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Differential equations satisfied by principal series Whittaker functions on $SU(2, 2)$

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Abstract

In this talk, we discuss about a holonomic system of differential equations for Whittaker functions associated with the principal series representation of $SU(2; 2)$ with higher dimensional minimal $K$-type.

1 Introduction

Throughout, let $G$ be the simple real Lie group $SU(2, 2)$ of rank two, and let $K = S(U(2) \times U(2))$ : the maximal compact subgroup of $G$ 
\begin{equation}
\pi : \text{ an irreducible representation of } G \text{ which is } K\text{-admissible.}
\end{equation}

For the representation $\pi$, there are two types of Whittaker model with respect to a character $\eta$ of $N$ (a spherical subgroup of $G$). One is the smooth model, and the other is the algebraic models induced by the space of algebraic Whittaker vectors:
\begin{equation}
W(\pi, \eta) := \text{Hom}_{(g, K)}(\pi |_{K}, C^{\infty}\text{-Ind}_{N}^{G}(\eta)),
\end{equation}

Here, $g$ is the Lie algebra of $G$, $\pi |_{K}$ is the subspace of $K$-finite vectors in $\pi$ and $C^{\infty}\text{-Ind}_{N}^{G}(\eta)$ is the right $G$-module smoothly induced from $\eta$.

Our aim is a characterization of the space $W(\pi, \eta)$ for the principal series representation $\pi$ of $G$ associated with a minimal parabolic subgroup, which leads to a description of the following challenging question associated to $\pi$.

Question. For each intertwiner $\Phi$ in $W(\pi, \eta)$, what is the image of $\Phi$ ? Equivalently, for each $K$-type $\tau$ occurring in $\pi$, one can ask the image of the $\tau$-isotypic component in $\pi$. The latter image is called the space of Whittaker functions of $\pi$ with respect to $\tau$.

The natural and classical approach. Let $\tau$ be a $K$-type belonging to $\pi$, and $f_{1}, ..., f_{n}$ be its a basis in $\pi$. Denote by $\phi_{j}(g)$ the image of $f_{j}$ under a fixed intertwiner $\Phi$. Then, for each $j$ and $k$ in $K$, the function $(k\phi_{j})(g) = \phi_{j}(gk)$ is a linear combination of the functions $\phi_{1}(g), ..., \phi_{n}(g)$. Thus, it is enough to consider the functions $\phi_{j}$ on $A$ for our purpose.
Assume that $C$ is a square matrix of size $\dim(\tau)$, with entries in the universal enveloping algebra of $g$, so that

$$\pi(C) \circ \left( \begin{array}{l} f_1 \\ f_2 \\ \vdots \\ f_n \end{array} \right) = \gamma \cdot \left( \begin{array}{l} f_1 \\ f_2 \\ \vdots \\ f_n \end{array} \right),$$

(1)

for some constant $\gamma \in \mathbb{C}$.

By applying $\Phi$ to the identity (1) we get the following system of differential equations (the $A$-radial part)

$$\mathcal{R}(C) \circ \left( \begin{array}{l} \phi_1(a) \\ \phi_2(a) \\ \vdots \\ \phi_n(a) \end{array} \right) = \gamma \cdot \left( \begin{array}{l} \phi_1(a) \\ \phi_2(a) \\ \vdots \\ \phi_n(a) \end{array} \right), \quad a \in A$$

where $\mathcal{R}$ denotes the infinitesimal action of $G$ on $C^\infty$-Ind$_{N}^{G}(\eta)$. Thus, one can regard the space $W(\pi, \eta)$ as a subset of the solution space $Sol(\mathcal{R}(C))$ of the system by sending $\Phi$ to the functions $\{\phi_1(a)\}$.

**Remark.** Recall that Whittaker functions satisfy differential equations with regular singularities at "0". The most important requirements for choosing a basis for $\tau$ are the simplicity and symmetricity of the series expansion of these functions $\phi_j(a)(a \in A)$ around 0 and of the system of differential equations arising from (1).

**Principal series $\pi_{s,\chi}$.** Let

$$a = \{a(t_1, t_2) = \left( \begin{array}{ccc} 0 & 0 & t_1 \\ 0 & 0 & 0 \\ t_1 & 0 & 0 \\ 0 & t_2 & 0 \end{array} \right) \mid t_1, t_2 \in \mathbb{R} \} \subset g,$$

$$M = \{\text{diag}(e^{i\theta}, e^{-i\theta}, e^{i\theta}, e^{-i\theta})\} \oplus \{1_{4}, \text{diag}(1, -1, 1, -1)\}.$$

Define linear functions $\lambda_1$ and $\lambda_2$ on $a$ by $\lambda_1(a(t_1, t_2)) = t_1$ and $\lambda_2(a(t_1, t_2)) = t_2$. Then the set $\{\pm \lambda_1 \pm \lambda_2, \pm 2\lambda_1, \pm 2\lambda_2\}$ forms the restricted root system of type $C_2$ for the pair $(g, a)$. Define $\lambda_1 \pm \lambda_2, 2\lambda_1$ and $2\lambda_2$ to be positive. Let $P_{\min}$ be the minimal parabolic subgroup of $G$ with Langlands decomposition $P_{\min} = MAN$, where $N$ is the unipotent subgroup defined by the root spaces corresponding to positive roots. For the character $s \otimes \chi$ of $M$, $s \in \mathbb{Z}$, and the $C$-valued real linear form $\mu = \mu_1 \lambda + \mu_2 \lambda_2$, one has the principal series representation

$$\pi_{s,\chi} := \text{Ind}_{\text{C}}^{G}(\langle s \otimes \chi \rangle_M \otimes e^{\mu+\rho} \otimes 1_N),$$

where $1_N$ is the trivial character of $N$.

The main object in the paper is the 8-dimensional space $W(\pi_{s,\chi}, \eta)$ of algebraic Whittaker vectors (see Kostant [2]) for non-degenerate character $\eta$ (unitary) of $N$. Note that it is sufficient for our purpose to assume that $s \geq 0$. 
1.1 Some previous results

Let us recall some known identities as in (1) and previous results for the space $W(\pi, \eta)$. The first example is the classical Casimir equation: let $\Omega$ be the Casimir operator of $G$. Then we have the following identity

$$\pi_{s,x}(\Omega)v = \chi_{\pi_{s,x}}(\Omega)v,$$

where $\chi_{\pi_{s,x}}$ is the infinitesimal character and $v$ is a differential vector. This identity gives us an injection of $W(\pi_{s,x}, \eta)$ into the solution space $\text{Sol}(\mathcal{R}(\Omega))$ of the above equation. Note that the space $\text{Sol}(\mathcal{R}(\Omega))$ is of infinite dimension.

Let $\pi$ be a discrete series representation of $G$ and $\tau$ be its minimal $K$-type. Then Yamashita [10] defined an operator $D_{\pi,\tau}$ on $\tau$ under $\pi$:

$$\pi(D_{\pi,\tau})\tau = 0.$$

This gives us an injection of $W(\pi, \eta)$ into the solution space $\text{Sol}(\mathcal{R}(D_{\pi,\tau}))$ of the operator $\mathcal{R}(D_{\pi,\tau})$. Moreover, under certain conditions, he showed that

$$W(\pi, \eta) \cong \text{Sol}(\mathcal{R}(D_{\pi,\tau}))$$

as vector spaces. This result is not just for the group $G$ (see [10] and [11]).

Let $\pi$ be the principal series representation of $G = Sp(2, \mathbb{R})$ as in [6], and $\tau$ be the minimal $K$-type of $\pi$. In [6], the authors obtained a matrix, of size $\dim(\tau)$, formula of the form $\pi(D)v = \gamma v$ which implies

$$W(\pi, \eta) \cong \text{Sol}(\mathcal{R}(\Omega), \mathcal{R}(\mathcal{D})), $$

where $\Omega$ stands for the Casimir operator of $Sp(2, \mathbb{R})$. Note that the possible value of $\dim(\tau)$ is 1 or 2. The degree of $D$ is 4 if $\dim(\tau) = 1$, and 2 for the case of dimension 2.

Remark. In the case $s = 0$ and $s = 1$, the corresponding spaces $W(\pi_{s,x}, \eta)$ behave quite similar to the above mentioned cases for $G = Sp(2, \mathbb{R})$, and are studied in [4].

2 Differential equations

We begin by providing some formulas for the multiplicity one $K$-types $\tau_{[0,s;l]}$ in the principal series $\pi_{s,x}$. These formulas come from the explicit $(\mathfrak{g}, K)$-module structure of $\pi_{s,x}$ which originally discussed by Oda [7].

Note that the space of the adjoint $K$-representation $(Ad, p_C)$ is generated by the matrix units $E_{ij}+2$ and $E_{i+2j}$ $(1 \leq i, j \leq 2)$ and denote by $E_{ij}+2$ and $E_{i+2j}$ their infinitesimal actions with respect to $\pi_s$. Let denote $F_{[s;l]}$ the transpose of the vector $(f_0, f_1, ..., f_s)$, where $\{f_j : 0 \leq j \leq s\}$ is the "nice" basis of $\tau_{[0,s;l]}$ introduced in [1] and $c_q := q/s$ for $0 \leq q \leq s$.

**Formula 1. (Casimir equation)** Let $\Omega$ be the Casimir operator. Then we have

$$\pi_{s,x}(\Omega) \cdot F_{[s;l]} = (\mu_1^2 + \mu_2^2 + \frac{1}{2} s^2 - 10)F_{[s;l]}.$$
Formula 2. (Shift equations) Set \( \nu_1 = \frac{1}{2}(s + l) \) and \( \nu_2 = \frac{1}{2}(s - l) \). Then we have

\[
\pi_{s, \chi}(\bar{Q}) \cdot F_{[s:l]} = \frac{1}{4}(\mu_1^2 - (\nu_1 + 1)^2)F_{[s:l]},
\]

and

\[
\pi_{s, \chi}(Q) \cdot F_{[s:l]} = \frac{1}{4}(\mu_2^2 - (\nu_2 - 1)^2)F_{[s:l]},
\]

where \( \bar{Q} = \{\bar{Q}_{ij}\}_{0 \leq i, j \leq s} \) and \( Q = \{Q_{ij}\}_{0 \leq i, j \leq s} \) are square matrices given by

\[
\bar{Q}_{qq-1} = -c_q(\mathcal{E}_{24}\mathcal{E}_{32} + \mathcal{E}_{14}\mathcal{E}_{31})
\]
\[
\bar{Q}_{qq+1} = -(1-c_q)(\mathcal{E}_{23}\mathcal{E}_{42} + \mathcal{E}_{13}\mathcal{E}_{41})
\]
\[
\bar{Q}_{qq} = (1-c_q)(\mathcal{E}_{23}\mathcal{E}_{32} + \mathcal{E}_{13}\mathcal{E}_{31}) + c_q(\mathcal{E}_{14}\mathcal{E}_{41} + \mathcal{E}_{24}\mathcal{E}_{42})
\]

and

\[
Q_{qq-1} = c_q(\mathcal{E}_{32}\mathcal{E}_{24} + \mathcal{E}_{31}\mathcal{E}_{14})
\]
\[
Q_{qq+1} = (1-c_q)(\mathcal{E}_{42}\mathcal{E}_{23} + \mathcal{E}_{41}\mathcal{E}_{13})
\]
\[
Q_{qq} = c_q(\mathcal{E}_{32}\mathcal{E}_{23} + \mathcal{E}_{31}\mathcal{E}_{13}) + (1-c_q)(\mathcal{E}_{41}\mathcal{E}_{14} + \mathcal{E}_{42}\mathcal{E}_{24})
\]

for \( 0 \leq q \leq s \), but all other entries are 0.

Formula 3. (Annihilation equations) We have

\[
\pi_{s, \chi}(A) \cdot F_{[s:l]} = 0,
\]

and

\[
\pi_{s, \chi}(\bar{A}) \cdot F_{[s:l]} = 0,
\]

where \( A = \{A_{ij}\} \) and \( \bar{A} = \{\bar{A}_{ij}\} \) are square matrices whose non-zero entries are given by

\[
A_{jj-1} = -\mathcal{E}_{31}\mathcal{E}_{14} - \mathcal{E}_{32}\mathcal{E}_{24},
\]
\[
A_{jj} = \mathcal{E}_{41}\mathcal{E}_{14} + \mathcal{E}_{42}\mathcal{E}_{24} - \mathcal{E}_{31}\mathcal{E}_{13} - \mathcal{E}_{32}\mathcal{E}_{23},
\]
\[
A_{jj+1} = \mathcal{E}_{41}\mathcal{E}_{13} + \mathcal{E}_{42}\mathcal{E}_{23},
\]

and

\[
\bar{A}_{jj-1} = -\mathcal{E}_{14}\mathcal{E}_{31} - \mathcal{E}_{24}\mathcal{E}_{32},
\]
\[
\bar{A}_{jj} = \mathcal{E}_{14}\mathcal{E}_{41} + \mathcal{E}_{24}\mathcal{E}_{42} - \mathcal{E}_{13}\mathcal{E}_{31} - \mathcal{E}_{23}\mathcal{E}_{32},
\]
\[
\bar{A}_{jj+1} = \mathcal{E}_{13}\mathcal{E}_{41} + \mathcal{E}_{23}\mathcal{E}_{42},
\]

for \( 1 \leq j \leq s - 1 \).

Proposition 2.1. On the K-type \( \tau_{[0,s;l]} \) with respect to the action \( \pi_{s, \chi} \) we have

\[ Q + \bar{Q} = \Omega/4. \]
2.1 A holonomic system of rank 8

**Coordinate system.** Since the $\mathbb{R}$-split torus $A$ for our case is two dimensional, one may choose the coordinate system $(y_1, y_2)$. Denote the Euler operators $y_1 \frac{\partial}{\partial y_1}$ and $y_2 \frac{\partial}{\partial y_2}$ with respect to this system by $\partial_1$ and $\partial_2$, respectively.

We now define the matrix differential operator $\overline{D}$ by

$$
\begin{pmatrix}
\overline{d}_{00} & \overline{d}_{01} & 0 & \cdots & 0 \\
0 & \overline{d}_{11} & \overline{d}_{12} & \cdots & 0 \\
0 & 0 & \overline{d}_{22} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \overline{d}_{s-2s-2} \\
0 & 0 & 0 & \cdots & \overline{d}_{s-1s-1} \\
0 & 0 & 0 & \cdots & \overline{d}_{ss-1} \\
0 & 0 & 0 & \cdots & \overline{d}_{ss}
\end{pmatrix}
$$

where

$$
d_{qq} = \frac{1}{4}((\partial_1 - q)^2 - \mu_1^2) - \xi \overline{\xi} y_1^2, \quad d_{qq+1} = \xi y_1 (\partial_2 + \frac{1}{2} s - q) + \xi y_1 y_2
$$

for $q = 0, \ldots, s - 1$ and

$$
d_{ss} = \frac{1}{4}((\partial_1 - 2\partial_2)^2 - \mu_2^2) - \xi \overline{\xi} y_1^2 - y_2^2 - \nu_1 y_2
$$

$$
d_{ss-1} = -\xi y_1 (\partial_2 + \frac{1}{2} s) + \xi y_1 y_2.
$$

We also define the matrix differential operator $D$ by

$$
\begin{pmatrix}
d_{00} & d_{01} & 0 & \cdots & 0 \\
d_{10} & d_{11} & 0 & \cdots & 0 \\
0 & a_{32} & d_{33} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & d_{s-2s-2} \\
0 & 0 & 0 & \cdots & d_{s-1s-1} \\
0 & 0 & 0 & \cdots & d_{ss-1} \\
0 & 0 & 0 & \cdots & d_{ss}
\end{pmatrix}
$$

where

$$
d_{00} = \frac{1}{4}((\partial_1 - 2\partial_2)^2 - \mu_2^2) - \xi \overline{\xi} y_1^2 - y_2^2 - \nu_2 y_2
$$

$$
d_{01} = -\xi y_1 (\partial_2 - \frac{1}{2} s) - \xi y_1 y_2
$$

and

$$
d_{qq} = \frac{1}{4}((\partial_1 - s + q)^2 - \mu_2^2) - \xi \overline{\xi} y_1^2, \quad d_{qq-1} = \xi y_1 (\partial_2 + q - \frac{1}{2} s) - \xi y_1 y_2
$$

for $q = 1, \ldots, s$. Here, the parameters $\xi$ and $\overline{\xi}$ are associated to the character $\eta$. 
By using Formulas 2 and 3, one can see that the Whittaker functions of π_{s,\chi} with respect to \tau_{[0,s;l]} satisfy the system of differential equations \mathcal{D} = 0 and \overline{D} = 0. Moreover, we have the following result which characterizes the Whittaker functions of π_{s,\chi} with respect to \tau_{[0,s;l]}.

**Theorem 2.2.** For s \geq 2, the natural map from W(π_{s,\chi}, \eta) into Ker(\overline{D}, D) is bijection if π_{s,\chi} is irreducible and \eta is a nondegenerate unitary character of N.

Here, we also have the following formula in the case s = 0, which is analogue to the class one case for \text{Sp}(2, \mathbb{R}) in [5]. Write W for the little Weyl group for (g, a), and (\rho_1, \rho_2) for the pair (3, 2) related to the half sum.

**Theorem 2.3.** Let π_{0,\chi} be an irreducible principal series with parameter \mu = (\mu_1, \mu_2) \in a_\mathbb{C}^*, and set \epsilon = \frac{1-\chi(-1)}{2}. Then the function \phi_{\mu} on A defined by

\[ \phi_{\mu}(y_1, y_2) = y_1^{\rho_1} y_2^{\rho_2} \sum_{m,n \geq 0} \frac{U_{m,n}^0}{2^{2n}(\frac{\mu_1-\epsilon}{2}+1)(\frac{\mu_2-\epsilon}{2}+1)n} \times y_1^{\mu_1+2m} y_2^{\mu_2+2n} 
+ \frac{\epsilon U_{m,n}^1}{2^{2n+1}(\frac{\mu_1-\epsilon}{2}+1)(\frac{\mu_2-\epsilon}{2}+1)n+1} \times y_1^{\mu_1+2m} y_2^{\mu_2+2n+1} \]

is a Whittaker function, on A, of π_{0,\chi} with the K-type \tau_{[0,0;2\epsilon]}. Moreover, the intertwiners \Phi_{\omega(\mu)} attached to the function \phi_{\omega(\mu)}(y_1, y_2) form a basis of the 8-dimensional space W(π_{0,\chi}, \eta). Here,

\[ U_{m,n}^t := \sum_{j=0}^{\min(m,n)} \frac{(\frac{\mu_1-\epsilon}{2} + n + 1 + t)_{m-j}}{(m-j)! (n-j)! j!(\frac{\mu_1+\mu_2}{2}+1)} \times (\frac{\mu_1-\mu_2}{2}+1)_{m-j} \]

for t = 0, 1.

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**References**


