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Numerical semigroups of double covering type and Hurwitz's problem [1]

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Abstract

We are interested in Hurwitz's Problem [2] posed in 1893. Buchweitz [1] and Torres [4] gave some essential statements related to this problem in 1980 and 1993 respectively. Moreover, recently significant examples were given by [3]. In this paper we show that solving Hurwitz's Problem is reduced to finding a necessary and sufficient condition for some kinds of symmetric numerical semigroups to be Weierstrass.

1 Hurwitz's Problem and Buchweitz's Answer

Let $\mathbb{N}_0$ be the additive monoid of non-negative integers. A submonoid H of $\mathbb{N}_0$ is called a *numerical semigroup* if the complement $\mathbb{N}_0\backslash H$ is finite. The cardinality of $\mathbb{N}_0\backslash H$ is called the *genus* of H, denoted by $g(H)$. In this paper a *curve* means a projective non-singular curve over an algebraically closed field k of characteristic 0. Let $k(C)$ be the field of rational functions on C. For a pointed curve (C, P) we set

$$H(P)=\{n\in \mathbb{N}_0 \mid \exists f\in k(C) \text{ with } (f)_{\infty}=nP\}.$$

A numerical semigroup H is said to be *Weierstrass* if there is a pointed curve (C, P) with $H=H(P)$. The following is the original question posed by Hurwitz in which we are interested:

Hurwitz's Problem (Original Version) (1893): *Is every numerical semigroup Weierstrass?*

This was a long-standing problem. Finally Buchweitz [1] found a non-Weierstrass numerical semigroup in 1980. Here, we will explain his example.

[1] This paper is an extended abstract and the details will appear elsewhere.

We consider the following condition: For a numerical semigroup H and any positive integer m we set

$$L_m(H) = \{l_1 + \cdots + l_m \mid l_i \in \mathbb{N}_0 \backslash H\}.$$

We say that the numerical semigroup H *satisfies the Buchweitz's condition* if $\sharp L_m(H) \leqq (2m-1)(g(H)-1)$ for all $m \geqq 2$.

Theorem 1.1 (Buchweitz) *Let H be a numerical semigroup. If it is Weierstrss, then it satisfies the Buchweitz's condition.*

Buchweitz gave a numerical semigroup of genus 16 which does not satisfy the Buchweitz's condition.

2 Non-Weierstrass semigroups satisfying the Buchweitz's condition

Theorem 1.1 posed the following problem:

Hurwitz's Problem (Second Version): *Is a numerical semigroup satisfying the Buchweitz's condition Weierstrass ?*

But Torres and Stöhr [4] found non-Weierstrass numerical semigroups which satisfy the Buchweitz's condition in 1994. We will introduce their method for constructing such numerical semigroups.

Let γ be a non-negative integer. A numerical semigroup H is said to be γ*-hyperelliptic* if it satisfies
i) $h_1, h_2, \ldots, h_\gamma$ are even where $H = \{0 < h_1 < h_2 < \cdots\}$,
ii) $h_\gamma = 4\gamma$,
iii) $4\gamma + 2 \in H$.

Theorem 2.1 (Torres [4]) *Let H be a γ-hyperelliptic numerical semigroup with $g(H) \geqq 6\gamma + 4$. If it is Weierstrass, then there exists a double covering $\pi : C \longrightarrow C'$ with a ramification point $P \in C$ such that $H(P) = H$.*

Remark 2.2 *For a numerical semigroup H we set*

$$d_2(H) = \left\{ \frac{h}{2} \;\middle|\; h \in H \text{ is even} \right\},$$

which is a numerical semigroup. Let $\pi : C \longrightarrow C'$ be a double covering with a ramification point P. Then we have $H(\pi(P)) = d_2(H(P))$.

Stöhr and Torres [4] gave γ-hyperelliptic numerical semigroups H satisfying the Buchweitz's condition with $g(H) \geqq 6\gamma + 4$ such that $d_2(H)$ is the non-Weierstass semigroup given by Buchweitz. By Torres' Thoerem these H are non-Weierstrass numerical semigroups satisfying the Buchweitz's condition.

3 Torres' Question

Torres [5] introduced the following notation including the notion of γ-hyperelliptic numerical semigroup.

Let γ and N be positive integers with $N \geqq 2$. A numerical semigroup $H = \{0 < h_1 < h_2 < \cdots\}$ is said to be *of type* (N, γ) if
i) $h_1, \ldots, h_\gamma$ are multiples of N,
ii) $h_\gamma = 2\gamma N$,
iii) $(2\gamma + 1)N \in H$.
In fact, type $(2, \gamma)$ means γ-hyperelliptic.

Torres [5] generalized Theorem 2.1.

Theorem 3.1 (Torres [5]) *Let H be a numerical semigroup of type (N, γ) with $g(H) > (2N - 1)(N\gamma + N - 1)$. If it is Weierstrass, then there exists a covering $\pi : C \longrightarrow C'$ of degree N with a total ramification point $P \in C$ such that $H(P) = H$ where the genus of C' is γ.*

We also generalize the notion of d_2 given in the previous section. Let N be an integer with $N \geqq 2$. For a numerical semigroup H we set

$$d_N(H) = \left\{ \frac{h}{N} \;\middle|\; h \in H \text{ is a multiple of } N \right\}.$$

Let $\pi : C \longrightarrow C'$ be a covering of degree N with a total ramification point P. Then we have $H(\pi(P)) \subseteqq d_N(H(P))$. Torres posed the following question in the end of his paper [5].

Hurwitz's Problem (Torres' Question): *Let H be a numerical semigroup satisfying the Buchweitz's condition. Then are the following equivalent ?*
i) *H is non-Weierstrass.*

ii) *There exists an integer* $N \geqq 2$ *with* $g(H) > (2N-1)(Ng(d_N(H))+N-1)$ *such that* H *is of type* $(N, g(d_N(H)))$ *and* $d_N(H)$ *is non-Weierstrass.*

We note that i) comes from ii) by Theorem 3.1.

4 Answer to Torres' Question

The aim of this section is to give a negative answer to Torres' Question in Section 3. We prepare some notation. A numerical semigroup H is said to be *of double covering type* if there exists a double covering $\pi : C \longrightarrow C'$ with a ramification point P such that $H(P) = H$. Using this notation we can restate Theorem 2.1 as follows:

Theorem 4.1 (Torres) *If* H *is a* γ*- hyperelliptic Weierstrass numerical semigroup with* $g(H) \geqq 6\gamma + 4$*, then it is of double covering type.*

We found crucial examples which give a negative answer to Torres' Question.

Theorem 4.2 ([3]) *For any* $\gamma \geqq 5$ *there are* γ*-hyperelliptic numerical semigroups* H *satisfying the Buchweitz's condition with* $g(H) \geqq 6\gamma + 4$ *which are not of double covering type such that* $d_2(H)$ *is Weierstrass. By Theorem* 4.1 *these* H *are non-Weierstrass.*

In fact, the following examples satisfy the conditions in Theorem 4.2.

Example 4.1 For any $l \geqq 2$ and any odd $n \geqq 4l + 3$ the submonoid of $\mathbb{N}_0$ generated by $8, 12, 8l + 2, 8l + 6, n$ and $n + 4$ is a non-Weierstrass numerical semigroup satisfying the Buchweitz's condition. Moreover, for any $N \geqq 2$ the semigroup $d_N(H)$ is Weierstrass.

Hence, Torres' Question has been solved negatively.

5 Hurwitz's Problem and Symmetric numerical semigroups

First we introduce one kind of numerical semigroup which plays an important role in rewriting Hurwitz's Problem. For a numerical semigroup H we set $c(H) = \min\{n \in \mathbb{N}_0 \mid n + \mathbb{N}_0 \subseteq H\}$, which is called the *conductor* of H. It is known that $c(H) \leqq 2g(H)$. A numerical semigroup H is said to be *symmetric* if $c(H) = 2g(H)$. We guess that some kinds of symmetric numerical semigroups hold the key to solving Hurwitz's Problem. In fact, we can prove the following theorem:

Theorem 5.1 *Let H be a symmetric numerical semigroup of genus $g \geqq 4g(d_2(H))$. If $d_2(H)$ is Weierstrass, then H is of double covering type. Hence, H is Weierstrass.*

Combining the above theorem with Theorem 2.1 we get the following:

Corollary 5.2 *Let H be a symmetric numerical semigroup of genus $g \geqq 6g(d_2(H)) + 4$. Then the following are equivalent:*
i) *$d_2(H)$ is Weierstrass.*
ii) *H is Weierstrass.*
In these cases, H is of double covering type.

Using Theorem 3.1 we obtain the following:

Corollary 5.3 *Let H be a symmetric numerical semigroup of genus $g \geqq 6g(d_2(H)) + 4$. Then Torres' Question in Section 3 is solved affirmatively.*

We can construct symmetric numerical semigroups from any numerical semigroups as follows:

Lemma 5.4 *Let H be a numerical semigroup. For $g \geqq 3g(H)$ we set*

$$S(H, g) = 2H \cup \{2g - 1 - 2t \mid t \in \mathbb{Z} \setminus H\}.$$

Then $S(H, g)$ is a symmetric numerical semigroup of genus g.

By Theorem 5.1 and Lemma 5.4 we get the main theorem in this paper.

Theorem 5.5 *Let H be a numerical semigroup satisfying the Buchweitz's condition and $g \geqq 3g(H)$. Then the following are equivalent:*
i) *H is Weierstrass.*
ii) *There exists an integer $g \geqq 6g(H) + 4$ such that $S(H, g)$ is Weierstrass, in this case it is of double covering type.*

By Theorem 5.5 Hurwitz's Problem is reduced to the following:

Problem Find a necessary and sufficient condition for a symmetric numerical semigroup S of sufficiently large genus compared with $g(d_2(S))$, at least $6g(d_2(S)) + 4$, to be Weierstrass.

References

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