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Differential equations satisfied by principal series Whittaker functions on $SU(2, 2)$

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Abstract

In this talk, we discuss about a holonomic system of differential equations for Whittaker functions associated with the principal series representation of $SU(2; 2)$ with higher dimensional minimal $K$-type.

1 Introduction

Throughout, let $G$ be the simple real Lie group $SU(2, 2)$ of rank two, and let

$$K = SU(2) \times U(2) : \text{the maximal compact subgroup of } G$$

$$\pi : \text{an irreducible representation of } G \text{ which is } K\text{-admissible.}$$

For the representation $\pi$, there are two types of Whittaker model with respect to a character $\eta$ of $N$ (a spherical subgroup of $G$). One is the smooth model, and the other is the algebraic models induced by the space of algebraic Whittaker vectors:

$$W(\pi, \eta) := \text{Hom}_{(g, K)}(\pi |_{K}, C^{\infty}\text{-Ind}_{N}^{G}(\eta)).$$

Here, $g$ is the Lie algebra of $G$, $\pi |_{K}$ is the subspace of $K$-finite vectors in $\pi$ and $C^{\infty}\text{-Ind}_{N}^{G}(\eta)$ is the right $G$-module smoothly induced from $\eta$.

Our aim is a characterization of the space $W(\pi, \eta)$ for the principal series representation $\pi$ of $G$ associated with a minimal parabolic subgroup, which leads to a description of the following challenging question associated to $\pi$.

Question. For each intertwiner $\Phi$ in $W(\pi, \eta)$, what is the image of $\Phi$? Equivalently, for each $K$-type $\tau$ occurring in $\pi$, one can ask the image of the $\tau$-isotypic component in $\pi$. The latter image is called the space of Whittaker functions of $\pi$ with respect to $\tau$.

The natural and classical approach. Let $\tau$ be a $K$-type belonging to $\pi$, and $f_{1}, ..., f_{n}$ be its a basis in $\pi$. Denote by $\phi_{j}(g)$ the image of $f_{j}$ under a fixed intertwiner $\Phi$. Then, for each $j$ and $k$ in $K$, the function $(k\phi_{j})(g) = \phi_{j}(gk)$ is a linear combination of the functions $\phi_{1}(g), ..., \phi_{n}(g)$. Thus, it is enough to consider the functions $\phi_{j}$ on $A$ for our purpose.
Assume that $C$ is a square matrix of size $\dim(\tau)$, with entries in the universal enveloping algebra of $\mathfrak{g}$, so that
\[
\pi(C) \circ \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} = \gamma \cdot \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix},
\]
for some constant $\gamma \in \mathbb{C}$.

By applying $\Phi$ to the identity (1) we get the following system of differential equations (the $A$-radial part)
\[
\mathcal{R}(C) \circ \begin{pmatrix} \phi_1(a) \\ \phi_2(a) \\ \vdots \\ \phi_n(a) \end{pmatrix} = \gamma \cdot \begin{pmatrix} \phi_1(a) \\ \phi_2(a) \\ \vdots \\ \phi_n(a) \end{pmatrix}, \quad a \in A
\]
where $\mathcal{R}$ denotes the infinitesimal action of $G$ on $C^\infty$-$Ind_N^G(\eta)$. Thus, one can regard the space $W(\pi, \eta)$ as a subset of the solution space $Sol(\mathcal{R}(C))$ of the system by sending $\Phi$ to the functions $\{\phi_j(a)\}$.

**Remark.** Recall that Whittaker functions satisfy differential equations with regular singularities at "0". The most important requirements for choosing a basis for $\tau$ are the simplicity and symmetricity of the series expansion of these functions $\phi_j(a)(a \in A)$ around 0 and of the system of differential equations arising from (1).

**Principal series $\pi_{s,\chi}$.** Let
\[
a = \{a(t_1, t_2) = \begin{pmatrix} 0 & 0 & t_1 & 0 \\ 0 & 0 & 0 & t_2 \\ t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \end{pmatrix} | t_1, t_2 \in \mathbb{R} \} \subset \mathfrak{g},
\]
\[
M = \{\text{diag}(e^{i\theta}, e^{-i\theta}, e^{i\theta}, e^{-i\theta})\} \oplus \{1_4, \text{diag}(1, -1, 1, -1)\}.
\]
Define linear functions $\lambda_1$ and $\lambda_2$ on $a$ by $\lambda_1(a(t_1, t_2)) = t_1$ and $\lambda_2(a(t_1, t_2)) = t_2$. Then the set $\{\pm \lambda_1 \pm \lambda_2, \pm 2\lambda_1, \pm 2\lambda_2\}$ forms the restricted root system of type $C_2$ for the pair $(\mathfrak{g}, a)$. Define $\lambda_1 \pm \lambda_2, 2\lambda_1$ and $2\lambda_2$ to be positive. Let $P_{\min}$ be the minimal parabolic subgroup of $G$ with Langlands decomposition $P_{\min} = MAN$, where $N$ is the unipotent subgroup defined by the root spaces corresponding to positive roots. For the character $s \otimes \chi$ of $M$, $s \in \mathbb{Z}$, and the $\mathbb{C}$-valued real linear form $\mu = \mu_1 \lambda + \mu_2 \lambda_2$, one has the principal series representation
\[
\pi_{s,\chi} := \text{Ind}_M^G((s \otimes \chi)_M \otimes e^{\mu + \rho} \otimes 1_N),
\]
where $1_N$ is the trivial character of $N$.

The main object in the paper is the 8-dimensional space $W(\pi_{s,\chi}, \eta)$ of algebraic Whittaker vectors (see Kostant [2]) for non-degenerate character $\eta$ (unitary) of $N$. Note that it is sufficient for our purpose to assume that $s \geq 0$. 
1.1 Some previous results

Let us recall some known identities as in (1) and previous results for the space $W(\pi, \eta)$. The first example is the classical Casimir equation: let $\Omega$ be the Casimir operator of $G$. Then we have the following identity

$$\pi_{s, \chi}(\Omega)v = \chi_{\pi_{s, \chi}}(\Omega)v,$$

where $\chi_{\pi_{s, \chi}}$ is the infinitesimal character and $v$ is a differential vector. This identity gives us an injection of $W(\pi_{s, \chi}, \eta)$ into the solution space $Sol(\mathcal{R}(\Omega))$ of the above equation. Note that the space $Sol(\mathcal{R}(\Omega))$ is of infinite dimension.

Let $\pi$ be a discrete series representation of $G$ and $\tau$ be its minimal $K$-type. Then Yamashita [10] defined an operator $D_{\pi, \tau}$ on $\tau$ under $\pi$:

$$\pi(D_{\pi, \tau})\tau = 0.$$ 

This gives us an injection of $W(\pi, \eta)$ into the solution space $Sol(\mathcal{R}(D_{\pi, \tau}))$ of the operator $\mathcal{R}(D_{\pi, \tau})$. Moreover, under certain conditions, he showed that

$$W(\pi, \eta) \cong Sol(\mathcal{R}(D_{\pi, \tau}))$$

as vector spaces. This result is not just for the group $G$ (see [10] and [11]).

Let $\pi$ be the principal series representation of $G = Sp(2, \mathbb{R})$ as in [6], and $\tau$ be the minimal $K$-type of $\pi$. In [6], the authors obtained a matrix, of size $\dim(\tau)$, formula of the form $\pi(D)v = \gamma v$ which implies

$$W(\pi, \eta) \cong Sol(\mathcal{R}(\Omega), \mathcal{R}(\mathcal{D})), $$

where $\Omega$ stands for the Casimir operator of $Sp(2, \mathbb{R})$. Note that the possible value of $\dim(\tau)$ is 1 or 2. The degree of $D$ is 4 if $\dim(\tau) = 1$, and 2 for the case of dimension 2.

**Remark.** In the case $s = 0$ and $s = 1$, the corresponding spaces $W(\pi_{s, \chi}, \eta)$ behave quite similar to the above mentioned cases for $G = Sp(2, \mathbb{R})$, and are studied in [4].

2 Differential equations

We begin by providing some formulas for the multiplicity one $K$-types $\tau_{[0, s; l]}$ in the principal series $\pi_{s, \chi}$. These formulas come from the explicit $(\mathfrak{g}, K)$-module structure of $\pi_{s, \chi}$ which originally discussed by Oda [7].

Note that the space of the adjoint $K$-representation $(Ad, \mathfrak{p}_{\mathbb{C}})$ is generated by the matrix units $E_{ij+2}$ and $E_{i+2j}$ ($1 \leq i, j \leq 2$) and denote by $\mathcal{E}_{ij+2}$ and $\mathcal{E}_{i+2j}$ their infinitesimal actions with respect to $\pi_{s}$. Let denote $F_{[s;l]}$ the transpose of the vector $(f_{0}, f_{1}, ..., f_{s})$, where $\{f_{j} : 0 \leq j \leq s\}$ is the "nice" basis of $\tau_{[0, s; l]}$ introduced in [1] and $c_{q} := q/s$ for $0 \leq q \leq s$.

**Formula 1. (Casimir equation)** Let $\Omega$ be the Casimir operator. Then we have

$$\pi_{s, \chi}(\Omega) \cdot F_{[s;l]} = (\mu_{1}^{2} + \mu_{2}^{2} + \frac{1}{2}s^{2} - 10)F_{[s;l]}.$$
Formula 2. (Shift equations) Set $\nu_1 = \frac{1}{2}(s + l)$ and $\nu_2 = \frac{1}{2}(s - l)$. Then we have
\[
\pi_{s,\chi}(\bar{Q}) \cdot F_{[s;l]} = \frac{1}{4}(\mu_1^2 - (\nu_1 + 1)^2)F_{[s;l]},
\]
and
\[
\pi_{s,\chi}(Q) \cdot F_{[s;l]} = \frac{1}{4}(\mu_2^2 - (\nu_2 - 1)^2)F_{[s;l]},
\]
where $\bar{Q} = \{\bar{Q}_{ij}\}_{0 \leq i,j \leq s}$ and $Q = \{Q_{ij}\}_{0 \leq i,j \leq s}$ are square matrices given by
\[
\begin{align*}
\bar{Q}_{qq-1} &= -c_q(\mathcal{E}_{24}\mathcal{E}_{32} + \mathcal{E}_{14}\mathcal{E}_{31}) \\
\bar{Q}_{qq+1} &= -(1-c_q)(\mathcal{E}_{23}\mathcal{E}_{42} + \mathcal{E}_{13}\mathcal{E}_{41}) \\
\bar{Q}_{qq} &= (1-c_q)(\mathcal{E}_{23}\mathcal{E}_{32} + \mathcal{E}_{13}\mathcal{E}_{31}) + c_q(\mathcal{E}_{14}\mathcal{E}_{14} + \mathcal{E}_{24}\mathcal{E}_{32})
\end{align*}
\]
and
\[
\begin{align*}
Q_{qq-1} &= c_q(\mathcal{E}_{32}\mathcal{E}_{24} + \mathcal{E}_{31}\mathcal{E}_{14}) \\
Q_{qq+1} &= (1-c_q)(\mathcal{E}_{42}\mathcal{E}_{23} + \mathcal{E}_{41}\mathcal{E}_{13}) \\
Q_{qq} &= c_q(\mathcal{E}_{32}\mathcal{E}_{23} + \mathcal{E}_{31}\mathcal{E}_{13}) + (1-c_q)(\mathcal{E}_{41}\mathcal{E}_{14} + \mathcal{E}_{42}\mathcal{E}_{24})
\end{align*}
\]
for $0 \leq q \leq s$, but all other entries are 0.

Formula 3. (Annihilation equations) We have
\[
\pi_{s,\chi}(A) \cdot F_{[s;l]} = 0,
\]
and
\[
\pi_{s,\chi}(\bar{A}) \cdot F_{[s;l]} = 0,
\]
where $A = \{A_{ij}\}$ and $\bar{A} = \{\bar{A}_{ij}\}$ are square matrix whose non-zero entries are given by
\[
\begin{align*}
A_{jj-1} &= -\mathcal{E}_{31}\mathcal{E}_{14} - \mathcal{E}_{32}\mathcal{E}_{24}, \\
A_{jj} &= \mathcal{E}_{41}\mathcal{E}_{14} + \mathcal{E}_{42}\mathcal{E}_{24} - \mathcal{E}_{31}\mathcal{E}_{13} - \mathcal{E}_{32}\mathcal{E}_{23}, \\
A_{jj+1} &= \mathcal{E}_{41}\mathcal{E}_{13} + \mathcal{E}_{42}\mathcal{E}_{23},
\end{align*}
\]
and
\[
\begin{align*}
\bar{A}_{jj-1} &= -\mathcal{E}_{14}\mathcal{E}_{31} - \mathcal{E}_{24}\mathcal{E}_{32}, \\
\bar{A}_{jj} &= \mathcal{E}_{14}\mathcal{E}_{41} + \mathcal{E}_{24}\mathcal{E}_{42} - \mathcal{E}_{13}\mathcal{E}_{31} - \mathcal{E}_{23}\mathcal{E}_{32}, \\
\bar{A}_{jj+1} &= \mathcal{E}_{13}\mathcal{E}_{41} + \mathcal{E}_{23}\mathcal{E}_{42},
\end{align*}
\]
for $1 \leq j \leq s - 1$.

Proposition 2.1. On the $K$-type $\tau_{[0,s;l]}$ with respect to the action $\pi_{s,\chi}$ we have
\[
Q + \bar{Q} = \Omega/4.
\]
2.1 A holonomic system of rank 8

**Coordinate system.** Since the $\mathbb{R}$-split torus $A$ for our case is two dimensional, one may choose the coordinate system $(y_1, y_2)$. Denote the Euler operators $y_1 \frac{\partial}{\partial y_1}$ and $y_2 \frac{\partial}{\partial y_2}$ with respect to this system by $\partial_1$ and $\partial_2$, respectively.

We now define the matrix differential operator $\overline{D}$ by

$$
\begin{pmatrix}
\overline{d}_{00} & \overline{d}_{01} & 0 & \cdots & 0 \\
0 & \overline{d}_{11} & \overline{d}_{12} & \cdots & 0 \\
0 & 0 & \overline{d}_{22} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \overline{d}_{s-2,s-2} \\
0 & 0 & 0 & \cdots & \overline{d}_{s-1,s-1} \\
0 & 0 & 0 & \cdots & \overline{d}_{s,s}
\end{pmatrix}
$$

where

$$
d_{qq} = \frac{1}{4}((\partial_1 - q)^2 - \mu_1^2) - \xi \overline{\xi} y_1^2, \quad d_{q,q+1} = \overline{\xi} y_1 (\partial_2 + \frac{1}{2}s - q) + \xi y_1 y_2
$$

for $q = 0, \ldots, s - 1$ and

$$
d_{ss} = \frac{1}{4}((\partial_1 - 2\partial_2)^2 - \mu_2^2) - \xi \overline{\xi} y_1^2 - y_2^2 - \nu_1 y_2
$$

$$
d_{s,s-1} = -\xi y_1 (\partial_2 + \frac{1}{2}s) + \xi y_1 y_2.
$$

We also define the matrix differential operator $D$ by

$$
\begin{pmatrix}
d_{00} & d_{01} & 0 & \cdots & 0 \\
d_{10} & d_{11} & 0 & \cdots & 0 \\
0 & a_{32} & d_{33} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & d_{s-2,s-2} \\
0 & 0 & 0 & \cdots & d_{s-1,s-1} \\
0 & 0 & 0 & \cdots & d_{s,s}
\end{pmatrix}
$$

where

$$
d_{00} = \frac{1}{4}((\partial_1 - 2\partial_2)^2 - \mu_2^2) - \xi \overline{\xi} y_1^2 - y_2^2 - \nu_2 y_2
$$

$$
d_{01} = -\xi y_1 (\partial_2 - \frac{1}{2}s) + \overline{\xi} y_1 y_2
$$

and

$$
d_{qq} = \frac{1}{4}((\partial_1 - s + q)^2 - \mu_2^2) - \xi \overline{\xi} y_1^2, \quad d_{q,q-1} = \xi y_1 (\partial_2 + q - \frac{1}{2}s) - \xi y_1 y_2
$$

for $q = 1, \ldots, s$. Here, the parameters $\xi$ and $\overline{\xi}$ are associated to the character $\eta$. 
By using Formulas 2 and 3, one can see that the Whittaker functions of \( \pi_{s, \chi} \) with respect to \( \tau_{[0, s;l]} \) satisfy the system of differential equations \( \mathcal{D} = 0 \) and \( \overline{D} = 0 \). Moreover, we have the following result which characterizes the Whittaker functions of \( \pi_{s, \chi} \) with respect to \( \tau_{[0, s;l]} \).

**Theorem 2.2.** For \( s \geq 2 \), the natural map from \( W(\pi_{s, \chi}, \eta) \) into \( \text{Ker}(\overline{D}, D) \) is bijection if \( \pi_{s, \chi} \) is irreducible and \( \eta \) is a nondegenerate unitary character of \( N \).

Here, we also have the following formula in the case \( s = 0 \), which is analogue to the class one case for \( \text{Sp}(2, \mathbb{R}) \) in [5]. Write \( W \) for the little Weyl group for \((g, a)\), and \((\rho_1, \rho_2)\) for the pair \((3, 2)\) related to the half sum.

**Theorem 2.3.** Let \( \pi_{0, \chi} \) be an irreducible principal series with parameter \( \mu = (\mu_1, \mu_2) \in a^*_c \), and set \( \epsilon = \frac{1-\chi(-1)}{2} \). Then the function \( \phi_\mu \) on \( A \) defined by

\[
\phi_\mu(y_1, y_2) = y_1^{\rho_1} y_2^{\rho_2} \sum_{m,n \geq 0} \frac{U_{m,n}^0}{2^{2n} (\frac{\mu_1-\epsilon}{2} + 1)^m (\frac{\mu_2-\epsilon}{2} + 1)^n} \times y_1^{\mu_1+2m} y_2^{\mu_2+2n}
+ \frac{\epsilon U_{m,n}^1}{2^{2n+1} (\frac{\mu_1-\epsilon}{2} + 1)^m (\frac{\mu_2-\epsilon}{2} + 1)^{n+1}} \times y_1^{\mu_1+2m} y_2^{\mu_2+2n+1},
\]

is a Whittaker function, on \( A \), of \( \pi_{0, \chi} \) with the \( K \)-type \( \tau_{[0,0;2\epsilon]} \). Moreover, the intertwiners \( \Phi_{\omega(\mu)} \) attached to the function \( \phi_{\omega(\mu)}(y_1, y_2) \) form a basis of the 8-dimensional space \( W(\pi_{0, \chi}, \eta) \). Here,

\[
U_{m,n}^t := \sum_{j=0}^{\min(m,n)} \frac{(\frac{\mu_1-\epsilon}{2} + n + 1 + t)_{m-j}}{(m-j)! (n-j)! j!(\frac{\mu_1+\mu_2}{2} + 1)^j (\frac{\mu_1-\mu_2}{2} + 1)_{m-j}}
\]

for \( t = 0, 1 \).

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**References**

[1] Bayarmagnai, G. The \((g, K)\)-module structure of principal series of \( SU(2, 2) \), J. Math. Soc, Vol. 61, No. 3 (2009), 661-686


