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On the depth of edge rings

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1. INTRODUCTION

This article is a summary of the papers [3], [4].

Let $G$ be a finite connected graph with no loop and no multiple edge, on the vertex set $V(G) = \{1, 2, \ldots, d\}$ and the edge set $E(G) = \{e_1, e_2, \ldots, e_r\}$. Let $K$ be a field and $K[t] = K[t_1, t_2, \ldots, t_d]$ the polynomial ring in $d = \#V(G)$ variables. We consider the subring of $K[t]$ generated by squarefree quadratic monomials $t^e = t_i t_j$ where $e = \{i, j\} \in E(G)$. This semigroup ring is called the edge ring of $G$ denoted by $K[G]$. Let $K[x] = K[x_1, x_2, \ldots, x_r]$ be the polynomial ring in $r = \#E(G)$ variables. The kernel of the surjective homomorphism $\pi: K[x] \to K[G]$ defined by setting $\pi(x_i) = t^e_i$ for $i = 1, 2, \ldots, r$ is called the toric ideal of $G$, denoted by $I_G$. Then we have $K[G] \cong K[x]/I_G$.

Ohsugi and Hibi [6, Corollary 2.3] gave the criterion of the normality of edge rings: $K[G]$ is normal if and only if $G$ satisfies the odd cycle condition, i.e., for any two odd cycles $C_1, C_2$ in $G$ with no common vertex, there exist $i \in V(C_1)$ and $j \in V(C_2)$ such that $\{i, j\} \in E(G)$, which is called a bridge between $C_1$ and $C_2$. It is known that a normal semigroup ring is Cohen–Macaulay. Hence it is natural to ask when $K[G]$ is Cohen–Macaulay. Here $K[G]$ is said to be Cohen–Macaulay if Krull-dim $K[G] = \text{depth} K[G]$, where Krull-dim $K[G]$ denotes the Krull dimension of $K[G]$ and depth $K[G]$ denotes the depth of $K[G]$. The Krull dimension of $K[G]$ is known: Krull-dim $K[G] = d$ if $G$ is a connected non-bipartite graph; Krull-dim $K[G] = d - 1$ if $G$ is a connected bipartite graph. Therefore we concentrate our attention on the depth of $K[G]$.

We have known that for an arbitrary bipartite graph and any graph with $d \leq 6$, the edge ring is normal by virtue of the odd cycle condition. When $d = 7$, there exists a finite graph $G$ for which $K[G]$ is non-normal. However all of these are Cohen-Macaulay and thus the depth of the edge rings is 7. From
our computational experiment, we give the following conjecture though it is completely open:

**Conjecture 1.1.** Let $G$ be a finite connected non-bipartite graph on $[d]$ with $d \geq 7$. Then $	ext{depth } K[G] \geq 7$.

On the other hand, we have found a family of graphs $G_{k+6}$, $k \geq 1$ (Figure 1), whose edge rings always have depth 7 (Lemma 2.1). As the result, we have the following theorem.

**Theorem 1.2.** Let $f, d$ be integers with $7 \leq f \leq d$. Then there exists a finite graph $G$ on $[d]$ with depth $K[G] = f$ and with Krull-dim $K[G] = d$.

This theorem also means that there exists a graph for which the edge ring is far from the Cohen–Macaulay property. We will prove Theorem 1.2 in Section 2 and show the outline of our proof of Lemma 2.1 which is a key lemma.

In general, the inequality $\text{depth } K[G]/\text{in}_{<}(I_G) \leq \text{depth } K[G]/I_G$ holds for an arbitrary monomial order $<$, where $\text{in}_{<}(I_G)$ denotes the initial ideal of $I_G$ with respect to $<$. We use this fact in the proof of Lemma 2.1. Actually, the equality holds for $G_{k+6}$ with the lexicographic order induced by $x_1 > x_2 > \cdots > x_r$. We are interested in the behavior of the depth when we take the initial ideal of a toric ideal. Computational experience yields the following conjecture:

**Conjecture 1.3.** Let $G$ be a finite connected non-bipartite graph on $[d]$ with $d \geq 6$ and suppose that its edge ring $K[G]$ is normal. Then $\text{depth } K[x]/\text{in}_{<}(I_G) \geq 6$ for any monomial order $<$ on $K[x]$.

Let $<_{\text{rev}}$ (resp. $<_{\text{lex}}$) denote a reverse lexicographic order (resp. a lexicographic order) on $K[x]$. Even though Conjecture 1.3 is completely open, the main result of this part is the following theorem.

**Theorem 1.4.** Let $f, d$ be integers with $6 \leq f \leq d$. Then there exists a finite connected non-bipartite graph $G$ on $[d]$ with the following properties:

1. $K[G]$ is normal;
2. $\text{depth } K[x]/\text{in}_{<\text{rev}}(I_G) = f$;
3. $K[x]/\text{in}_{<\text{lex}}(I_G)$ is Cohen–Macaulay.

Similarly to Theorem 1.2, the family of the graphs $H_{k+5}$, $k \geq 1$ (which is obtained by adding a bridge between 2 triangles to $G_{k+5}$; see Figure 3) plays

![Figure 1. The finite graph $G_{k+6}$](image-url)
an essential role in our proof of Theorem 1.4; see Lemma 3.1. In Section 3, we will state the outline of the proofs of Theorem 1.4 and Lemma 3.1.

2. THE DEPTH OF THE EDGE RING OF $G_{k+6}$

This section is devoted to proving the following lemma.

**Lemma 2.1.** Let $k \geq 1$ be an integer and let $G_{k+6}$ be the graph as in Figure 1. Then

$$\text{depth } K[G_{k+6}] = \text{depth } K[x]/I_{G_{k+6}} = 7.$$ 

Once we establish this lemma, we can prove Theorem 1.2 easily. In fact, the graph obtained from $G_{d-f+7}$ by adding $f - 7$ edges $\{1, d-f+8\}, \{1, d-f+9\}, \ldots, \{1, d\}$ satisfies the required properties.

Let $G$ be a graph. We associate each edge $e_i = \{i_1, j_i\} \in E(G)$ with the vector $a_i \in \mathbb{Z}^d$ whose $i_{th}$ and $j_{th}$ entries are 1 and the others are 0. Set $S_G = Na_1 + Na_2 + \cdots + Na_r$. Then $K[G] \cong K[S_G]$. We consider $S_G$-grading on $K[x]$ and $K[G]$.

Now we prove Lemma 2.1. We set $G = G_{k+6}$ and $r = \#E(G) = 2(k-1)+8$. The proof of Lemma 2.1 is divided into two parts: a proof of depth $K[G] \leq 7$ and that of depth $K[G] \geq 7$.

**(Step 1):** First we prove that depth $K[G] \leq 7$. By the Auslander–Buchsbaum formula, we have

$$\text{depth } K[G] + \text{pd } K[G] = \text{depth } K[x] = \#E(G) = 2(k-1)+8,$$

where $\text{pd } K[G]$ denotes the projective dimension of $K[G]$. Thus we may prove that $\text{pd } K[G] \geq 2k - 1$. Since $\text{pd } K[G] = \max \{i : \beta_{i,s}(K[G]) \neq 0\}$, where $\beta_{i,s}(K[G]) = \dim_K \text{Tor}_i(K[G], K)_s$ is the $i$th Betti number of $K[G]$ in degree $s \in S_G$, it is sufficient to prove that $\beta_{2k-1,s}(K[G]) \neq 0$ for some $s \in S_G$. For $s \in S_G$, let $\Delta_s$ be the simplicial complex defined by

$$\Delta_s := \left\{ F \subset [r] : s - \sum_{i \in F} a_i \in S_G \right\}.$$

We use the following result due to Briales, Campillo, Marijuán, and Pisón [1].

**Lemma 2.2 ([1, Theorem 2.1]).** Let $G$ be a finite simple graph. Then

$$\beta_{i+1,s}(K[G]) = \dim_K \tilde{H}_i(\Delta_s; K).$$

Let us consider the simplicial complex $\Delta_s$ with

$$s = (1,1,k+1,k+1,1,1,2,2,\ldots,2) \in S_G.$$

Then we can prove that $\tilde{H}_{2k-2}(\Delta_s; K) \neq 0$ and can conclude that $\text{pd } K[G] \geq 2k - 1$, as desired.
(Step 2): Next we prove that depth $K[G] \geq 7$. Since the inequality
\[ \text{depth } K[x]/I_G \geq K[x]/\text{in}_{<}(I_G) \]
holds for an arbitrary monomial order $<$, we may prove $K[x]/\text{in}_{<}(I_G) \geq 7$ for the lexicographic order $< \text{ induced by } x_1 > x_2 > \cdots > x_r$. To compute $\text{in}_{<}(I_G)$, we first find the generators of $I_G$. Ohsugi and Hibi [7, Lemma 3.1] proved that a toric ideal of a finite simple graph is generated by binomials corresponding to primitive even closed walks of the graph. By [7, Lemma 3.2], there are 2 kinds of such walks in $G$ (see Figure 2):

(I) 4-cycles: $\{e_{2i+7}, e_{2i+8}, e_{2j+8}, e_{2j+7}\}$, $0 \leq i < j \leq k-1$;
(II) the 2 triangles with two length 2 walks connecting the triangles:
$\{e_2, e_1, e_3, e_{2p+7}, e_{2p+8}, e_4, e_6, e_5, e_{2q+8}, e_{2q+7}\}$, $0 \leq p \leq q \leq k-1$.

Hence $I_G$ is generated by the following binomials:
\[ x_{2i+7}x_{2j+8} - x_{2i+8}x_{2j+7}, \quad 0 \leq i < j \leq k-1, \]
\[ x_1x_4x_5x_{2p+7}x_{2q+7} - x_2x_3x_6x_{2p+8}x_{2q+8}, \quad 0 \leq p \leq q \leq k-1. \]

We can prove that the set of these binomials forms a Gröbner basis of $I_G$ by a straightforward application of Buchberger's criterion. Thus $\text{in}_{<}(I_G)$ is generated by

(2.1) $x_{2i+7}x_{2j+8}$, $0 \leq i < j \leq k-1$,
(2.2) $x_1x_4x_5x_{2p+7}x_{2q+7}$, $0 \leq p \leq q \leq k-1$.

Now we prove depth $K[x]/\text{in}_{<}(I_G) \geq 7$. Let $I'$ be the ideal generated by monomials (2.1). Then
\[ \text{in}_{<}(I_G) = x_1x_4x_5(x_7, x_9, \ldots, x_{2(k-1)+7})^2 + I', \]
\[ = ((x_7, x_9, \ldots, x_{2(k-1)+7})^2 + I') \cap ((x_1x_4x_5) + I'). \]

We set
\[ I_1 = ((x_7, x_9, \ldots, x_{2(k-1)+7})^2 + I'), \quad I_2 = (x_1x_4x_5) + I'. \]

By the short exact sequence
\[ 0 \to K[x]/I_1 \cap I_2 \to K[x]/I_1 \oplus K[x]/I_2 \to K[x]/(I_1 + I_2) \to 0, \]
we may prove that depth $K[x]/I_1 \geq 7$, depth $K[x]/I_2 \geq 7$, and depth $K[x]/(I_1 + I_2) \geq 6$. Since $x_1, x_2, x_3, x_4, x_5, x_6, x_8$ is a $K[x]/I_1$-regular sequence, we have depth $K[x]/I_1 \geq 7$. Because $x_1x_4x_5$ is a $K[x]/I'$-regular element, we have depth $K[x]/I_2 = \text{depth } K[x]/I' - 1$. Then the sequence $x_1, x_2, \ldots, x_6, x_8, x_{2(k-1)+7}$
FIGURE 3. The finite graph $H_{k+5}$

is $K[x]/I'$-regular and we have depth $K[x]/I' \geq 8$. Similarly, we have that depth $K[x]/(I_1 + I_2) \geq 6$.

3. THE DEPTH OF INITIAL IDEALS OF NORMAL EDGE RINGS

In this section, we state the outline of our proof of Theorem 1.4. We consider the family of graphs $H_{k+5}, k \geq 1$ of Figure 3. The following lemma is a key in the proof of Theorem 1.4.

**Lemma 3.1.** Let $k \geq 1$ be an arbitrary integer and $H_{k+5}$ the graph of Figure 3. Then

1. $K[H_{k+5}]$ is normal;
2. depth $K[x]/in_{rev}(I_{H_{k+5}}) = 6$;
3. depth $K[x]/in_{lex}(I_{H_{k+5}})$ is Cohen-Macaulay.

Once we prove Lemma 3.1, we can prove Theorem 1.4 by a similar way to Theorem 1.2.

The rest of this section is devoted to the proof of Lemma 3.1.

We set $H = H_{k+5}$. First, we find Gröbner bases of $I_H$ with respect to the monomial orders $<_{rev}, <_{lex}$. Similarly to the proof of Lemma 2.1, we list the primitive even closed walks of $H$; there are 5 kinds of such walks:

(I) 4-cycles: $\{e_i, e_{k+1+i}, e_{k+1+j}, e_j\}, 2 \leq i < j \leq k$;

(II) the 2 triangles with the bridge: $\{e_1, e_{k+1}, e_{2k+4}, e_{2k+3}, e_{k+2}, e_{2k+2}, e_{2k+5}, e_{2k+3}\}$;

(III) 6-cycles: $\{e_{k+1}, e_r, e_{k+1+r}, e_{k+2}, e_{2k+3}, e_{2k+4}\}, 2 \leq r \leq k$;

(IV) 6-cycles: $\{e_1, e_r, e_{k+1+r}, e_{2k+2}, e_{2k+5}, e_{2k+3}\}, 2 \leq r \leq k$;

(V) the 2 triangles with two length 2 walks connecting the triangles $\{e_1, e_{2k+4}, e_{k+1}, e_p, e_{k+1+p}, e_{2k+2}, e_{2k+5}, e_{k+2}, e_{k+1+q}, e_q\}, 2 \leq p \leq q \leq k$;

Similarly to Lemma 2.1, we have the following lemma from a straightforward application of Buchberger's criterion.

**Lemma 3.2.** The set of binomials corresponding to primitive even closed walks (I), (II), (III), (IV), and (V) is a Gröbner basis of $I_H$ with respect to both $<_{rev}$ and $<_{lex}$. 


By virtue of Lemma 3.2, we obtain the generators of $\text{in}_{\text{rev}}(I_H)$ and $\text{in}_{\text{lex}}(I_H)$.

**Corollary 3.3.** The initial ideal in $\text{in}_{\text{rev}}(I_{H_{k+5}})$ is generated by the following monomials:

$$x_jx_{k+1+i}, \quad 2 \leq i < j \leq k,$$

$$x_{k+1}x_{2k+2}x_{2k+3},$$

$$x_{k+1}x_{2k+1+r}x_{2k+3}, \quad x_rx_{2k+2}x_{2k+3}, \quad 2 \leq r \leq k,$$

$$x_px_qx_{k+2}x_{2k+4}, \quad 2 \leq p \leq q \leq k.$$

**Corollary 3.4.** The initial ideal in $\text{in}_{\text{lex}}(I_{H_{k+5}})$ is generated by the following monomials:

$$x_ix_{k+1+j}, \quad 2 \leq i < j \leq k,$$

$$x_1x_{k+2}x_{2k+4}x_{2k+5},$$

$$x_rx_{k+2}x_{2k+4}, \quad x_1x_{k+1+r}x_{2k+5}, \quad 2 \leq r \leq k.$$  

In particular, $\text{in}_{\text{lex}}(I_{H_{k+5}})$ is a squarefree monomial ideal.

Now we state the outline of our proof of Lemma 3.1.

**Proof of Lemma 3.1 (1).** Since $H$ satisfies the odd cycle condition, the edge ring $K[H]$ is normal.
**Proof of Lemma 3.1 (2).** We prove that depth $K[x]/\text{in}_{<rev}(I_H) = 6$. Set $I = \text{in}_{<rev}(I_H)$. Similar to the proof of depth $K[G_{k+6}] = 7$ in the previous section, we will first prove depth $K[x]/I \leq 6$ and then that depth $K[x]/I \geq 6$.

To prove depth $K[x]/I \leq 6$, it is enough to show that $pd K[x]/I \geq 2k - 1$ by the Auslander–Buchsbaum formula. We prove this by showing that the $(2k - 1)$th Betti number of $K[x]/I$ does not vanish. For a monomial ideal, the Betti number is described in terms of the Koszul simplicial complex; the Koszul simplicial complex of $I$ in degree $a \in \mathbb{Z}_{\geq 0}$ is defined by

$$K^a(I) := \{ \alpha \in \{0,1\}^r : x^{\alpha-a} \in I \}.$$ 

**Lemma 3.5 ([5, Theorem 1.34]).** Let $S$ be a polynomial ring over $K$ and $I$ squarefree monomial ideal of $S$. Then

$$\beta_{i+1,a}(S/I) = \dim_K \tilde{H}_{i-1}(K^a(I);K).$$

We set

$$a = \sum_{j=2}^{k}(e_j + e_{k+1+j}) + e_{k+1} + e_{2k+2} + 2e_{2k+3},$$

where $e_i$ is the $i$th unit vector of $\mathbb{R}^{2k+5}$. Then we can show $\tilde{H}_{2k-3}(K^a(I);K) \neq 0$.

The proof of depth $K[x]/I \geq 6$ is similar to that of depth $K[x]/\text{in}_{<}(I_G) \geq 7$ in the previous section. We rewrite the ideal $I$ as the intersection of ideals for each of which it is easy to estimate the depth, though the method of division is technical.

**Proof of Lemma 3.1 (3).** Finally, we prove that $K[x]/\text{in}_{<lex}(I_H)$ is Cohen–Macaulay. We set $J = \text{in}_{<lex}(I_H)$. Since $J$ is a squarefree monomial ideal, $J$ is the Stanley–Reisner ideal $I_\Delta$ of some simplicial complex $\Delta$. It is known that the Stanley–Reisner ideal $K[\Delta] = K[x]/I_\Delta$ is Cohen–Macaulay if $\Delta$ is shellable. Our proof is done by showing that $\Delta$ is shellable.

**References**


