2}) \otimes X(F).$$

We realize the Weil representation $r_\psi, r'_\psi$ of $\text{Sp}(Z)$ in $S(Z^+(F)), S(Z'^+(F))$, respectively. Put

$$R(F) = \{(g, h) \in \text{GSp}_2(F) \times \text{GU}_{2,2}(K_v) \mid \nu(g) = \nu(h)\}$$
where \( \nu \) indicates the similitude norm. We embed \( \mathcal{R}(F) \) into \( \text{Sp}(Z(F)) \) through the action \( z \rightarrow \varrho(h^{-1})zg \), and denote

\[
\theta_{\psi}(g, h)\phi(z) = r_{\psi}(g, \varrho(h^{-1}))\phi(z),
\]
\[
r'_{\psi}(g, h)\phi(z) = r'_{\psi}(g, \varrho(h^{-1}))\phi(z).
\]

Let \( \psi \) be a nontrivial additive character of \( \mathbb{Q}\backslash A \) and \( r_{\psi} = \bigotimes_{v} r_{\psi_{v}}, r'_{\psi} = \bigotimes_{v} r'_{\psi_{v}} \).

Let \( \tau \) be an automorphic form on \( \text{PGSp}_{2}(\mathbb{A}) \). For \( \phi \in \mathcal{S}(Z^{+}(\mathbb{A})) \) and \( h \in \text{GU}_{2,2}(K_{A}) \), we define

\[
\theta_{\psi}(\phi, \tau)(h) = \int_{\text{Sp}_{2}(\mathbb{Q})\backslash\text{Sp}_{2}(\mathbb{A})} \sum_{z \in Z^{+}(\mathbb{Q})} r_{\psi^{-1}}(g_{1}g, h)\phi(z)\tau(g_{1}g)dg_{1},
\]

where \( g \in \text{GSp}_{2}(\mathbb{A}) \) is taken so that \( (g, h) \in \mathcal{R}(A) \). Then, the value \( \theta_{\psi}(\phi, \tau)(h) \) does not depend on the choice of \( g \), and \( \theta_{\psi}(\phi, \tau) \) is an automorphic form on \( \text{GU}_{2,2}(K_{A}) \). If \( \tau \) has the trivial central character, then so does \( \theta_{\psi}(\phi, \tau) \). For an irreducible cuspidal automorphic representation \( \pi \) of \( \text{PGSp}_{2}(\mathbb{A}) \), we define \( \Theta_{\psi}(\pi) \) the space of these automorphic forms on \( \text{PGU}_{2,2}(K_{A}) \) obtained from all \( \tau \in \pi \) and all \( \phi \in \mathcal{S}(Z^{+}(\mathbb{A})) \). For an irreducible cuspidal automorphic representation \( \sigma \) of \( \text{GU}_{2,2}(K_{A}) \), we define \( \Theta_{\psi}(\sigma) \) the space of these automorphic forms on \( \text{PGSp}_{2}(\mathbb{A}) \) using \( \mathcal{S}(Z^{+}(\mathbb{A})) \) and \( r'_{\psi} \), similarly.

For \( s, x, y, z \in \mathbb{Q} \), let

\[
n(s, x, y, z) = \begin{bmatrix}1 & s & 1 & 1 \\ 1 & x & y & 1 \\ -s & 1 & z & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \in \text{GSp}_{2}(\mathbb{Q}).
\]

Let \( N(\mathbb{Q}) = \{n(s, x, y, z) \mid s, x, y, z \in \mathbb{Q}\} \). On \( N(\mathbb{Q}) \), for a nontrivial additive character \( \psi \), we define \( \psi_{N}(n(s, x, y, z)) = \psi(s + z) \), and the Whittaker function \( W_{\psi}(f; g) \) of an automorphic form \( f \) with respect to \( \psi \) by

\[
W_{\psi}(f; g) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \psi_{N}(n)f(ng)dn.
\]

We say \( \pi \) is globally generic, if there is an \( f \in \mathcal{R} \) having a nontrivial Whittaker function with respect to some nontrivial \( \psi \). For \( x, z \in \mathbb{Q}, s, y \in K \), let

\[
n_{K}(s, x, y, z) = \begin{bmatrix}1 & s & 1 & 1 \\ 1 & x & y & 1 \\ -s & 1 & z & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \in \text{GU}_{2,2}(K).
\]

Let \( N_{K}(K) = \{n_{K}(s, x, y, z) \mid x, z \in \mathbb{Q}, s, y \in K\} \) and, for a nontrivial \( \psi \) on \( \mathbb{Q} \), we define \( \psi_{N_{K}}(n_{K}(s, x, y, z)) = \psi(\text{Re}(s) + z) \). We define Whittaker functions of automorphic forms on \( \text{GU}_{2,2}(K_{A}) \), and globally generic representation, similarly. Further, we define \( \psi'_{N_{K}}(n_{K}(0, x, y, z)) = \psi(\text{Im}(y)) \) on the subgroup composed of \( n_{K}(0, x, y, z) \). For an automorphic form \( f \) on \( \text{GU}_{2,2}(K_{A}) \), the Shalike model of \( f \) with respect to \( \psi \) is defined by

\[
\int_{N'_{K}(K)\backslash N'_{K}(K_{A})} \psi'_{N_{K}}(n)f(ng)dn.
\]

First, we recall Ranakrishnan, Shahidi's result in [21].
**Proposition 1.1** (Ranakrishnan-Shahidi). Let $\rho, \mu$ be as in the conjecture. Then, there is an irreducible, globally, generic, cuspidal, automorphic representation $\pi^{gn}$ such that

$$L_S(s, \pi^{gn}; \text{spin}) := \prod_{v \notin S} L(s, \pi_v^{gn}; \text{spin}) = L_S(s, \rho \otimes \pi(\mu)).$$

We start an argument from this $\pi^{gn}$.

**Proposition 1.2.** Let $\psi_0$ be the standard additive character on $\mathbb{Q} \backslash A$. If $\pi$ is an irreducible, globally generic, cuspidal representation of $\text{PGSp}_2(A)$, then $\Theta_{\psi_0}(\pi)$ is nontrivial and a globally generic representation.

**Proof.** Through a computation similar to used in the proof of Proposition 2.2 of Piatetski-Shapiro, Soudry [16], we get

$$\int_{N(\mathbb{Q}) \backslash N(A)} \psi_0(s) \theta_{\psi_0}(\phi, f) \phi(n(s)h) ds$$

$$= \int_{N(A) \backslash \text{Sp}_2(A)} r_{\psi_0}(g, h) \phi(\epsilon_{-1} \otimes e_1 + \epsilon_{-2} \otimes e_2 - \epsilon_+ \otimes e) W_{\psi_0}(f; g) dg.$$

It is possible to choose $\phi$ so that the right hand side of (1.1) is not zero at $h = 1$ if $W_\psi(f; 1) \neq 0$. Thus the assertion. \hfill \Box

Let $\sigma$ be an irreducible constituent of the above nontrivial $\Theta_{\psi_0}(\pi)$. Thanks to the next result due to Furusawa and Morimoto announced in the last year,

**Theorem 1.3.** An irreducible, globally generic, cuspidal automorphic representation $\Pi$ of $\text{PGU}_{2,2}(K_A)$ has a Shalike model, if and only if $L_S(s, \Pi; \Lambda_t^{2}) = 1$ (a partial $L$-function of $\Pi$ with respect to outer exterior representation $\Lambda_t^{2}$ (c.f. [9])) has a simple pole at $s = 1$.

and the observation that, if an irreducible cuspidal automorphic representation $\pi$ of $\text{PGSp}_2(A)$ has a partial spinor $L$-function $L_S(s, \rho \otimes \pi(\mu))$ for some finite set $S$ of places, then

$$L_S(s, \pi; \Lambda_t^{2}) = \zeta(s)L_S(s, \pi, \chi_{-4}; r_5)$$

$$= \zeta(s)L_S(s, \rho, \chi_{-4}; \text{sym}_2)L_S(s, \mu^2)$$

has a simple pole at $s = 1$, we find that $\sigma$ has a Shalike model, where $L_S(s, \pi, \chi_{-4}; r_5)$ indicates the $L$-function of $\pi$ of degree 5 twisted by the quadratic character $\chi_{-4}$.

Further,

**Proposition 1.4.** An irreducible, globally generic, cuspidal automorphic representation $\Pi$ of $\text{PGU}_{2,2}(K_A)$ has a nontrivial $\theta$-lift $\Theta_{\psi_0}(\Pi)$ to $\text{PGSp}_2(A)$, if and only if $\Pi$ has a Shalike model with respect to $\psi_0$.

**Proof.** Let $\tau$ be an automorphic form of $\Pi$, and $B_{\psi_0}(\tau; *)$ the Shalike model of $\tau$. Then, the Whittaker function of $F = \theta_{\psi_0}(\varphi, \tau)$ with respect to $\psi_0$ is

$$\int_{N_K(\mathbb{Q}) \backslash \text{SU}_{2,2}(K_A)} r_{\psi_0}(g, h) \varphi(\epsilon_1 \otimes e_{+1} + \epsilon_2 \otimes e_{-2}) B_{\psi_0}(h) dh.$$ \hfill (1.1)

It is possible to choose $\varphi$ so that this function of $g$ is nontrivial. Hence the assertion. \hfill \Box
Therefore, we conclude that an irreducible, globally generic, cuspidal, automorphic representation $\pi^{\text{gen}}$ of $\mathrm{PGSp}_2(A)$ with $L_S(\pi;\text{spin}) = L_S(\rho \otimes \pi(\mu))$ can come back through these $\theta$-lifts $\Theta_{\psi_0}, \Theta'_{\psi_0}$.

Now then, we will observe the levels of these automorphic representations. First, $\pi^{\text{gen}}$ has the spinor $L$-function, from the functional equation of the $L$-function and the result of Roberts-Schmidt [22], one can estimate the parameter level of $\pi^{\text{gen}}$ divides $2^{10}$. More precisely, $\pi^{\text{gen}}$ has a right $K^{\text{param}}(2^{10})$ (paramodular group) invariant Whittaker function $W_{\psi_0}$ such that $W_{\psi_0}(1) \neq 0$. Let $\mathcal{O}_K$ be the ring of integers of $K$ and

$$\Gamma_0(2^5)^K = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{GU}_{2,2}(\mathcal{O}_K) \mid C \equiv 0 \pmod{2^5\mathcal{O}_K} \right\}.$$  

Setting a $\phi$ suitably in (1.1), we can construct a right $\Gamma_0(2^5)^K$-invariant Shalike model $B_{\psi_0}$ of $\Theta_{\psi_0}(\pi^{\text{gen}})$ such that $B_{\psi_0}(1) \neq 0$. Setting a $\varphi$ suitably in (1.1), we can construct a right $\Gamma(4,8)_2$-invariant Whittaker model of $\Theta'_{\psi_0}(\Theta_{\psi_0}(\pi^{\text{gen}}))$. Thus, by the strong multiplicity one theorem for globally generic representation of $\mathrm{GSp}_2(A)$, due to Soudry [23], $\pi^{\text{gen}}$ has a right $\Gamma(4,8)_2$-invariant vector. One can deduce the following from Weissauer's result

**Proposition 1.5** (Proposition 1.5 of [25]). *If an irreducible globally generic cuspidal automorphic representation $\pi$ of $\mathrm{GSp}_2(A)$ has a cohomological weight, then there is an irreducible cuspidal automorphic representation $\pi^{\text{hol}}$ such that*

- $\pi^{\text{hol}} \simeq \pi_v$ for all nonarchimedean places $v$.
- $\pi^{\text{hol}}_\infty$ is a holomorphic discrete series with a cohomological weight.

**Remark 1.** Ramakrishnan, Shahidi [21] showed the existence of some holomorphic Siegel modular cusp forms of degree 2 with interesting spinor $L$-functions, using this result.

Applying this, and looking the $\Gamma$-factor of $L(s, \rho \otimes \pi(\mu)) = L(s, B\mathcal{C}_K(\rho) \otimes \mu)$ ($B\mathcal{C}_K(\rho)$ indicates the base change lift of $\rho$ to $\mathrm{GL}_2(K_\Lambda)$), one finds that there is an eigenform $F \in S_3(\Gamma(4,8))$ such that $L(s, F;\text{spin}) = L(s, \rho \otimes \pi(\mu))$. In [15], as conjectured by van Geemen, van Straten [2], we showed that all irreducible cuspidal automorphic representation $\pi_f$, ($1 \leq i \leq 6$) have different spinor $L$-functions from $L(s, \rho \otimes \pi(\mu))$. Hence, the conjecture is true.

**References**


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