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<th>Numerical semigroups of double covering type and Hurwitz's problem (Algebras, Languages, Algorithms and Computations)</th>
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<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 1769: 60-65</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2011-10</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/171483">http://hdl.handle.net/2433/171483</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Numerical semigroups of double covering type and Hurwitz's problem

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Abstract

We are interested in Hurwitz's Problem [2] posed in 1893. Buchweitz [1] and Torres [4] gave some essential statements related to this problem in 1980 and 1993 respectively. Moreover, recently significant examples were given by [3]. In this paper we show that solving Hurwitz’s Problem is reduced to finding a necessary and sufficient condition for some kinds of symmetric numerical semigroups to be Weierstrass.

1 Hurwitz's Problem and Buchweitz's Answer

Let \(\mathbb{N}_0\) be the additive monoid of non-negative integers. A submonoid \(H\) of \(\mathbb{N}_0\) is called a numerical semigroup if the complement \(\mathbb{N}_0 \setminus H\) is finite. The cardinality of \(\mathbb{N}_0 \setminus H\) is called the genus of \(H\), denoted by \(g(H)\). In this paper a curve means a projective non-singular curve over an algebraically closed field \(k\) of characteristic 0. Let \(k(C)\) be the field of rational functions on \(C\). For a pointed curve \((C, P)\) we set

\[ H(P) = \{n \in \mathbb{N}_0 \mid \exists f \in k(C) \text{ with } (f)_\infty = nP\}. \]

A numerical semigroup \(H\) is said to be Weierstrass if there is a pointed curve \((C, P)\) with \(H = H(P)\). The following is the original question posed by Hurwitz in which we are interested:

Hurwitz's Problem (Original Version) (1893): Is every numerical semigroup Weierstrass?

This was a long-standing problem. Finally Buchweitz [1] found a non-Weierstrass numerical semigroup in 1980. Here, we will explain his example.

\[ ^1\text{This paper is an extended abstract and the details will appear elsewhere.} \]
We consider the following condition: For a numerical semigroup $H$ and any positive integer $m$ we set

$$L_m(H) = \{l_1 + \cdots + l_m \mid l_i \in \mathbb{N}_0 \setminus H\}.$$ 

We say that the numerical semigroup $H$ satisfies the Buchweitz's condition if $\#L_m(H) \leq (2m - 1)(g(H) - 1)$ for all $m \geq 2$.

**Theorem 1.1** (Buchweitz) *Let $H$ be a numerical semigroup. If it is Weierstrass, then it satisfies the Buchweitz's condition.*

Buchweitz gave a numerical semigroup of genus 16 which does not satisfy the Buchweitz's condition.

## 2 Non-Weierstrass semigroups satisfying the Buchweitz's condition

Theorem 1.1 posed the following problem:

**Hurwitz's Problem (Second Version):** Is a numerical semigroup satisfying the Buchweitz's condition Weierstrass?

But Torres and Stöhr [4] found non-Weierstrass numerical semigroups which satisfy the Buchweitz's condition in 1994. We will introduce their method for constructing such numerical semigroups.

Let $\gamma$ be a non-negative integer. A numerical semigroup $H$ is said to be $\gamma$-hyperelliptic if it satisfies

i) $h_1, h_2, \ldots, h_\gamma$ are even where $H = \{0 < h_1 < h_2 < \cdots\}$,

ii) $h_\gamma = 4\gamma$,

iii) $4\gamma + 2 \in H$.

**Theorem 2.1** (Torres [4]) *Let $H$ be a $\gamma$-hyperelliptic numerical semigroup with $g(H) \geq 6\gamma + 4$. If it is Weierstrass, then there exists a double covering $\pi : C \rightarrow C'$ with a ramification point $P \in C$ such that $H(P) = H.*

**Remark 2.2** For a numerical semigroup $H$ we set

$$d_2(H) = \left\{ \frac{h}{2} \mid h \in H \text{ is even} \right\},$$
which is a numerical semigroup. Let \( \pi : C \to C' \) be a double covering with a ramification point \( P \). Then we have \( H(\pi(P)) = d_2(H(P)) \).

Stöhr and Torres [4] gave \( \gamma \)-hyperelliptic numerical semigroups \( H \) satisfying the Buchweitz's condition with \( g(H) \geq 6\gamma + 4 \) such that \( d_2(H) \) is the non-Weierstrass semigroup given by Buchweitz. By Torres' Theorem these \( H \) are non-Weierstrass numerical semigroups satisfying the Buchweitz's condition.

3 Torres' Question

Torres [5] introduced the following notation including the notion of \( \gamma \)-hyperelliptic numerical semigroup.

Let \( \gamma \) and \( N \) be positive integers with \( N \geq 2 \). A numerical semigroup \( H = \{0 < h_1 < h_2 < \cdots \} \) is said to be of type \((N, \gamma)\) if

i) \( h_1, \ldots, h_\gamma \) are multiples of \( N \),

ii) \( h_\gamma = 2\gamma N \),

iii) \((2\gamma + 1)N \in H\).

In fact, type \((2, \gamma)\) means \( \gamma \)-hyperelliptic.


**Theorem 3.1** (Torres [5]) Let \( H \) be a numerical semigroup of type \((N, \gamma)\) with \( g(H) > (2N - 1)(N\gamma + N - 1) \). If it is Weierstrass, then there exists a covering \( \pi : C \to C' \) of degree \( N \) with a total ramification point \( P \in C \) such that \( H(P) = H \) where the genus of \( C' \) is \( \gamma \).

We also generalize the notion of \( d_2 \) given in the previous section. Let \( N \) be an integer with \( N \geq 2 \). For a numerical semigroup \( H \) we set

\[
d_N(H) = \left\{ \frac{h}{N} \biggm| h \in H \text{ is a multiple of } N \right\}.
\]

Let \( \pi : C \to C' \) be a covering of degree \( N \) with a total ramification point \( P \). Then we have \( H(\pi(P)) \subseteq d_N(H(P)) \). Torres posed the following question in the end of his paper [5].

**Hurwitz's Problem (Torres' Question)**: Let \( H \) be a numerical semigroup satisfying the Buchweitz's condition. Then are the following equivalent?

i) \( H \) is non-Weierstrass.
ii) There exists an integer $N \geq 2$ with $g(H) > (2N-1)(Ng(d_N(H)) + N-1)$ such that $H$ is of type $(N, g(d_N(H)))$ and $d_N(H)$ is non-Weierstrass.

We note that i) comes from ii) by Theorem 3.1.

4 Answer to Torres' Question

The aim of this section is to give a negative answer to Torres' Question in Section 3. We prepare some notation. A numerical semigroup $H$ is said to be of double covering type if there exists a double covering $\pi : C \rightarrow C'$ with a ramification point $P$ such that $H(P) = H$. Using this notation we can restate Theorem 2.1 as follows:

**Theorem 4.1** (Torres) If $H$ is a $\gamma$-hyperelliptic Weierstrass numerical semigroup with $g(H) \geq 6\gamma + 4$, then it is of double covering type.

We found crucial examples which give a negative answer to Torres' Question.

**Theorem 4.2** ([3]) For any $\gamma \geq 5$ there are $\gamma$-hyperelliptic numerical semigroups $H$ satisfying the Buchweitz's condition with $g(H) \geq 6\gamma + 4$ which are not of double covering type such that $d_2(H)$ is Weierstrass. By Theorem 4.1 these $H$ are non-Weierstrass.

In fact, the following examples satisfy the conditions in Theorem 4.2.

**Example 4.1** For any $l \geq 2$ and any odd $n \geq 4l + 3$ the submonoid of $\mathbb{N}_0$ generated by $8, 12, 8l + 2, 8l + 6, n$ and $n + 4$ is a non-Weierstrass numerical semigroup satisfying the Buchweitz's condition. Moreover, for any $N \geq 2$ the semigroup $d_N(H)$ is Weierstrass.

Hence, Torres' Question has been solved negatively.

5 Hurwitz's Problem and Symmetric numerical semigroups

First we introduce one kind of numerical semigroup which plays an important role in rewriting Hurwitz's Problem. For a numerical semigroup $H$ we
set \( c(H) = \min\{n \in \mathbb{N}_0 \mid n + \mathbb{N}_0 \subseteq H\} \), which is called the conductor of \( H \). It is known that \( c(H) \leqq 2g(H) \). A numerical semigroup \( H \) is said to be symmetric if \( c(H) = 2g(H) \). We guess that some kinds of symmetric numerical semigroups hold the key to solving Hurwitz's Problem. In fact, we can prove the following theorem:

**Theorem 5.1** Let \( H \) be a symmetric numerical semigroup of genus \( g \geqq 4g(d_2(H)) \). If \( d_2(H) \) is Weierstrass, then \( H \) is of double covering type. Hence, \( H \) is Weierstrass.

Combining the above theorem with Theorem 2.1 we get the following:

**Corollary 5.2** Let \( H \) be a symmetric numerical semigroup of genus \( g \geqq 6g(d_2(H)) + 4 \). Then the following are equivalent:

i) \( d_2(H) \) is Weierstrass.

ii) \( H \) is Weierstrass.

In these cases, \( H \) is of double covering type.

Using Theorem 3.1 we obtain the following:

**Corollary 5.3** Let \( H \) be a symmetric numerical semigroup of genus \( g \geqq 6g(d_2(H)) + 4 \). Then Torres' Question in Section 3 is solved affirmatively.

We can construct symmetric numerical semigroups from any numerical semigroups as follows:

**Lemma 5.4** Let \( H \) be a numerical semigroup. For \( g \geqq 3g(H) \) we set

\[
S(H, g) = 2H \cup \{2g - 1 - 2t \mid t \in \mathbb{Z}\backslash H\}.
\]

Then \( S(H, g) \) is a symmetric numerical semigroup of genus \( g \).

By Theorem 5.1 and Lemma 5.4 we get the main theorem in this paper.

**Theorem 5.5** Let \( H \) be a numerical semigroup satisfying the Buchweitz's condition and \( g \geqq 3g(H) \). Then the following are equivalent:

i) \( H \) is Weierstrass.

ii) There exists an integer \( g \geqq 6g(H) + 4 \) such that \( S(H, g) \) is Weierstrass, in this case it is of double covering type.
By Theorem 5.5 Hurwitz's Problem is reduced to the following:

**Problem** Find a necessary and sufficient condition for a symmetric numerical semigroup $S$ of sufficiently large genus compared with $g(d_2(S))$, at least $6g(d_2(S)) + 4$, to be Weierstrass.

**References**


