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On the depth of edge rings

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1. Introduction

This article is a summary of the papers [3], [4].

Let G be a finite connected graph with no loop and no multiple edge, on the vertex set $V(G) = [d] := \{1, 2, \ldots, d\}$ and the edge set $E(G) = \{e_1, e_2, \ldots, e_r\}$. Let K be a field and $K[\mathbf{t}] = K[t_1, t_2, \ldots, t_d]$ the polynomial ring in $d = \#V(G)$ variables. We consider the subring of $K[\mathbf{t}]$ generated by squarefree quadratic monomials $t^e = t_i t_j$ where $e = \{i, j\} \in E(G)$. This semigroup ring is called the *edge ring* of G denoted by $K[G]$. Let $K[\mathbf{x}] = K[x_1, x_2, \ldots, x_r]$ be the polynomial ring in $r = \#E(G)$ variables. The kernel of the surjective homomorphism $\pi\colon K[\mathbf{x}] \to K[G]$ defined by setting $\pi(x_i) = t^{e_i}$ for $i = 1, 2, \ldots, r$ is called the *toric ideal* of G, denoted by I_G. Then we have $K[G] \cong K[\mathbf{x}]/I_G$.

Ohsugi and Hibi [6, Corollary 2.3] gave the criterion of the normality of edge rings: $K[G]$ is normal if and only if G satisfies the *odd cycle condition*, i.e., for any two odd cycles C_1, C_2 in G with no common vertex, there exist $i \in V(C_1)$ and $j \in V(C_2)$ such that $\{i, j\} \in E(G)$, which is called a *bridge* between C_1 and C_2. It is known that a normal semigroup ring is Cohen–Macaulay. Hence it is natural to ask when $K[G]$ is Cohen–Macaulay. Here $K[G]$ is said to be Cohen–Macaulay if $\text{Krull-dim}\, K[G] = \text{depth}\, K[G]$, where $\text{Krull-dim}\, K[G]$ denotes the Krull dimension of $K[G]$ and $\text{depth}\, K[G]$ denotes the depth of $K[G]$. The Krull dimension of $K[G]$ is known: $\text{Krull-dim}\, K[G] = d$ if G is a connected non-bipartite graph; $\text{Krull-dim}\, K[G] = d - 1$ if G is a connected bipartite graph. Therefore we concentrate our attention on the depth of $K[G]$.

We have known that for an arbitrary bipartite graph and any graph with $d \leq 6$, the edge ring is normal by virtue of the odd cycle condition. When $d = 7$, there exists a finite graph G for which $K[G]$ is non-normal. However all of these are Cohen-Macaulay and thus the depth of the edge rings is 7. From

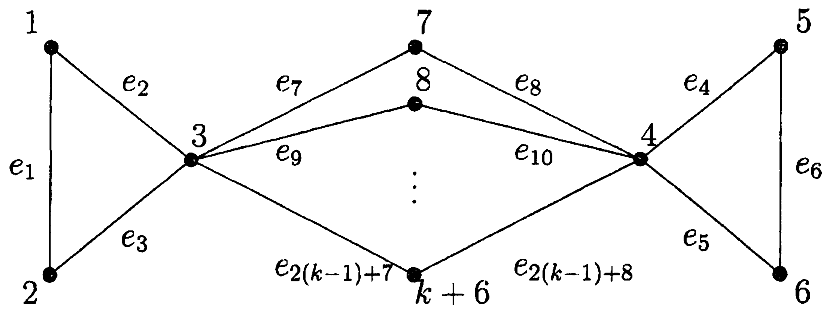

FIGURE 1. The finite graph G_{k+6}

our computational experiment, we give the following conjecture though it is completely open:

Conjecture 1.1. Let G be a finite connected non-bipartite graph on $[d]$ with $d \geq 7$. Then $\operatorname{depth} K[G] \geq 7$.

On the other hand, we have found a family of graphs G_{k+6}, $k \geq 1$ (Figure 1), whose edge rings always have depth 7 (Lemma 2.1). As the result, we have the following theorem.

Theorem 1.2. *Let f, d be integers with $7 \leq f \leq d$. Then there exists a finite graph G on $[d]$ with* $\operatorname{depth} K[G] = f$ *and with* $\text{Krull-dim}\, K[G] = d$.

This theorem also means that there exists a graph for which the edge ring is far from the Cohen–Macaulay property. We will prove Theorem 1.2 in Section 2 and show the outline of our proof of Lemma 2.1 which is a key lemma.

In general, the inequality $\operatorname{depth} K[G]/\operatorname{in}_<(I_G) \leq \operatorname{depth} K[G]/I_G$ holds for an arbitrary monomial order $<$, where $\operatorname{in}_<(I_G)$ denotes the initial ideal of I_G with respect to $<$. We use this fact in the proof of Lemma 2.1. Actually, the equality holds for G_{k+6} with the lexicographic order induced by $x_1 > x_2 > \cdots > x_r$. We are interested in the behavior of the depth when we take the initial ideal of a toric ideal. Computational experience yields the following conjecture:

Conjecture 1.3. Let G be a finite connected non-bipartite graph on $[d]$ with $d \geq 6$ and suppose that its edge ring $K[G]$ is normal. Then $\operatorname{depth} K[\mathbf{x}]/\operatorname{in}_<(I_G) \geq 6$ for any monomial order $<$ on $K[\mathbf{x}]$.

Let $<_{\text{rev}}$ (resp. $<_{\text{lex}}$) denote a reverse lexicographic order (resp. a lexicographic order) on $K[\mathbf{x}]$. Even though Conjecture 1.3 is completely open, the main result of this part is the following theorem.

Theorem 1.4. *Let f, d be integers with $6 \leq f \leq d$. Then there exists a finite connected non-bipartite graph G on $[d]$ with the following properties:*

(1) *$K[G]$ is normal;*
(2) $\operatorname{depth} K[\mathbf{x}]/\operatorname{in}_{<_{\text{rev}}}(I_G) = f$*;*
(3) *$K[\mathbf{x}]/\operatorname{in}_{<_{\text{lex}}}(I_G)$ is Cohen-Macaulay.*

Similarly to Theorem 1.2, the family of the graphs H_{k+5}, $k \geq 1$ (which is obtained by adding a bridge between 2 triangles to G_{k+5}; see Figure 3) plays an essential role in our proof of Theorem 1.4; see Lemma 3.1. In Section 3, we will state the outline of the proofs of Theorem 1.4 and Lemma 3.1.

2. THE DEPTH OF THE EDGE RING OF G_{k+6}

This section is devoted to proving the following lemma.

Lemma 2.1. *Let $k \geq 1$ be an integer and let G_{k+6} be the graph as in Figure 1. Then*

$$\operatorname{depth} K[G_{k+6}] = \operatorname{depth} K[\mathbf{x}]/I_{G_{k+6}} = 7.$$

Once we establish this lemma, we can prove Theorem 1.2 easily. In fact, the graph obtained from G_{d-f+7} by adding $f-7$ edges

$$\{1, d-f+8\}, \{1, d-f+9\}, \dots, \{1, d\}$$

satisfies the required properties.

Let G be a graph. We associate each edge $e_l = \{i_l, j_l\} \in E(G)$ with the vector $a_l \in \mathbb{Z}^d$ whose i_lth and j_lth entries are 1 and the others are 0. Set $S_G = \mathbb{N}a_1 + \mathbb{N}a_2 + \cdots + \mathbb{N}a_r$. Then $K[G] \cong K[S_G]$. We consider S_G-grading on $K[\mathbf{x}]$ and $K[G]$.

Now we prove Lemma 2.1. We set $G = G_{k+6}$ and $r = \#E(G) = 2(k-1)+8$. The proof of Lemma 2.1 is divided into two parts: a proof of $\operatorname{depth} K[G] \leq 7$ and that of $\operatorname{depth} K[G] \geq 7$.

(Step 1): First we prove that $\operatorname{depth} K[G] \leq 7$. By the Auslander–Buchsbaum formula, we have

$$\operatorname{depth} K[G] + \operatorname{pd} K[G] = \operatorname{depth} K[\mathbf{x}] = \#E(G) = 2(k-1)+8,$$

where $\operatorname{pd} K[G]$ denotes the projective dimension of $K[G]$. Thus we may prove that $\operatorname{pd} K[G] \geq 2k-1$. Since $\operatorname{pd} K[G] = \max\{i : \beta_{i,s}(K[G]) \neq 0\}$, where $\beta_{i,s}(K[G]) = \dim_K \operatorname{Tor}_i(K[G], K)_s$ is the ith Betti number of $K[G]$ in degree $s \in S_G$, it is sufficient to prove that $\beta_{2k-1,s}(K[G]) \neq 0$ for some $s \in S_G$. For $s \in S_G$, let Δ_s be the simplicial complex defined by

$$\Delta_s := \left\{ F \subset [r] \ : \ s - \sum_{l \in F} a_l \in S_G \right\}.$$

We use the following result due to Briales, Campillo, Marijuán, and Pisón [1].

Lemma 2.2 ([1, Theorem 2.1]). *Let G be a finite simple graph. Then*

$$\beta_{i+1,s}(K[G]) = \dim_K \tilde{H}_i(\Delta_s; K).$$

Let us consider the simplicial complex Δ_s with

$$s = (1, 1, k+1, k+1, 1, 1, 1, 2, 2, \dots, 2) \in S_G.$$

Then we can prove that $\tilde{H}_{2k-2}(\Delta_s; K) \neq 0$ and can conclude that $\operatorname{pd} K[G] \geq 2k-1$, as desired.

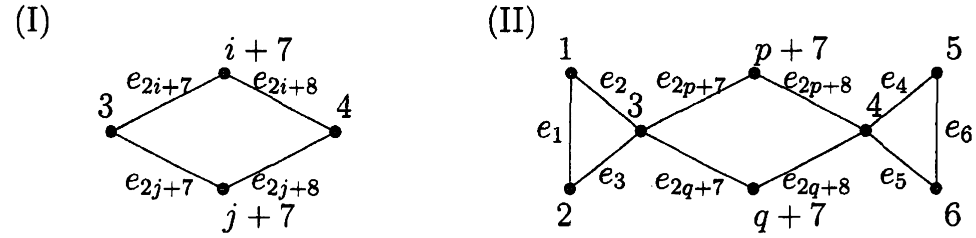

FIGURE 2. Primitive even closed walks of G_{k+6}

(Step 2): Next we prove that $\operatorname{depth} K[G] \geq 7$. Since the inequality

$$\operatorname{depth} K[\mathbf{x}]/I_G \geq K[\mathbf{x}]/\mathrm{in}_<(I_G)$$

holds for an arbitrary monomial order $<$, we may prove $K[\mathbf{x}]/\mathrm{in}_<(I_G) \geq 7$ for the lexicographic order $<$ induced by $x_1 > x_2 > \cdots > x_r$. To compute $\mathrm{in}_<(I_G)$, we first find the generators of I_G. Ohsugi and Hibi [7, Lemma 3.1] proved that a toric ideal of a finite simple graph is generated by binomials corresponding to *primitive even closed walks* of the graph. By [7, Lemma 3.2], there are 2 kinds of such walks in G (see Figure 2):

(I) 4-cycles: $\{e_{2i+7}, e_{2i+8}, e_{2j+8}, e_{2j+7}\}$, $0 \leq i < j \leq k-1$;

(II) the 2 triangles with two length 2 walks connecting the triangles: $\{e_2, e_1, e_3, e_{2p+7}, e_{2p+8}, e_4, e_6, e_5, e_{2q+8}, e_{2q+7}\}$, $0 \leq p \leq q \leq k-1$;

Hence I_G is generated by the following binomials:

$$x_{2i+7}x_{2j+8} - x_{2i+8}x_{2j+7}, \qquad 0 \leq i < j \leq k-1,$$
$$x_1x_4x_5x_{2p+7}x_{2q+7} - x_2x_3x_6x_{2p+8}x_{2q+8}, \quad 0 \leq p \leq q \leq k-1.$$

We can prove that the set of these binomials forms a Gröbner basis of I_G by a straightforward application of Buchberger's criterion. Thus $\mathrm{in}_<(I_G)$ is generated by

$$x_{2i+7}x_{2j+8}, \qquad 0 \leq i < j \leq k-1, \tag{2.1}$$
$$x_1x_4x_5x_{2p+7}x_{2q+7}, \quad 0 \leq p \leq q \leq k-1. \tag{2.2}$$

Now we prove $\operatorname{depth} K[\mathbf{x}]/\mathrm{in}_<(I_G) \geq 7$. Let I' be the ideal generated by monomials (2.1). Then

$$\begin{aligned}\mathrm{in}_<(I_G) &= x_1x_4x_5(x_7, x_9, \ldots, x_{2(k-1)+7})^2 + I' \\ &= ((x_7, x_9, \ldots, x_{2(k-1)+7})^2 + I') \cap ((x_1x_4x_5) + I').\end{aligned}$$

We set

$$I_1 = ((x_7, x_9, \ldots, x_{2(k-1)+7})^2 + I'), \qquad I_2 = (x_1x_4x_5) + I'.$$

By the short exact sequence

$$0 \to K[\mathbf{x}]/I_1 \cap I_2 \to K[\mathbf{x}]/I_1 \oplus K[\mathbf{x}]/I_2 \to K[\mathbf{x}]/(I_1 + I_2) \to 0,$$

we may prove that $\operatorname{depth} K[\mathbf{x}]/I_1 \geq 7$, $\operatorname{depth} K[\mathbf{x}]/I_2 \geq 7$, and $\operatorname{depth} K[\mathbf{x}]/(I_1 + I_2) \geq 6$. Since $x_1, x_2, x_3, x_4, x_5, x_6, x_8$ is a $K[\mathbf{x}]/I_1$-regular sequence, we have $\operatorname{depth} K[\mathbf{x}]/I_1 \geq 7$. Because $x_1x_4x_5$ is a $K[\mathbf{x}]/I'$-regular element, we have $\operatorname{depth} K[\mathbf{x}]/I_2 = \operatorname{depth} K[\mathbf{x}]/I' - 1$. Then the sequence $x_1, x_2, \ldots, x_6, x_8, x_{2(k-1)+7}$

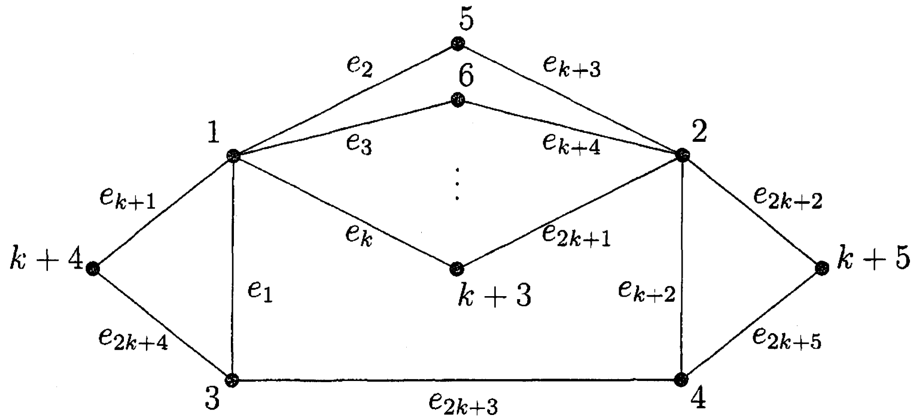

FIGURE 3. The finite graph H_{k+5}

is $K[\mathbf{x}]/I'$-regular and we have $\operatorname{depth} K[\mathbf{x}]/I' \geq 8$. Similarly, we have that $\operatorname{depth} K[\mathbf{x}]/(I_1 + I_2) \geq 6$.

3. The depth of initial ideals of normal edge rings

In this section, we state the outline of our proof of Theorem 1.4. We consider the family of graphs H_{k+5}, $k \geq 1$ of Figure 3. The following lemma is a key in the proof of Theorem 1.4.

Lemma 3.1. *Let $k \geq 1$ be an arbitrary integer and H_{k+5} the graph of Figure 3. Then*

(1) $K[H_{k+5}]$ *is normal;*
(2) $\operatorname{depth} K[\mathbf{x}]/\operatorname{in}_{<_{\text{rev}}}(I_{H_{k+5}}) = 6$;
(3) $\operatorname{depth} K[\mathbf{x}]/\operatorname{in}_{<_{\text{lex}}}(I_{H_{k+5}})$ *is Cohen–Macaulay.*

Once we prove Lemma 3.1, we can prove Theorem 1.4 by a similar way to Theorem 1.2.

The rest of this section is devoted to the proof of Lemma 3.1.

We set $H = H_{k+5}$. First, we find Gröbner bases of I_H with respect to the monomial orders $<_{\text{rev}}, <_{\text{lex}}$. Similarly to the proof of Lemma 2.1, we list the primitive even closed walks of H; there are 5 kinds of such walks:

(I) 4-cycles: $\{e_i, e_{k+1+i}, e_{k+1+j}, e_j\}$, $2 \leq i < j \leq k$;
(II) the 2 triangles with the bridge: $\{e_1, e_{k+1}, e_{2k+4}, e_{2k+3}, e_{k+2}, e_{2k+2}, e_{2k+5}, e_{2k+3}\}$;
(III) 6-cycles: $\{e_{k+1}, e_r, e_{k+1+r}, e_{k+2}, e_{2k+3}, e_{2k+4}\}$, $2 \leq r \leq k$;
(IV) 6-cycles: $\{e_1, e_r, e_{k+1+r}, e_{2k+2}, e_{2k+5}, e_{2k+3}\}$, $2 \leq r \leq k$;
(V) the 2 triangles with two length 2 walks connecting the triangles $\{e_1, e_{2k+4}, e_{k+1}, e_p, e_{k+1+p}, e_{2k+2}, e_{2k+5}, e_{k+2}, e_{k+1+q}, e_q\}$, $2 \leq p \leq q \leq k$;

Similarly to Lemma 2.1, we have the following lemma from a straightforward application of Buchberger's criterion.

Lemma 3.2. *The set of binomials corresponding to primitive even closed walks (I), (II), (III), (IV), and (V) is a Gröbner basis of I_H with respect to both $<_{\text{rev}}$ and $<_{\text{lex}}$.*

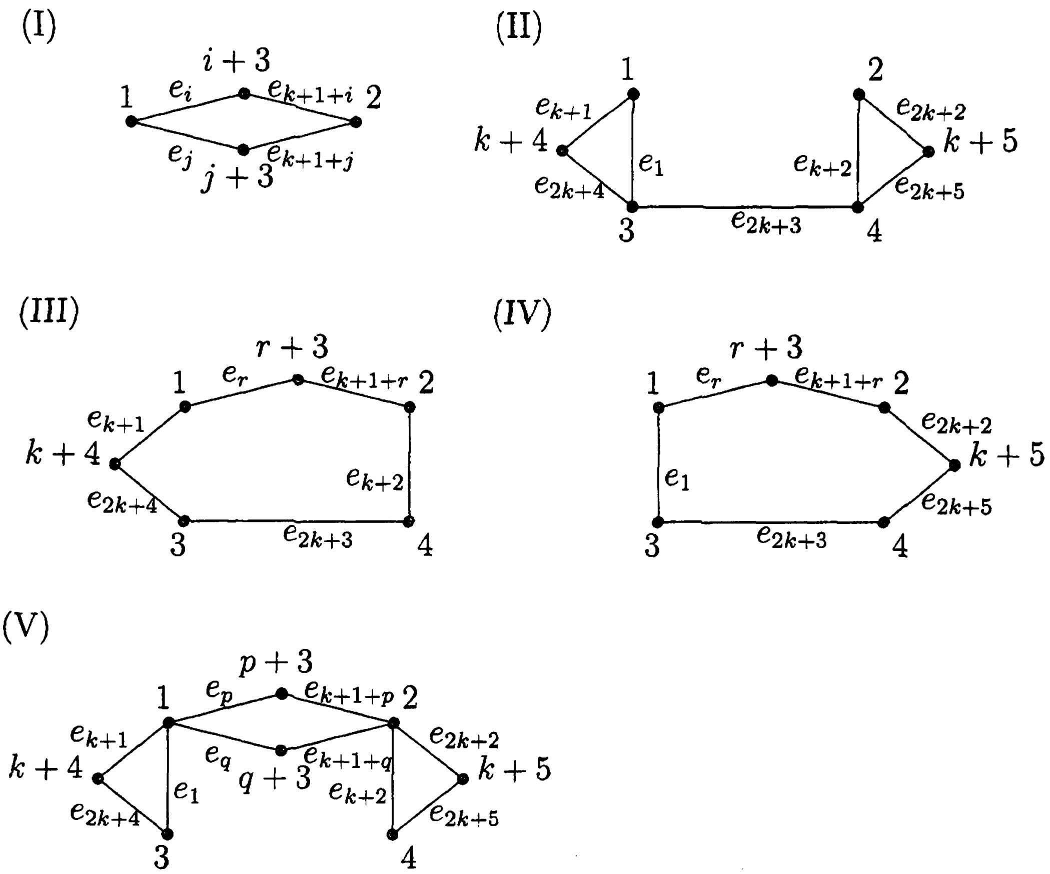

FIGURE 4. Primitive even closed walks of H_{k+5}

By virtue of Lemma 3.2, we obtain the generators of $\text{in}_{<_{\text{rev}}}(I_H)$ and $\text{in}_{<_{\text{lex}}}(I_H)$.

Corollary 3.3. *The initial ideal* $\text{in}_{<_{\text{rev}}}(I_{H_{k+5}})$ *is generated by the following monomials:*

$$x_j x_{k+1+i}, \qquad 2 \leq i < j \leq k,$$

$$x_{k+1} x_{2k+2} x_{2k+3}^2,$$

$$x_{k+1} x_{k+1+r} x_{2k+3}, \ x_r x_{2k+2} x_{2k+3}, \qquad 2 \leq r \leq k,$$

$$x_p x_q x_{k+2} x_{2k+2} x_{2k+4}, \qquad 2 \leq p \leq q \leq k.$$

Corollary 3.4. *The initial ideal* $\text{in}_{<_{\text{lex}}}(I_{H_{k+5}})$ *is generated by the following monomials:*

$$x_i x_{k+1+j}, \qquad 2 \leq i < j \leq k,$$

$$x_1 x_{k+2} x_{2k+4} x_{2k+5},$$

$$x_r x_{k+2} x_{2k+4}, \ x_1 x_{k+1+r} x_{2k+5}, \qquad 2 \leq r \leq k.$$

In particular, $\text{in}_{<_{\text{lex}}}(I_{H_{k+5}})$ *is a squarefree monomial ideal.*

Now we state the outline of our proof of Lemma 3.1.

Proof of Lemma 3.1 (1). Since H satisfies the odd cycle condition, the edge ring $K[H]$ is normal.

Proof of Lemma 3.1 (2). We prove that depth $K[\mathbf{x}]/\operatorname{in}_{<_{\mathrm{rev}}}(I_H) = 6$. Set $I = \operatorname{in}_{<_{\mathrm{rev}}}(I_H)$. Similar to the proof of depth $K[G_{k+6}] = 7$ in the previous section, we will first prove depth $K[\mathbf{x}]/I \leq 6$ and then that depth $K[\mathbf{x}]/I \geq 6$.

To prove depth $K[\mathbf{x}]/I \leq 6$, it is enough to show that pd $K[\mathbf{x}]/I \geq 2k-1$ by the Auslander–Buchsbaum formula. We prove this by showing that the $(2k-1)$th Betti number of $K[\mathbf{x}]/I$ does not vanish. For a monomial ideal, the Betti number is described in terms of the Koszul simplicial complex; the Koszul simplicial complex of I in degree $a \in \mathbb{Z}^r_{\geq 0}$ is defined by

$$\mathbf{K}^a(I) := \{\alpha \in \{0,1\}^r \ : \ \mathbf{x}^{a-\alpha} \in I\}.$$

Lemma 3.5 ([5, Theorem 1.34]). *Let S be a polynomial ring over K and I squarefree monomial ideal of S. Then*

$$\beta_{i+1,a}(S/I) = \dim_K \tilde{H}_{i-1}(\mathbf{K}^a(I); K).$$

We set

$$a = \sum_{j=2}^{k}(\mathbf{e}_j + \mathbf{e}_{k+1+j}) + \mathbf{e}_{k+1} + \mathbf{e}_{2k+2} + 2\mathbf{e}_{2k+3},$$

where $\mathbf{e}_i$ is the ith unit vector of $\mathbb{R}^{2k+5}$. Then we can show $\tilde{H}_{2k-3}(\mathbf{K}^a(I); K) \neq 0$.

The proof of depth $K[\mathbf{x}]/I \geq 6$ is similar to that of depth $K[\mathbf{x}]/\operatorname{in}_<(I_{G_{k+6}}) \geq 7$ in the previous section. We rewrite the ideal I as the intersection of ideals for each of which it is easy to estimate the depth, though the method of division is technical.

Proof of Lemma 3.1 (3). Finally, we prove that $K[\mathbf{x}]/\operatorname{in}_{<_{\mathrm{lex}}}(I_H)$ is Cohen–Macaulay. We set $J = \operatorname{in}_{<_{\mathrm{lex}}}(I_H)$. Since J is a squarefree monomial ideal, J is the Stanley–Reisner ideal I_Δ of some simplicial complex Δ. It is known that the Stanley–Reisner ideal $K[\Delta] = K[\mathbf{x}]/I_\Delta$ is Cohen–Macaulay if Δ is shellable. Our proof is done by showing that Δ is shellable.

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