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THE ORBIT DECOMPOSITION AND ORBIT TYPE OF THE AUTOMORPHISM GROUP OF CERTAIN EXCEPTIONAL JORDAN ALGEBRA AND ITS APPLICATIONS

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ABSTRACT. Let $\mathcal{J}^{1}$ be the real form of complex simple Jordan algebra with the automorphism group $F_{4(-20)}$. The classification of $F_{4(-20)}$-orbits and the stabilizer groups of $F_{4(-20)}$-orbit on $\mathcal{J}^{1}$ are determined. As applications, for $F_{4(-20)}$, the Bruhat and Gauss decomposition, the Iwasawa decomposition and also the Iwasawa decomposition with respect to $K_{\epsilon}$ in sense of T. Oshima and J. Sekiguchi are given concretely.

1. THE EXCEPTIONAL JORDAN ALGEBRA $\mathcal{J}^{1}$ AND THE AUTOMORPHISM GROUP $F_{4(-20)}$.

Denote the cartesian $n$-power of a set $X$ as $X^{n} := X \times \cdots \times X$ ($n$ times). For $F = \mathbb{R}$ or $\mathbb{C}$, let $V$ be a $F$-linear space, $GL_{F}(V)$ the group of $F$-linear automorphism of $V$, and $End_{F}(V)$ the linear space of $F$-linear endomorphisms on $V$. A subset $C$ is said to be a cone if $x \in V$ and $\lambda > 0$ imply that $\lambda x \in C$. For a mapping $f : V \to V$ and $c \in F$, put $V_{f,c} := \{v \in V \mid f(v) = cv\}$ and $V_{f} := V_{f,1}$. Let $G$ be a subgroup of $GL_{F}(V)$, $\phi$ an automorphism on $G$ and $v, v_{i} \in V$. Then denote the subgroups $G^{\phi} := \{g \in G \mid \phi g = g\}$, the stabilizer of $v$ as $G_{v} := \{g \in G \mid gv = v\}$ and $G_{v_{1}, \ldots, v_{n}} := \cap_{i=1}^{n}G_{v_{i}}$. And denote the $G$-orbit of $v$ as $\text{Orb}_{G}(v) := \{gv \mid g \in G\}$.

For $H$ (Quaternions), the $O$ (Octonions) is defined as $O := H \oplus He = \{m + ae \mid m, a \in H\}$, the conjugation, the multiplication, the inner product and the quadratic form as $\overline{m + ae} := \overline{m} - ae$, $(m + ae)(n + be) := (mn - ba) + (a\overline{n} + b\overline{m})e$ (especially, $e^{2} = -1$), $(m + ae)(n + be) := (m|n) + (a|b)$ and $n(x) := (x|x)$, respectively. For $x \in O$, the scalar part and the vector part of $x$ and the set $\text{Im}O$ are defined by $\text{Re}(x) := \frac{1}{2}(x + \overline{x})$, $\text{Im}(x) := \frac{1}{2}(x - \overline{x})$ and $\text{Im}O := \{x \in O \mid \overline{x} = -x\}$, respectively.

For $\xi = (\xi_{1}, \xi_{2}, \xi_{3}) \in \mathbb{R}^{3}$ and $x = (x_{1}, x_{2}, x_{3}) \in O^{3}$, denote

$$h^{1}(\xi; x) := \begin{pmatrix} \xi_{1} & \sqrt{-1}x_{3} & \sqrt{-1}\overline{x}_{2} \\ \sqrt{-1}\overline{x}_{3} & \xi_{2} & x_{1} \\ \sqrt{-1}x_{2} & \overline{x}_{1} & \xi_{3} \end{pmatrix}$$
and
\[ J^1 := \{ h^1(\xi; x) | \xi \in \mathbb{R}^3, x \in O^3 \}. \]

The **Jordan product** is defined by
\[ X \circ Y := \frac{1}{2}(XY + YX) \quad \text{for } X, Y \in J^1. \]

Then the identity element of the Jordan product is \( E := \text{diag}(1, 1, 1) \).

For \( X = h^1(\xi_1, \xi_2, \xi_3; x_1, x_2, x_3) \) and \( Y = h^1(\eta_1, \eta_2, \eta_3; y_1, y_2, y_3) \in J^1 \),
the **trace** and the **inner product** are defined as
\[ \text{tr}(X) = \xi_1 + \xi_2 + \xi_3, \]
\[ (X|Y) = \text{tr}(X \circ Y) = \left( \sum_{k=1}^{3} \xi_k \eta_k \right) + 2(x_1|y_1) - 2(x_2|y_2) - 2(x_3|y_3), \]
respectively. Hereafter we denote \( X \times Y := X \times X \). The **characteristic polynomial** \( \Phi_X(\lambda) \) of \( X \in J^1 \) is defined by
\[ \Phi_X(\lambda) := \det(\lambda E - X) = \frac{1}{3}(\lambda E - X|\lambda E - X)^x_2) \]
\[ = \lambda^3 - \text{tr}(X)\lambda^2 + \text{tr}(X^2)\lambda - \det(X). \]

For \( i \in \{1, 2, 3\} \) and \( x \in O \), denote
\[ E_i := h^1(\delta_{i1}, \delta_{i2}, \delta_{i3}; 0, 0, 0), \quad F^1_i(x) := h^1(0, 0, 0; \delta_{i1}x, \delta_{i2}x, \delta_{i3}x), \]
\[ P^+ := h^1(1, -1, 0; 0, 0, 1), \quad P^- := h^1(-1, 1, 0; 0, 0, 1), \]
\[ Q^+(x) := h^1(0, 0, 0; x, \overline{x}, 0), \quad Q^-(x) := h^1(0, 0, 0; x, -\overline{x}, 0) \]
where \( \delta_{ij} \) is the Kronecker's delta. Then \( X \in J^1 \) can be expressed by
\[ X = h^1(\xi_1, \xi_2, \xi_3; x_1, x_2, x_3) = \sum_{i=1}^{3} (\xi_i E_i + F^1_i(x_i)) \]
for some \( \xi_i \in \mathbb{R} \) and \( x_i \in O \), and denote
\[ (X)_{E_i} := \xi_i = (X|E_i), \quad (X)_{F^1_i} := x_i. \]

**Lemma 1.1.** (cf. [25, Lemma 1.6 with \( J^1 \subset J^C \)]) For all \( X \in J^1 \),
\[ (X^\times_2)^x_2 = \det(X)X. \]
The linear Lie group $F_{4(-20)}$ is defined by

$$F_{4(-20)} := \text{Aut}(\mathcal{J}^1) = \{g \in GL_{\mathbb{R}}(\mathcal{J}^1) \mid g(X \circ Y) = gX \circ gY\}.$$ 

The following result is proved after [34, 35], [39, Lemma 2.1.2, Proposition 2.1.3] and [33, p.159, Proposition 5.9.4, §5.10].

**Proposition 1.2.** (cf. [24, Theorem 1.4], [25, Proposition 0.1(1)])

$$F_{4(-20)} = \{g \in F_{4(-20)} \mid \text{tr}(gX) = \text{tr}(X)\} = \{g \in GL_{\mathbb{R}}(\mathcal{J}^1) \mid \det(gX) = \det(X), gE = E\} = \{g \in GL_{\mathbb{R}}(\mathcal{J}^1) \mid \Phi_{gX}(\lambda) = \Phi_X(\lambda)\} = \{g \in GL_{\mathbb{R}}(\mathcal{J}^1) \mid g(E \times Y) = gE \times gY\}.$$ 

A characteristic root of $X \in \mathcal{J}^1$ is said to be a solution of $\Phi_X(\lambda) = 0$ over $\mathbb{C}$. By Proposition 1.2, the trace, the inner product, the determinant, the identity element, the cross product and the characteristic polynomial are invariant under the action of $F_{4(-20)}$. Moreover the set of all characteristic roots and those multiplicities are invariant under the action of $F_{4(-20)}$.

**Proposition 1.3.** ([39]) $F_{4(-20)}$ is a connected and simply connected non-compact simple real Lie group of type $F_{4(-20)}$.

2. **The Orbit Decomposition of $F_{4(-20)}$-Orbits on $\mathcal{J}^1$.**

The subset $\mathcal{H} \subset \mathcal{J}^1$ and the Cayley hyperbolic planes $\mathcal{H}(O)$ and $\mathcal{H}'(O)$ of $\mathcal{J}^1$ are defined as

$$\mathcal{H} := \{X \in \mathcal{J}^1 \mid X^2 = 0, \text{tr}(X) = 1\},$$

$$\mathcal{H}(O) := \{X \in \mathcal{J}^1 \mid X^2 = 0, \text{tr}(X) = 1, (X|E_1) \geq 1\},$$

$$\mathcal{H}'(O) := \{X \in \mathcal{J}^1 \mid X^2 = 0, \text{tr}(X) = 1, (X|E_1) \leq 0\},$$

respectively.

**Proposition 2.1.** (cf. [24, Propositions 1.6(1) and 2.10])

1. $\mathcal{H} = \mathcal{H}(O) \bigcup \mathcal{H}'(O)$.
2. $\mathcal{H}(O) = \text{Orb}_{F_{4(-20)}}(E_1)$.
3. $\mathcal{H}'(O) = \text{Orb}_{F_{4(-20)}}(E_2) = \text{Orb}_{F_{4(-20)}}(E_3)$.

The cone $\mathcal{N}$ of $\mathcal{J}^1$ is defined by

$$\mathcal{N} = \{X \in \mathcal{J}^1 \mid \text{tr}(X) = \text{tr}(X^2) = \det(X) = 0\}.$$
Then using Lemma 1.1, \( \mathcal{N} \) contains the following cones:

\[
\mathcal{N}_{1}(O) := \{X \in \mathcal{J}^{1} | X^{x2} = 0, \text{tr}(X) = 0, X \neq 0\},
\]
\[
\mathcal{N}_{1}^{+}(O) := \{X \in \mathcal{J}^{1} | X^{x2} = 0, \text{tr}(X) = 0, (X|E_{1}) > 0\},
\]
\[
\mathcal{N}_{1}^{-}(O) := \{X \in \mathcal{J}^{1} | X^{x2} = 0, \text{tr}(X) = 0, (X|E_{1}) < 0\},
\]
\[
\mathcal{N}_{2}(O) := \{X \in \mathcal{J}^{1} | \text{tr}(X) = \text{tr}(X^{x2}) = \det(X) = 0, X^{x2} \neq 0\},
\]
\[
\mathcal{N}_{0}(O) := \{0\}.
\]

**Proposition 2.2.** (cf. [24, Propositions 1.6(2), 2.10(2) and 4.3(4)])

(1) \( \mathcal{N}_{1}(O) = \mathcal{N}_{1}^{+}(O) \setminus \mathcal{N}_{1}^{-}(O) \).

(2) \( \mathcal{N} = \mathcal{N}_{0}(O) \prod \mathcal{N}_{1}^{+}(O) U \mathcal{N}_{1}^{-}(O) \).

(3) \( \mathcal{N}_{1}^{+}(O) = \text{Orb}_{F_{4(-20)}}(P^{+}) \).

(4) \( \mathcal{N}_{1}^{-}(O) = \text{Orb}_{F_{4(-20)}}(P^{-}) \).

(5) \( \mathcal{N}_{2}(O) = \text{Orb}_{F_{4(-20)}}(Q^{+}(1)) \).

For \( X \in \mathcal{J}^{1} \), denote \( L^{x}(X) \in \text{End}_{\mathbb{R}}(\mathcal{J}^{1}) \) as

\[
L^{x}(X)Y := X \times Y \quad \text{for} \ Y \in \mathcal{J}^{1}
\]

and the minimal space of \( X \) as

\[
V_{X} := \{aX^{x2} + bX + cE | a, b, c \in \mathbb{R}\}.
\]

Then \( V_{X} \) is closed under the cross product ([25, Lemma 1.6(3)]). And for \( \lambda_{0} \in \mathbb{R} \), denote the elements \( p(X), E_{X,\lambda_{0}}, W_{X,\lambda_{0}} \in V_{X} \) as

\[
p(X) := X - \frac{1}{3} \text{tr}(X)E,
\]
\[
E_{X,\lambda_{0}} := \frac{1}{\text{tr}((\lambda_{0}E - X)^{x2})}(\lambda_{0}E - X)^{x2},
\]
\[
W_{X,\lambda_{0}} := X - (\lambda_{0}E_{X,\lambda_{0}} + \frac{\text{tr}(X) - \lambda_{0}}{2}(E - E_{X,\lambda_{0}}))
\]

respectively. If \( E_{X,\lambda_{1}} \) is well-defined (ie, \( \text{tr}((\lambda_{1}E - X)^{x2}) \neq 0 \)), then

\[
X = \lambda_{0}E_{X,\lambda_{0}} + \frac{\text{tr}(X) - \lambda_{0}}{2}(E - E_{X,\lambda_{0}}) + W_{X,\lambda_{0}}.
\]

For \( r \in \mathbb{R} \), consider the eigenspace \( \mathcal{J}^{1}_{L^{x}(2E_{\lambda_{1}}),r} \). Then we have the following two lemmas (cf. [24]):

**Lemma 2.3.** Let \( X \in \mathcal{J}^{1} \). Then for all \( g \in F_{4(-20)} \),

\[
g(V_{X}) = V_{gX}, \ gE_{X,\lambda_{1}} = E_{gX,\lambda_{1}}, \ gW_{X,\lambda_{1}} = W_{gX,\lambda_{1}}, \ gp(X) = p(gX).
\]

**Lemma 2.4.** Assume that \( X \in \mathcal{J}^{1} \) has a characteristic root \( \lambda_{1} \in \mathbb{R} \) of multiplicity \( 1 \).

(1) \( E_{X,\lambda_{1}} \) is well-defined (ie, \( \text{tr}((\lambda_{1}E - X)^{x2}) \neq 0 \)), and \( E_{X,\lambda_{1}} \in \mathcal{H} \cap V_{X} \).
(2) $E_{X,\lambda_1} \in J^1_{L^*(2E_{X,\lambda_1}),0}$, $E - E_{X,\lambda_1} \in J^1_{L^*(2E_{X,\lambda_1}),1} \cap V_X$ and $W_{X,\lambda_1} \in J^1_{L^*(2E_{X,\lambda_1}),-1} \cap V_X$.

**Main Theorem 1.** ( $F_{4(-20)}$-orbits on $\mathcal{J}^1$ [24, Main Theorem] )

$F_{4(-20)}$-orbits on $\mathcal{J}^1$ are classified as follows.

(I) Assume that $X \in \mathcal{J}^1$ admits the characteristic roots $\lambda_1 > \lambda_2 > \lambda_3$. Then there exists the unique $i \in \{1, 2, 3\}$ such that $\mathcal{H}(O) \cap V_X = \{E_{X,\lambda_i}\}$ and $\mathcal{H}'(O) \cap V_X = \{E_{X,\lambda_{i+1}}, E_{X,\lambda_{i+2}}\}$ where $i, i+1, i+2$ are counted modulo 3. In this case, $X$ can be transformed to one of the following canonical forms by $F_{4(-20)}$.

<table>
<thead>
<tr>
<th>Cases</th>
<th>The canonical forms of $X$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $E_{X,\lambda_1} \in \mathcal{H}(O)$</td>
<td>$\text{diag}(\lambda_1, \lambda_2, \lambda_3)$</td>
</tr>
<tr>
<td>2. $E_{X,\lambda_2} \in \mathcal{H}(O)$</td>
<td>$\text{diag}(\lambda_2, \lambda_3, \lambda_1)$</td>
</tr>
<tr>
<td>3. $E_{X,\lambda_3} \in \mathcal{H}(O)$</td>
<td>$\text{diag}(\lambda_3, \lambda_1, \lambda_2)$</td>
</tr>
</tbody>
</table>

(II) Assume that $X \in \mathcal{J}^1$ admits the characteristic roots $\lambda_1 \in \mathbb{R}$, $p \pm \sqrt{-1}q$ with $p \in \mathbb{R}$ and $q > 0$. Then $X$ can be transformed to the following canonical form by $F_{4(-20)}$.

<table>
<thead>
<tr>
<th>The characteristic roots of $X$</th>
<th>The canonical form of $X$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_1 \in \mathbb{R}$, $p \pm \sqrt{-1}q$</td>
<td>$\text{diag}(p, p, \lambda_1) + F_3^1(q)$</td>
</tr>
</tbody>
</table>

(III) Assume that $X \in \mathcal{J}^1$ admits the characteristic roots $\lambda_1$ of multiplicity 1 and $\lambda_2$ of multiplicity 2. Then $W_{X,\lambda_1} \in \mathcal{N}_1(O) \coprod \{0\}$. In this case, $X$ can be transformed to one of the following canonical forms by $F_{4(-20)}$.

<table>
<thead>
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</tr>
</thead>
<tbody>
<tr>
<td>5. $E_{X,\lambda_1} \in \mathcal{H}(O)$</td>
<td>$\text{diag}(\lambda_1, \lambda_2, \lambda_2)$</td>
</tr>
<tr>
<td>6. $E_{X,\lambda_1} \in \mathcal{H}'(O)$, $W_{X,\lambda_1} = 0$</td>
<td>$\text{diag}(\lambda_2, \lambda_2, \lambda_1)$</td>
</tr>
<tr>
<td>7. $W_{X,\lambda_1} \in \mathcal{N}^+_1(O)$</td>
<td>$\text{diag}(\lambda_2, \lambda_2, \lambda_1) + P^+$</td>
</tr>
<tr>
<td>8. $W_{X,\lambda_1} \in \mathcal{N}_1^-(O)$</td>
<td>$\text{diag}(\lambda_2, \lambda_2, \lambda_1) + P^-$</td>
</tr>
</tbody>
</table>

(IV) Assume that $X \in \mathcal{J}^1$ admits the characteristic root of multiplicity 3. Then $p(X) \in \mathcal{N}$. In this case, $X$ can be transformed to one of the following canonical forms by $F_{4(-20)}$.

<table>
<thead>
<tr>
<th>Cases</th>
<th>The canonical form of $X$</th>
</tr>
</thead>
<tbody>
<tr>
<td>9. $p(X) = 0$</td>
<td>$\frac{1}{3}\text{tr}(X)E$</td>
</tr>
<tr>
<td>10. $p(X) \in \mathcal{N}_1^+(O)$</td>
<td>$\frac{2}{3}\text{tr}(X)E + P^+$</td>
</tr>
<tr>
<td>11. $p(X) \in \mathcal{N}_1^-(O)$</td>
<td>$\frac{1}{3}\text{tr}(X)E + P^-$</td>
</tr>
<tr>
<td>12. $p(X) \in \mathcal{N}_2(O)$</td>
<td>$\frac{2}{3}\text{tr}(X)E + Q^+(1)$</td>
</tr>
</tbody>
</table>

(V) By $F_{4(-20)}$, the above canonical forms cannot be transformed from each other.
3. The stabilizer groups of Spin group type.

Let $G$ be a topological group with identity element 1. Then $G^0$ denotes the identity connected component. Denote the quadratic form $Q_{p,q}$ on $\mathbb{R}^{p,q}$ as $Q_{p,q}(x):=-(x_1^2+\cdots+x_p^2)+(x_{p+1}^2+\cdots+x_{p+q}^2)$ for $x=(x_1,\ldots,x_{p+q})$, the quadratic space as $(\mathbb{R}^{p,q},Q_{p,q})$, the set of all orthogonal transformations as $O(\mathbb{R}^{p,q},Q_{p,q})$ and $SO(\mathbb{R}^{p,q},Q_{p,q}) := \{g \in O(\mathbb{R}^{p,q},Q_{p,q})|\det(g)=1\}$ where $\det(g)$ is the determinant of $g \in \text{End}_\mathbb{R}(\mathbb{R}^{p,q})$. Then $O(\mathbb{R}^{p,q},Q_{p,q})$ and $SO(\mathbb{R}^{p,q},Q_{p,q})$ are linear Lie groups. Denote the quadratic form $Q$ on $J^1$ as $Q(X) := -\text{tr}(X^{\times 2})$ for $X \in J^1$ and consider the subspace $J^1_{0,9}$, $J^1_{8,1}$ and $J^1_{7,1}$ of eigenspace of $L^x(2E_i)$ with eigenvalue $-1$ as

\[
J^1_{0,9} := J^1_{L^x(2E_1),-1}, \quad J^1_{8,1} := J^1_{L^x(2E_3),-1}, \quad J^1_{7,1} := \{X \in J^1_{8,1} | (F^1_3(1)|X)=0\}.
\]

Then $J^1_{0,9} = \{\xi(E_2-E_3)+F^1_1(x)|\xi \in \mathbb{R}, x \in O\}$, $J^1_{8,1} = \{\xi(E_1-E_2)+F^1_3(x)|\xi \in \mathbb{R}, x \in O\}$ and $J^1_{7,1} = \{\xi(E_1-E_2)+F^1_3(x)|\xi \in \mathbb{R}, x \in \text{Im}O\}$.

Since $Q(\xi(E_2-E_3)+F^1_1(x)) = \xi^2+n(x)$ and $Q(\xi(E_1-E_2)+F^1_3(x)) = \xi^2-n(x)$, we see that $(J^1_{0,9},Q)$, $(J^1_{8,1},Q)$ and $(J^1_{7,1},Q)$ are isomorphic to $(\mathbb{R}^8,|\cdot|^9)$, $(\mathbb{R}^8,1,Q_{8,1})$ and $(\mathbb{R}^7,1,Q_{7,1})$, respectively. Moreover, denote

\[
S^8 := \{X \in J^1_{0,9} | Q(X)=1\}, \quad S^8_{+} := \{X \in J^1_{8,1} | Q(X)=1, (E_3|X)>0\}, \quad S^7_{+} := \{X \in J^1_{7,1} | Q(X)=1, (E_3|X)>0\}.
\]

From now on, the groups $SO(8)$ and $SO(7)$ are identified with the groups $SO(8) = \{g \in \text{GL}_\mathbb{R}(O) | (gx|gy)=(x|y), \det(g)=1\}$ and $SO(7) = \{g \in SO(8) | g1=1\}$, respectively. The subgroup $T(O)$ of $SO(8)^3$ is defined as

\[
T(O) := \{(g_1,g_2,g_3) \in SO(8)^3 | (g_1x)(g_2y) = g_3(xy) \text{ for all } x,y \in O\}
\]

(cf. [2], [9, (2.4.6)], [22], [33], [43]), and the subgroup $\tilde{D}_4$ of $SO(8)^3$ as

\[
\tilde{D}_4 := \{(g_1,g_2,g_3) \in SO(8)^3 | (g_1x)(g_2y) = \overline{g_3(x\overline{y})} \text{ for all } x,y \in O\}.
\]

For $i \in \{1,2,3\}$, the homomorphism $p_i : \tilde{D}_4 \to SO(8)$ is defined by

\[
p_i(g_1,g_2,g_3) := g_i \text{ for } (g_1,g_2,g_3) \in \tilde{D}_4.
\]

The subgroup $\tilde{B}_3$ of $\tilde{D}_4$ is defined as

\[
\tilde{B}_3 := \{(g_1,g_2,g_3) \in \tilde{D}_4 | g_31=1\}
\]

and the homomorphism $q : \tilde{B}_3 \to SO(7)$ as $q := p_3|\tilde{B}_3$. Denote $\epsilon_i(j) := (-1)^{1+\delta_{ij}}$ where $\delta_{ij}$ is the Kronecker delta. Thus if $i=j$, then $\epsilon_i(j) = 1$, else $\epsilon_i(j) = -1$. 

Lemma 3.1.
(1) ([43, Theorems 1.15.1 and 1.16.1]) $\tilde{D}_4$ and $\tilde{B}_3$ are connected.
(2) (The principle of triality: [2], [9, (2.4.6)], cf. [43, Theorem 1.14.2])
The following sequence is exact:
$$1 \rightarrow \{(1,1,1), (\epsilon_i(1), \epsilon_i(2), \epsilon_i(3))\} \rightarrow \tilde{D}_4 \xrightarrow{p_i} \text{SO}(8) \rightarrow 1.$$ 
(3) ([43, Theorem 1.15.2])
The following sequence is exact:
$$1 \rightarrow \{(1,1,1), (-1,-1,1)\} \rightarrow \tilde{B}_3 \xrightarrow{q} \text{SO}(7) \rightarrow 1.$$ 

By Lemma 3.1, we see that $\tilde{D}_4$ is connected and a two-fold covering group of SO(8), and $\tilde{B}_3$ is connected and a two-fold covering group of SO(7). So denote
$$\text{Spin}(8) := \tilde{D}_4, \quad \text{Spin}(7) := \tilde{B}_3.$$ 

Lemma 3.2. ([22], cf. [43, Theorem 2.7.1], [26, lemma 3.2])
The following homomorphisms are group isomorphisms:
(1) $\varphi_0 : \text{Spin}(8) \rightarrow (F_{4(-20)})_{E_1,E_2,E_3};$
$$\varphi_0(g_1,g_2,g_3)(\sum (\xi_i E_i + F^1_i(x_i))) = \sum (\xi_i E_i + F^1_i(g_i x_i)),$$
(2) $\varphi_0 : \text{Spin}(7) \rightarrow (F_{4(-20)})_{E_1,E_2,F_3^1(1)};\quad \varphi_0 \simeq \varphi_0|_{\text{Spin}(7)}.$$

Hereafter $\text{Spin}(8)$ and $\text{Spin}(7)$ are identified with $(F_{4(-20)})_{E_1,E_2,E_3}$ and $(F_{4(-20)})_{E_1,E_2,F_3^1(1)}$ via $\varphi_0$, respectively.

Lemma 3.3. ([38], [39], cf. [26, Lemmas 3.9 and 3.12])

(1) $(F_{4(-20)})_{E_1}/\text{Spin}(8) \simeq S^8_1,$
(2) $(F_{4(-20)})_{E_3}/\text{Spin}(8) \simeq S^8_{+1},$
(3) $(F_{4(-20)})_{F_3^1(1)}/\text{Spin}(7) \simeq S^7_{+1}.$

Furthermore, $(F_{4(-20)})_{E_1}, (F_{4(-20)})_{E_3}$ and $(F_{4(-20)})_{F_3^1(1)}$ are connected.

Lemma 3.4. ([38], [39], cf. [26, Lemmas 3.10 and 3.13])

(1) The following sequence is exact.
$$1 \rightarrow \mathbb{Z}_2 \rightarrow (F_{4(-20)})_{F_3^1(1)} \xrightarrow{f} O^0(J^1_{7,1},Q) \rightarrow 1$$
where $f(g) = g|_{J^1_{7,1}}.$
(2) The following sequence is exact.
$$1 \rightarrow \mathbb{Z}_2 \rightarrow (F_{4(-20)})_{E_1} \xrightarrow{f} \text{SO}(J^1_{0,9},Q) \rightarrow 1$$
where $f(g) = g|_{J^1_{0,9}}.$
(3) The following sequence is exact.
$$1 \rightarrow \mathbb{Z}_2 \rightarrow (F_{4(-20)})_{E_3} \xrightarrow{f} O^0(J^1_{8,1},Q) \rightarrow 1$$
where $f(g) = g|_{J^1_{8,1}}.$
Since Lemmas 3.3, 3.4 and \( \pi_1(\SO(n)) = \mathbb{Z}_2 = \pi_1(\O^0(n, 1)) \) \((n \geq 3)\), we can put

\[
\Spin^0(7, 1) := (F_{4(-20)})_{F_3^0(1)}, \quad \Spin(9) := (F_{4(-20)})_{E_1},
\]

\[
\Spin^0(8, 1) := (F_{4(-20)})_{E_3} \cong (F_{4(-20)})_{E_2}.
\]

The element \( \sigma_i \in F_{4(-20)} \) is defined by

\[
\sigma_i \left( \sum_{j=1}^{3} (\xi_j E_j + F_j^1(x_j)) \right) := \sum_{j=1}^{3} (\xi_j E_j + \epsilon_i(j) F_j^1(x_j))
\]

[38] (cf. [39]) where indices are counted modulo 3. The involutive automorphism \( \tilde{\sigma}_i \) of \( F_{4(-20)} \) is defined as

\[\tilde{\sigma}_i(g) := \sigma_i g \sigma_i \quad \text{for} \quad g \in F_{4(-20)},\]

and the subgroup \( K \) of \( F_{4(-20)} \) as

\[K := (F_{4(-20)})_{\sigma_1} = \{ g \in F_{4(-20)} \mid \sigma_1 g = g \sigma_1 \}.
\]

**Proposition 3.5.** ([38, Theorem 8],[39, Theorem 2.4.4], cf. [26, Proposition 3.16]).

1. \( (F_{4(-20)})_{\tilde{\sigma}_1} = (F_{4(-20)})_{E_1} \).
2. \( K = (F_{4(-20)})_{E_1} = \Spin(9). \)
3. \( (F_{4(-20)})_{\overline{\sigma}_2} = (F_{4(-20)})_{E_2} \cong \Spin^0(8, 1). \)

4. **THE STABILIZER GROUPS OF SEMIDIRECT PRODUCT GROUP TYPE.**

Denote the Lie algebras \( \mathfrak{o}(8) = \text{Lie}(\O(8)) \) and \( f_{4(-20)} = \text{Lie}(F_{4(-20)}) \). Since \( \varphi_0 : D_4 \rightarrow (F_{4(-20)})_{E_1,E_2,E_3} \) is an isomorphism by Lemma 3.2, the Lie subalgebra \( \mathfrak{o}_4 \) of \( f_{4(-20)} \) is defined by

\[
\mathfrak{o}_4 := \left\{ d\varphi_0(D_1, D_2, D_3) \mid \begin{array}{c}
(D_1, D_2, D_3) \in \mathfrak{o}(8)^3, \\
(D_1 x)y + x(D_2 y) = D_3(xy)
\end{array} \right\}.
\]

Then

\[d\varphi_0(D_1, D_2, D_3)(\sum (\xi_i E_i + F_i^1(x_i))) = \sum F_i^1(D_i x_i).
\]

For \( a \in \O \), denote

\[
A_1^1(a) := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & -\overline{a} & 0 \end{pmatrix}, \quad A_2^1(a) := \begin{pmatrix} 0 & 0 & \sqrt{-1}a \\ 0 & 0 & 0 \\ -\sqrt{-1}a & 0 & 0 \end{pmatrix},
\]

\[
A_3^1(a) := \begin{pmatrix} 0 & -\sqrt{-1}a & 0 \\ \sqrt{-1}a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

\( \tilde{A}_i^1(a) \in \text{End}_\mathbb{R}(\mathcal{J}^1) \) is defined as

\[\tilde{A}_i^1(a) := [A_i^1(a), X] \quad \text{for} \quad X \in \mathcal{J}^1.
\]
and the subspaces $u_{i}^{1}$ of $\text{End}_{\mathbb{R}}(\mathcal{J}^{1})$ as $u_{i}^{1} := \{ \tilde{A}_{i}^{1}(a) \mid a \in \mathcal{O} \}$. The differential $d\tilde{\sigma}_{i}$ of the involutive automorphism $\tilde{\sigma}_{i}$ is written by same letter $\tilde{\sigma}_{i}$. Then $\tilde{\sigma}_{i}(\phi) = \sigma_{i}\phi\sigma_{i}$ for $\phi \in f_{4(-20)}$.

**Lemma 4.1.**

1. ([9], cf. [24, Proposition 2.1]) $f_{4(-20)} = \mathfrak{d}_{4} \oplus u_{1}^{1} \oplus u_{2}^{1} \oplus n_{3}^{1}$.

2. ([43], cf. [26, Lemma 4.2]) $\tilde{\sigma}_{1}$ is a Cartan involution.

3. If $f_{4(-20)} = \mathfrak{t} \oplus \mathfrak{p}$ is a Cartan decomposition with respect to $\tilde{\sigma}_{1}$, then

$$\mathfrak{t} = \mathfrak{d}_{4} \oplus u_{1}^{1}, \quad \mathfrak{p} = u_{2}^{1} \oplus u_{3}^{1}.$$  

Now $\tilde{A}_{3}^{1}(1) \in \mathfrak{p}$. Let us define the abelian subspace $a$ of $\mathfrak{p}$, the 1-parameter subgroup $A$, and $\alpha \in a^{*}$ as

$$a := \{ t\tilde{A}_{3}^{1}(1) \mid t \in \mathbb{R} \}, \quad A := \{ \exp(t\tilde{A}_{3}^{1}(1)) \mid t \in \mathbb{R} \}, \quad \alpha(\tilde{A}_{3}^{1}(1)) := 1$$

respectively. Denote

$$\mathfrak{g}_{\lambda} := \{ \phi \in f_{4(-20)} \mid [H, \phi] = \lambda(H)\phi \text{ for all } H \in a \},$$

$$\Sigma := \{ \lambda \in a^{*} \mid \lambda \neq 0, \mathfrak{g}_{\lambda} \neq \{0\} \},$$

and the centralizer $a$ of the group $K$ and its Lie algebra as

$$M := Z_{K}(a) = \{ k \in K \mid k\tilde{A}_{3}^{1}(1)k^{-1} = \tilde{A}_{3}^{1}(1) \},$$

$$m := Z_{\mathfrak{e}}(a) = \{ \phi \in \mathfrak{e} \mid [\phi, \tilde{A}_{3}^{1}(1)] = 0 \}$$

respectively. For $p \in \text{ImO}$, $l_{p}, r_{p}, t_{p} \in \text{End}_{\mathbb{R}}(\mathcal{O})$ are defined by

$$l(p)x := px, \quad r(p)(x) := xp, \quad t(p)x := px + xp$$

for $x \in \mathcal{O}$ respectively. Then we see that

$$\delta(p) := d\varphi_{0}(l_{p}, r_{p}, t_{-p}) \in \mathfrak{d}_{4}.$$  

For $p \in \text{ImO}$ and $x \in \mathcal{O}$, denote

$$G_{1}(x) := \tilde{A}_{1}^{1}(x) + \tilde{A}_{2}^{1}(-\overline{x}), \quad G_{2}(p) := -\tilde{A}_{3}^{1}(p) - \delta(p),$$

$$G_{-1}(x) := \tilde{A}_{1}^{1}(x) + \tilde{A}_{2}^{1}(\overline{x}), \quad G_{-2}(p) := \tilde{A}_{3}^{1}(p) - \delta(p)$$

For $i = \pm 1$ and $j = \pm 2$, denote the subspaces $\mathfrak{g}_{i}$ and $\mathfrak{g}_{j}$ as $f_{4(-20)}$

$$\mathfrak{g}_{i} := \{ G_{i}(p) \mid p \in \text{ImO} \}, \quad \mathfrak{g}_{j} := \{ G_{j}(x) \mid x \in \mathcal{O} \}$$

respectively.

**Proposition 4.2.** (cf. [26, Proposition 4.4])

$$M = (F_{4(-20)})_{E_{1},F_{3}^{1}(1)} = (F_{4(-20)})_{E_{2},F_{3}^{1}(1)} = (F_{4(-20)})_{E_{1},E_{2},E_{3},F_{3}^{1}(1)} = \varphi_{0}(\text{Spin}(7)).$$
Lemma 4.3. (cf. [26, Lemma 4.5])

$a$ is a maximal abelian subspace of $p$,

$g_{\pm \alpha} = g_{\pm 1}, \ g_{\pm 2} = g_{\pm 2}$ (resp),

and $(f_{4(-20)}, a)$-root space decomposition of $f_{4(-20)}$ is given by

$f_{4(-20)} = g_{-2} \oplus g_{-\alpha} \oplus a \oplus m \oplus g_{\alpha} \oplus g_{2} = g_{-2} \oplus g_{-1} \oplus a \oplus m \oplus g_{1} \oplus g_{2}$.

So the nilpotent subalgebras $n^{\pm}$ are defined as

$n^{+} := g_{2} \oplus g_{\alpha} = \{G_{2}(p) + G_{1}(x) \mid p \in \text{Im}O, x \in O\},$

$n^{-} := g_{-2} \oplus g_{-\alpha} = \{G_{-2}(p) + G_{-1}(x) \mid p \in \text{Im}O, x \in O\}$ (resp).

Then

$[n^{+}, [n^{+}, n^{+}]] = [n^{-}, [n^{-}, n^{-}]] = 0.$

And the nilpotent subgroups $N^{\pm}$ of $F_{4(-20)}$ are defined as

$N^{+} := \exp n^{+} = \{\exp(G_{2}(p) + G_{1}(x)) \mid p \in \text{Im}O, x \in O\},$

$N^{-} := \exp n^{-} = \{\exp(G_{-2}(p) + G_{-1}(x)) \mid p \in \text{Im}O, x \in O\}$ (resp).

Lemma 4.4.

(1) $\exp G_{2}(p) \exp G_{1}(x) = \exp(G_{2}(p) + G_{1}(x)) = \exp G_{1}(x) \exp G_{2}(p)$.

(2) $\bar{\sigma}_{1} n^{+} = n^{-}$ and $\bar{\sigma}_{1} n^{-} = n^{+}$. Furthermore,

$\bar{\sigma}_{1}(G_{\pm 2}(p) + G_{\pm 1}(x)) = G_{\mp 2}(p) + G_{\mp 1}(x)$ (resp).

(3) $\bar{\sigma}_{1}(N^{+}) = N^{-}$ and $\bar{\sigma}_{1}(N^{-}) = N^{+}$. Furthermore,

$\bar{\sigma}_{1}(\exp(G_{\pm 2}(p) + G_{\pm 1}(p))) = \exp(G_{\mp 2}(p) + G_{\mp 1}(p))$ (resp).

Lemma 4.5. ([26, Lemma 5.3])

Let $g = (g_{1}, g_{2}, g_{3}), h \in \text{Spin}(7), p, q \in \text{Im}O, x, y \in O$.

$\exp(G_{2}(p) + G_{1}(x))\varphi_{0}(g) \exp(G_{2}(q) + G_{1}(y))\varphi_{0}(h)$

$= \exp(G_{2}(p + g_{3}q + \text{Im}(x\overline{(g_{1}y)})) + G_{1}(x + g_{1}y))\varphi_{0}(gh)$.

Let us consider $G := \text{Spin}(7) \times \text{Im}O \times O$ in which multiplication is defined by

$(g, p, x)(h, q, y) := (gh, p + g_{3}q + \text{Im}(x\overline{(g_{1}y)}), x + g_{1}y)$

where $p, q \in \text{Im}O, x, y \in O$ and $g = (g_{1}, g_{2}, g_{3}), h \in \text{Spin}(7)$. Denote

$H := \{(g, 0, 0) \mid g \in \text{Spin}(7)\},$

$N := \{(1, p, x) \mid p \in \text{Im}O, x \in O\},$

$G' := \{(g, p, 0) \mid g \in \text{Spin}(7), p \in \text{Im}O\}, \ N_{1} := \{(1, p, 0) \mid p \in \text{Im}O\},$

$G'' := \{(g, p, q) \mid g \in G_{2}, p, q \in \text{Im}O\},$

$H'' := \{(g, 0, 0) \mid g \in G_{2}\}, \ N_{2} := \{(1, p, q) \mid p, q \in \text{Im}O\}.$
Lemma 4.6. ([26, Lemma 5.2])
(1) $G$ is a group with respect to the multiplication.
(2) $H, N, G', N_1, G'', H''$ and $N_2$ are subgroups of $G$.
(3) We have
\[ G = H \ltimes N, \quad G' = H \ltimes N_1, \quad G'' = H'' \ltimes N_2. \]

Denote $\text{Spin}(7) := H$, $\text{Im}O \times O := N$, $\text{Im}O = N_1$, $G_2 := H''$ and $\text{Im}O \times \text{Im}O := N''$ so that
\[ \text{Spin}(7) \ltimes (\text{Im}O \times O) = G, \quad \text{Spin}(7) \ltimes \text{Im}O = G'. \]

The homomorphisms $\varphi : \text{Spin}(7) \times (\text{Im}O \times O) \to (F_{4(-20)})_{P-}$, $\varphi_1 : \text{Spin}(7) \times \text{Im}O \to (F_{4(-20)})_{E_3,P-}$ and $\varphi_2 : G_2 \times (\text{Im}O \times \text{Im}O) \to (F_{4(-20)})_Q$ are defined as
\[ \varphi(g, p, x) = \exp(G_2(p) + G_1(x)) \varphi(g), \quad \varphi_1(g, p) = \exp(G_2(p)) \varphi(g) \]
\[ \varphi_2(g, p, q) = \exp(G_2(p) + G_1(q)) \varphi(g) \quad \text{for} \ p, q \in \text{Im}O \text{ and } x \in O \]
respectively.

Proposition 4.7. ([26, Proposition 5.6])
(1) $\varphi_1$ is an isomorphism onto $(F_{4(-20)})_{E_3,P-}$.
(2) $\varphi$ is an isomorphism onto $(F_{4(-20)})_{P-}$.
(3) $\varphi_2$ is an isomorphism onto $(F_{4(-20)})_Q$.

The key of proof of (2): By direct calculation,
\[ \text{Orb}_{N^+}(E_3) = \{ X \in J^{1} | P^- \times X = -\frac{1}{2}P^-, X^{x^2} = 0, \text{tr}(X) = 1 \}. \]

Then this equation deduces $\text{Orb}_{N^+}(E_3) = \text{Orb}_{(F_{4(-20)})_{P-}}(E_3)$.

The mappings $\psi_1 : F_{4(-20)} \to \text{O}_1$, $\psi_2 : F_{4(-20)} \to \text{Im}O$ and $\psi_3 : F_{4(-20)} \to F_{4(-20)}$ are defined as for $g \in F_{4(-20)}$,
\[ \psi_1(g) := \frac{1}{2}((gE_3)_{F_1} + (gE_3)_{F_2}), \]
\[ \psi_2(g) := -\frac{1}{2} \text{Im} \left( (g(-E_1 + E_2))_{F_3} \right), \]
\[ \psi_3(g) := \exp(-G_1(\psi_1(g)) - G_2(\psi_2(g))) g \]
respectively.

Proposition 4.8. ([26, Proposition 5.7])
(1) Let $g \in (F_{4(-20)})_{P-}$. Then $\psi_3(g) \in M$ and
\[ g = \exp(G_1(\psi_1(g)) + G_2(\psi_2(g))) \psi_3(g) \in N^+M. \]
(2) We have
\[ (F_{4(-20)})_{P-} = N^+M = MN^+. \]
5. The Orbit Types of $F_{4(-20)}$-Orbits on $\mathcal{J}^{1}$.

Main Theorem 2. (The orbit types of $F_{4(-20)}$-orbits on $\mathcal{J}^{1}$ [26, Main Theorem 1])

The orbit types of $F_{4(-20)}$-orbits on $\mathcal{J}^{1}$ are given as follows.

1. Assume that $X \in \mathcal{J}^{1}$ admits the characteristic roots $\lambda_1 > \lambda_2 > \lambda_3$. Then $X$ can be transformed to the following canonical forms by $F_{4(-20)}$ with the following type of stabilizer group.

   \[
   \begin{array}{ll}
   \text{The canonical forms of } X & \text{The type of stabilizer group} \\
   1. \, \text{diag}(\lambda_1, \lambda_2, \lambda_3) & \text{Spin}(8) \\
   2. \, \text{diag}(\lambda_2, \lambda_3, \lambda_1) & \text{Spin}(8) \\
   3. \, \text{diag}(\lambda_3, \lambda_1, \lambda_2) & \text{Spin}(8) \\
   \end{array}
   \]

2. Assume that $X \in \mathcal{J}^{1}$ admits the characteristic roots $\lambda_1 \in \mathbb{R}$, $p \pm \sqrt{-1}q$ with $p \in \mathbb{R}$ and $q > 0$. Then $X$ can be transformed to the following canonical form by $F_{4(-20)}$ with the following type of stabilizer group.

   \[
   \begin{array}{ll}
   \text{The canonical forms of } X & \text{The type of stabilizer group} \\
   4. \, \text{diag}(p, p, \lambda_1) + F_{3}^{1}(q) & \text{Spin}^0(7, 1) \\
   \end{array}
   \]

3. Assume that $X \in \mathcal{J}^{1}$ admits the characteristic roots $\lambda_1$ of multiplicity 1 and $\lambda_2$ of multiplicity 2. Then $X$ can be transformed to the following canonical forms by $F_{4(-20)}$ with the following types of stabilizer group.

   \[
   \begin{array}{ll}
   \text{The canonical forms of } X & \text{The type of stabilizer group} \\
   5. \, \text{diag}(\lambda_1, \lambda_2, \lambda_2) & \text{Spin}(9) \\
   6. \, \text{diag}(\lambda_2, \lambda_2, \lambda_1) & \text{Spin}^0(8, 1) \\
   7. \, \text{diag}(\lambda_2, \lambda_2, \lambda_1) + P^+ & \text{Spin}(7) \ltimes \text{ImO} \\
   8. \, \text{diag}(\lambda_2, \lambda_2, \lambda_1) + P^- & \text{Spin}(7) \ltimes \text{ImO} \\
   \end{array}
   \]

4. Assume that $X \in \mathcal{J}^{1}$ admits the characteristic root of multiplicity 3. Then $X$ can be transformed to the following canonical forms by $F_{4(-20)}$ with the following types of stabilizer group.

   \[
   \begin{array}{ll}
   \text{The canonical forms of } X & \text{The type of stabilizer group} \\
   9. \, \frac{1}{3}\text{tr}(X)E & F_{4(-20)} \\
   10. \, \frac{1}{3}\text{tr}(X)E + P^+ & \text{Spin}(7) \ltimes (\text{ImO} \times \text{O}) \\
   11. \, \frac{1}{3}\text{tr}(X)E + P^- & \text{Spin}(7) \ltimes (\text{ImO} \times \text{O}) \\
   12. \, \frac{1}{3}\text{tr}(X)E + Q^+(1) & G_2 \ltimes (\text{ImO} \times \text{ImO}) \\
   \end{array}
   \]


Let $G$ be a linear connected semisimple Lie group with its Lie algebra $\mathfrak{g}$ over $\mathbb{R}$. Let $\theta$ be a Cartan involution of $\mathfrak{g}$, $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$ a Cartan decomposition, $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$, $\mathfrak{m} = Z_\theta(\mathfrak{a})$. $\mathfrak{a}^*$
denotes the dual space of $\mathfrak{a}$. For any element $\lambda \in \mathfrak{a}^*$, let $\mathfrak{g}_\lambda := \{X \in \mathfrak{g} \mid [H, X] = \lambda(H)X \text{ for all } H \in \mathfrak{a}\}$. $\lambda$ is called a root of $(\mathfrak{g}, \mathfrak{a})$ if $\lambda \neq 0$ and $\mathfrak{g}_\lambda \neq \{0\}$. The set of roots of $(\mathfrak{g}, \mathfrak{a})$ is denoted by $\Sigma$. Then $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{m} \oplus \sum_{\lambda \in \Sigma} \mathfrak{g}_\lambda$ follows. Denote by $\Sigma^+$ a set of positive root of $(\mathfrak{g}, \mathfrak{a})$ with respect to the same ordering in $\mathfrak{a}^*$, $\Sigma^- := \{-\lambda \mid \lambda \in \Sigma^+\}$, $\mathfrak{n}^+ := \sum_{\lambda \in \Sigma^+} \mathfrak{g}_\lambda$ and $\mathfrak{n}^- := \sum_{\lambda \in \Sigma^-} \mathfrak{g}_\lambda$. Then $\mathfrak{n}^+$ and $\mathfrak{n}^-$ are nilpotent subalgebras, $\theta \mathfrak{n}^\pm = \mathfrak{n}^\mp$ (resp), and $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{n}^+$ follow. Suppose that there exists an involutive automorphism $\Theta$ on $G$ such that the differential $d\Theta = \theta$, and the center $Z(G)$ of $G$ is finite. Denote the subgroup $K := G^\Theta$ of $G$. Then $\text{Lie}(K) = \mathfrak{t}$ and $K$ is connected, closed, and $K$ is a maximal compact subgroup of $G$ since $Z(G)$ of $G$ is finite.

Set $A := \exp \mathfrak{a}$, $M := Z_K(\mathfrak{a}) = \{k \in K \mid kXk^{-1} = X \text{ for all } X \in \mathfrak{a}\}$ and $N^\pm := \exp \mathfrak{n}^\pm$ (resp). Then the identity connected component $M^0$ of $M$ is the analytic subgroup corresponding to $\mathfrak{m}$, and $\Theta N^\pm = N^\mp$ (resp). Denote the normalizer of $\mathfrak{a}$ of the group $K$ as $M^\ast := N_K(\mathfrak{a}) = \{k \in K \mid kak^{-1} \subset \mathfrak{a}\}$ and the finite factor group $W := M^\ast/M$.

For all $w \in W$, we fix a representative $\bar{w} \in M^\ast$. Then the following decompositions:

\[ (1) \quad G = \coprod_{w \in W} MAN^+ \bar{w}N^- \quad \text{(Bruhat decomposition)}, \]
\[ (1)' \quad G = MAN^+ N^- \quad \text{(Gauss decomposition)}, \]
\[ (2) \quad G = KAN^+ \quad \text{(Iwasawa decomposition)}. \]

(cf. [15],[18], [27],[23]). In (1)', the set $MAN^+ N^-$ is open dense in $G$, and so almost any $g \in G$ can be expressed by

\[ g = m_G(g)a_G(g)n_G(g)\bar{n}_G(g) \]

for some $m_G(g) \in M, \quad a_G(g) \in A, \quad n_G(g) \in N^+$ and $\bar{n}_G(g) \in N^-$ with uniquely determined factors. In (2), any $g \in G$ can be uniquely expressed by

\[ g = k(g)(\exp H(g))n(g) \]

for some $k(g) \in K, H(g) \in \mathfrak{a}$ and $n(g) \in N$.

A signature of roots is defined by the mapping $\epsilon$ of $\Sigma$ to $\{-1, 1\}$ such that $\epsilon$ satisfies the conditions:

(i) $\epsilon(\lambda) = \epsilon(-\lambda)$ \hspace{1cm} for all $\lambda \in \Sigma$,

(ii) $\epsilon(\lambda + \mu) = \epsilon(\lambda)\epsilon(\mu)$ \hspace{1cm} if $\lambda, \mu, \lambda + \mu \in \Sigma$

[27, Definition 1.1]. For the Cartan involution $\theta$ and any signature $\epsilon$ of roots, let us define an involutive automorphism $\theta_\epsilon$ of $\mathfrak{g}$ such that

(i) $\theta_\epsilon(X) := \epsilon(\lambda)\theta(X)$ \hspace{1cm} for all $\lambda \in \Sigma$ and $X \in \mathfrak{g}_\lambda$,

(ii) $\theta_\epsilon(X) := \theta(X)$ \hspace{1cm} for all $X \in \mathfrak{a} \oplus \mathfrak{m}$

[27, Definition 1.2]. $\theta_\epsilon$ is called the $(\theta, \epsilon)$-involutive of $\mathfrak{g}$. Set

\[ \mathfrak{t}_\epsilon := \{X \in \mathfrak{g} \mid \theta_\epsilon X = X\}, \quad \mathfrak{p}_\epsilon := \{X \in \mathfrak{g} \mid \theta_\epsilon X = -X\}. \]
Then \( g = \mathfrak{t}_\epsilon \oplus \mathfrak{p}_\epsilon \). Let \((K_\epsilon)_0\) be the analytic subgroup of \( G \) with the Lie algebra \( \mathfrak{t}_\epsilon \) and the subgroup \( K_\epsilon \) of \( G \) as \( K_\epsilon := (K_\epsilon)_0 M \). In fact, since all elements of \( M \) normalize \((K_\epsilon)_0\) by [27, Lemma 1.4(i)], \( K_\epsilon \) is a subgroup of \( G \). Denote \( M^*_\epsilon := K_\epsilon \cap M^* \) and \( W_\epsilon := M^*_\epsilon \backslash M \).

**Proposition 6.1.** (Iwasawa decomposition with respect to \( K_\epsilon \) in sense of T. Oshima and J. Sekiguchi [27, Proposition 1.10])

Let the factor set \( W_\epsilon \backslash W = \{w_1 = 1, w_2, \cdots, w_r\} \) where \( r = [W : W_\epsilon] \).

Fix representatives \( \overline{w}_1 = 1, \overline{w}_2, \cdots, \overline{w}_r \in M^*_\epsilon = K_\epsilon \cap M^* \) for \( w_1 = 1, w_2, \cdots, w_r \). Then the decomposition \( G \supset \bigcup_{i=1}^{r} K_\epsilon \overline{w}_i A N^+ \)

has the following properties.

1. If \( k \overline{w}_i a n = k' \overline{w}_j a' n' \) with \( k, k' \in K_\epsilon, a, a' \in A \) and \( n, n' \in N^+ \), then \( k = k', i = j, a = a' \) and \( n = n' \).

2. The map \( (k, a, n) \mapsto k \overline{w}_i a n \) defines an analytic diffeomorphism of the product manifold \( K_\epsilon \times A \times N^+ \) onto the open submanifold \( K_\epsilon \overline{w}_i A N^+ \) of \( G \) (\( i = 1, \cdots r \)).

3. The submanifolds \( \bigcup_{i=1}^{r} K_\epsilon \overline{w}_i A N^+ \) is open dense in \( G \).

7. **The Gauss decomposition of \( F_{4(-20)} \)**

We have

\[ \mathcal{N}_1^{-}(O) = \text{Orb}_{F_{4(-20)}}(P^-) \cong F_{4(-20)}/(F_{4(-20)})_{P^-} = F_{4(-20)}/N^+ M. \]

So considering \( AN^- \)-orbits on \( \mathcal{N}_1^{-}(O) \), we obtain:

**Main Theorem 3.** (The Bruhat and Gauss decomposition of \( F_{4(-20)} \)[26, Main Theorem 2])

1. Assume that \( g \in F_{4(-20)} \) and \( (gP^+ | P^-) \neq 0 \). Let

\[ t := -\frac{1}{2} \log \left( \frac{1}{4} (gP^+ | P^-) \right) \in \mathbb{R}, \]

\[ a_G(g) := \exp(t \tilde{A}_3^1(1)) \in A, \]

\[ \overline{n}_G(g) = \tilde{\sigma}_1(\exp(-\mathcal{G}_1(\frac{\sigma_1 g^{-1}P_F - (\sigma_1 g^{-1}P^-)_{F_3}^-}{gP^+ | P^-})) - \mathcal{G}_2(\frac{\text{Im}((\sigma_1 g^{-1}P^-)_{F_3}^-)}{(gP^+ | P^-)} I)) \in N^- , \]

\[ n_G(g) := \exp(t(\mathcal{G}_1(\tilde{\psi}_1(a_G(g) \overline{n}_G(g)) g^{-1}))) + 2 \mathcal{G}_2(\psi_2(a_G(g) \overline{n}_G(g) g^{-1}))) \in N^+ , \]

\[ m_G(g) := \psi_3(a_G(g) \overline{n}_G(g) g^{-1})^{-1}. \]

Then
(i) \((gP^+|P^-) < 0\), and \(\alpha_G(g), \, n_G(g), \, m_G(g)\) are well-defined,

(ii) \(m_G(g) \in M\) and
\[
g = m_G(g)\alpha_G(g)n_G(g)\overline{n}_G(g) \in MAN^+N^-.
\]

(2) Assume \(g \in F_4(-20)\) and \((gP^+|P^-) = 0\). Let
\[
t := -\frac{1}{2}\log(-(gE_1|P^-)) \in \mathbb{R},
\]
\[
a'(g) = \exp(t\tilde{A}^1_3(1)) \in A,
\]
\[
n'(g) := \exp(t(\mathcal{G}_1(\psi_1(\sigma_1 a'(g)g^{-1}) + 2\mathcal{G}_2(\psi_2(\sigma_1 a'(g)g^{-1})))) \in N^+,
\]
\[
m'(g) := \psi_3(\sigma_1 a'(g)g^{-1})^{-1}.
\]

Then
(i) \((gE_1|P^-) < 0\), and \(a'(g), \, n'(g), \, m'(g)\) are well-defined,
(ii) \(m'(g) \in M\) and
\[
g = m'(g)a'(g)n^l(g)\sigma_1 \in MAN^+\sigma_1 = MAN^+\sigma_1N^-.
\]

(3) The following equations hold.
\[
MAN^+N^- = \{g \in F_4(-20) \mid (gP^+|P^-) \neq 0\} \\
= \{g \in F_4(-20) \mid (gP^+|P^-) < 0\} \neq \emptyset,
\]
\[
MAN^+\sigma_1 = MAN^+\sigma_1N^- \\
= \{g \in F_4(-20) \mid (gP^+|P^-) = 0\} \neq \emptyset.
\]

Especially,
\[
F_4(-20) = MAN^+N^- \coprod MAN^+\sigma_1N^- \quad \text{(Bruhat decomposition)}
\]
\[
= MAN^+N^- \coprod MAN^+\sigma_1
\]

(4) \(MAN^+N^-\) is open dense in \(F_4(-20)\). Especially
\[
F_4(-20) = MAN^+N^- \quad \text{(Gauss decomposition)}.
\]

8. The Iwasawa decomposition of \(F_4(-20)\).

We have
\[
\mathcal{H}(O) = Orb_{F_4(-20)}(E_1) \simeq F_4(-20)/(F_4(-20))E_1 = F_4(-20)/K.
\]

So considering \(AN^+\)-orbits on \(\mathcal{H}(O)\), we obtain:

Main Theorem 4. (The Iwasaws decomposition of \(F_4(-20)[26, \text{Main Theorem 3}]\))
For any $g \in F_{4(-20)}$, let
\[
H(g) := \frac{1}{2} \log(-(gP^{-}|E_{1}))A_{3}^{1}(1) \in a,
\]
\[
n(g) := \exp(G_{1}\left(\frac{(g^{-1}E_{1})_{F_{1}} - (g^{-1}E_{1})_{F_{2}}}{(gP^{-}|E_{1})}\right) + G_{2}\left(\frac{{\rm Im}((g^{-1}E_{1})_{F_{3}})}{(gP^{-}|E_{1})}\right)) \in N^{+}
\]
\[
k(g) := gn(g)^{-1}\exp(-H(g)).
\]

Then
(1) $(gP^{-}|E_{1}) < 0$. Especially $H(g)$, $n(g)$ and $k(g)$ is well-defined.
(2) $k(g) \in K$ and
\[
g = k(g)(\exp H(g))n(g) \in KAN^{+}.
\]

9. THE IWASAWA DECOMPOSITION WITH RESPECT TO $K_{\epsilon}$.

For $G = F_{4(-20)}$, let $\epsilon$ be a signature of root defined by
\[
\epsilon(\alpha) = \epsilon(-\alpha) := -1, \; \epsilon(2\alpha) = \epsilon(-2\alpha) := 1.
\]

Denote the $(\tilde{\sigma}_{1}, \epsilon)$-involution by $(\tilde{\sigma}_{1})_{\epsilon}$, and use same notations $K_{\epsilon}$, $(K_{\epsilon})_{0}$, $K_{\epsilon}$, $M^{*}$, $M_{\epsilon}^{*}$, $W$ and $W_{\epsilon}$ corresponding to notations of general $G$ respectively.

**Proposition 9.1.** ([26, Lemma 6.2])

(1) $(\tilde{\sigma}_{1})_{\epsilon} = \tilde{\sigma}_{2}$.
(2) $K_{\epsilon} = (F_{4(-20)})_{E_{2}}$.
(3) $M^{*} = M \bigsqcup \sigma_{1} M$. Especially $W = \{M, \sigma_{1} M\} \cong \mathbb{Z}_{2}$.
(4) $M_{\epsilon}^{*} = M \bigsqcup \sigma_{1} M$. Especially $W_{\epsilon} = \{M, \sigma_{1} M\}$ and $[W : W_{\epsilon}] = 1$.

We have
\[
\mathcal{H}'(O) = Orb_{F_{4(-20)}}(E_{2}) \simeq F_{4(-20)}/(F_{4(-20)})_{E_{2}} = F_{4(-20)}/K_{\epsilon}.
\]

So considering $AN^{+}$-orbits on $\mathcal{H}'(O)$. we obtain:

**Main Theorem 5.** (The Iwasawa decomposition with respect to $K_{\epsilon}$[26, Main Theorem 4])

Let $\mathcal{D}$ be the domain of $F_{4(-20)}$ defined by
\[
\mathcal{D} := \{g \in F_{4(-20)} \mid (gP^{-}|E_{2}) > 0\}.
\]
For any $g \in \mathcal{D}$, let
\[
H_\epsilon(g) := \frac{1}{2} \log((gP^{-}|E_2))\tilde{A}_3^1(1) \in a,
\]
\[
n_\epsilon(g) := \exp(G_1 \left( \frac{(g^{-1}E_2)_{F_1} - (g^{-1}E_2)_{F_2}}{(gP^{-}|E_2)} \right) + G_2 \left( \frac{\text{Im}((g^{-1}E_2)_{F_3})}{(gP^{-}|E_2)} \right)) \in N^+.
\]
\[
k_\epsilon(g) := gn_\epsilon(g)^{-1} \exp(-H_\epsilon(g)).
\]

Then
(1) $k_\epsilon(g) \in K_\epsilon$ and
\[
g = k_\epsilon(g)(\exp H_\epsilon(g))n_\epsilon(g) \in K_\epsilon AN^+.
\]

(2) $\mathcal{D} = K_\epsilon AN^+ = \{g \in F_{4(-20)} \mid (gP^{-}|E_2) \neq 0\}$. Furthermore, $\mathcal{D}$ is open dense in $F_{4(-20)}$.

Moreover we have:

**Theorem 9.2.** ([26, Theorem 9.6])
(1) The following equations hold.
\[
K_\epsilon MAN^+ = \{g \in F_{4(-20)} \mid (gP^{-}|E_2) \neq 0\}
= \{g \in F_{4(-20)} \mid (gP^{-}|E_2) > 0\} = K_\epsilon AN^+.
\]

(2) $K_\epsilon \exp \left( -\tilde{A}_1^1(\frac{\pi}{2}) \right) MAN^+ = \{g \in F_{4(-20)} \mid (gP^{-}|E_2) = 0\}.$

(3) $F_{4(-20)} = K_\epsilon MAN^+ \coprod K_\epsilon \exp \left( -\tilde{A}_1^1(\frac{\pi}{2}) \right) MAN^+.$

**Remark 9.3.** Theorem 9.2(3) is a special case of [21, Theorems 3], so the decomposition in Theorem 9.2(3) is called a Matsuki decomposition of $F_{4(-20)}$.

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