数理解析研究所講究録
第 1772 巻 2011 年 109-113

ON THE COEFFICIENTS OF THE RIEMANN MAPPING FUNCTION FOR THE COMPLEMENT OF THE MANDELBROT SET

HIROKAZU SHIMAUCHI

ABSTRACT. We denote the Mandelbrot set by $\mathbb{M}$, the Riemann sphere by $\widehat{\mathbb{C}}$ and the unit disk by $\mathbb{D}$. Let $f:\mathbb{D}\to\mathbb{C}\setminus\{1/z:z\in\mathbb{M}\}$ and $\Psi:\widehat{\mathbb{C}}\setminus\overline{\mathbb{D}}\to\widehat{\mathbb{C}}\setminus\mathbb{M}$ be the Riemann mapping functions and let their expansions be $z+\sum_{m=2}^{\infty}a_m z^m$ and $z+\sum_{m=0}^{\infty}b_m z^{-m}$, respectively. We consider several interesting properties of the coefficients a_m and b_m. The detailed studies of these coefficients were given in [1, 3, 4, 5, 8]. This is a partial summary of [11], which contains Zagier's observations (see [1]).

1. INTRODUCTION

For $c\in\mathbb{C}$, let $P_c(z):=z^2+c$ and $P_c^{\circ n}(z)=P_c(P_c(\dots P_c(z)\dots))$ be the n-th iteration of $P_c(z)$ with $P_c^{\circ 0}(z)=z$. In the theory of one-dimensional complex dynamics, there is a detailed study of the dynamics of $P_c(z)$ on the Riemann sphere $\widehat{\mathbb{C}}$. For each fixed c, the *(filled in) Julia set* of $P_c(z)$ consists of those values z that remain bounded under iteration. The *Mandelbrot set* $\mathbb{M}$ consists of those parameter values c for which the Julia set is connected. It is known that $\mathbb{M}=\{c\in\mathbb{C}:\{P_c^{\circ n}(0)\}_{n=0}^{\infty}\text{ is bounded}\}$, compact and is contained in the closed disk of radius 2. Furthermore, $\mathbb{M}$ is connected. However, its local connectivity is still unknown, and there is a very important conjecture which states that $\mathbb{M}$ is locally connected (see [2]).

Let $G\subsetneq\mathbb{C}$ be a simply connected domain with $w_0\in G$. Furthermore let $G'\subsetneq\widehat{\mathbb{C}}$ be a simply connected domain with $\infty\in G'$ which has more than one boundary point. Due to the Riemann mapping theorem there exist unique conformal mappings $f:\mathbb{D}\to G$ such that $f(0)=0$ and $f'(0)>0$ and $g:\mathbb{D}^*\to G'$ such that $g(\infty)=\infty$ and $\lim_{z\to\infty}g(z)/z>0$ respectively, where $\mathbb{D}:=\{z\in\mathbb{C}:|z|<1\}$ and $\mathbb{D}^*:=\widehat{\mathbb{C}}\setminus\mathbb{D}$. We call f (and g) the Riemann mapping function of G (and G').

Douady and Hubbard demonstrated in [2] the connectedness of the Mandelbrot set by constructing a conformal isomorphism $\Phi:\widehat{\mathbb{C}}\setminus\mathbb{M}\to\mathbb{D}^*$. Note that $\Psi:=\Phi^{-1}$ is the Riemann mapping function of $\widehat{\mathbb{C}}\setminus\mathbb{M}$. We recall a lemma of Carathéodory.

Lemma 1 (Carathéodory's Continuity Lemma). *Let $G\subset\widehat{\mathbb{C}}$ be a simply connected domain and a function f maps $\mathbb{D}$ conformally onto G. Then f has a continuous extension to $\overline{\mathbb{D}}$ if and only if the boundary of G is locally connected.*

This implies if Ψ can be extended continuously to the unit circle, then the Mandelbrot set is locally connected. This is the motivation of our study.

Jungreis presented an algorithm to compute the coefficients b_m of the Laurent series expansion of $\Psi(z)$ at ∞ in [7].

Key words and phrases. Mandelbrot set; conformal mapping.

Several detailed studies of b_m are given in [1, 3, 4, 8] and remarkable empirical observations are mentioned in [1] by Zagier. Especially a formula for b_m is given in [3]. Many of these coefficients are shown to be zero and infinitely many non-zero coefficients are determined.

In addition, Ewing and Schober [5] studied the coefficients a_m of the Taylor series expansion of the function $f(z) := 1/\Psi(1/z)$ at the origin. Note that f is the Riemann mapping function of the bounded domain $\mathbb{C} \setminus \{1/z : z \in \mathbb{M}\}$ and f has a continuous extension to the boundary if and only if the Mandelbrot set is locally connected.

In [12], Komori and Yamashita studied a generalization of b_m. Let $P_{d,c}(z) = z^d + c$ with an integer $d \geq 2$ and let $\mathbb{M}_d := \{c \in \mathbb{C} : \{P_{d,c}^{\circ n}(0)\}_{n=0}^{\infty}$ is bounded $\}$. Constructing the Riemann mapping function Ψ_d of $\widehat{\mathbb{C}} \setminus \mathbb{M}_d$, they analyzed the coefficients $b_{d,m}$ of the Laurent series at ∞.

The author has been studying $b_{d,m}$ and the coefficients $a_{d,m}$ of the Taylor series at the origin of the function $f_d(z) := 1/\Psi_d(1/z)$ in [11].

In [12] and [11], there is a generalization of the results for $d = 2$, propositions for $d > 3$ and a verification of Zagier's observations.

In this paper, we focus on the case $d = 2$. Especially we mention the observations by Zagier and the asymptotic behavior of b_m.

2. Computing the Laurent Series of Ψ

Now we introduce how to construct Φ. This is established by Douady and Hubbard (see [1]).

Theorem 2. *Let $c \in \widehat{\mathbb{C}} \setminus \mathbb{M}$. Then*

$$\phi_c(z) := z \prod_{k=1}^{\infty} \left(1 + \frac{c}{P_c^{\circ k-1}(z)^2}\right)^{\frac{1}{2^k}}$$

is well-defined on some neighborhood of ∞ which includes c. Moreover, $\Phi(c) := \phi_c(c)$ maps $\widehat{\mathbb{C}} \setminus \mathbb{M}$ conformally onto $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, and satisfies $\Phi(c)/c \to 1$ as $c \to \infty$. Thus $\widehat{\mathbb{C}} \setminus \mathbb{M}$ is simply connected and $\mathbb{M}$ is connected.

Set $A_n(c) = P_c^{\circ n}(c)$ for simplicity. Applying the following proposition, we can calculate the coefficients b_m of Ψ.

Proposition 3 (see [1]).

$$A_n(\Psi(z)) = z^{2^n} + O(\frac{1}{z^{2^n-1}}).$$

Jungreis [7] presented an algorithm to compute b_m and calculated the first 4095 numerical values of b_m. Bielefeld, Fisher and Haeseler calculated the first 8000 terms in [1].

Ewing and Schober [4] computed the first 240000 numerical values of b_m, using an backward recursion formula in the following way.

Let n be a non-negative integer, and let

$$(1) \qquad A_n(\Psi(z)) = \sum_{m=0}^{\infty} \beta_{n,m} z^{2^n-m} \text{ for } |z| > 1.$$

Using proposition 3, $\beta_{n,m} = 0$ for $n \geq 1$ and $1 \leq m \leq 2^{n+1} - 2$. Furthermore $\beta_{n,0} = 1$ for all $n \in \mathbb{N} \cup \{0\}$. Since $P_0(\Psi(z)) = \Psi(z)$, obviously $\beta_{0,m} = b_{m-1}$ for $m \geq 1$. Applying the recursion $A_n(z) = A_{n-1}(z)^2 + z$ to equation (1), we get

$$\sum_{m=0}^{\infty} \beta_{n,m} z^{2^n - m} = \sum_{m=0}^{\infty} \sum_{k=0}^{m} \beta_{n-1,k}\beta_{n-1,m-k} z^{2^n - m} + \sum_{m=2^n-1}^{\infty} \beta_{0,m-2^n-1} z^{2^n-m}.$$

For $m \geq 2^n - 1$, we compare the coefficients of the right and left-hand side. Hence

$$\beta_{n,m} = \sum_{k=0}^{m} \beta_{n-1,k}\beta_{n-1,m-k} + \beta_{0,m-2^n-1}.$$

Since $\beta_{n-1,0} = 1$ and $\beta_{n,m} = 0$ for $n \geq 1$ and $1 \leq m \leq 2^{n+1} - 2$, we obtain the following formula:

$$\beta_{n,m} = 2\beta_{n-1,m} + \sum_{k=2^n-1}^{m-2^n+1} \beta_{n-1,k}\beta_{n-1,m-k} + \beta_{0,m-2^n-1} \text{ for } n \geq 1 \text{ and } m \geq 2^n - 1.$$

This is the forward recursion to determine $\beta_{n,m}$ in terms of $\beta_{j,k}$ with $j < n$. A corresponding backward recursion formula is derived to be

$$\beta_{n-1,m} = \frac{1}{2}\left(\beta_{n,m} - \sum_{k=2^n-1}^{m-2^n+1} \beta_{n-1,k}\beta_{n-1,m-k} - \beta_{0,m-2^n-1}\right).$$

The formula gives $\beta_{m,n}$ in terms of $\beta_{j,k}$ with $j > n, k \leq m$. If n is sufficiently large, then $\beta_{n,m} = 0$ for a fixed m. Hence, using this backward recursion formula, we can determine $\beta_{j,m}$ for all j.

Example 4. Considering $b_0 = \beta_{0,1} = (0 - \beta_{0,0})/2 = -1/2$, $b_1 = \beta_{0,2} = (0 - \beta_{0,1}^2 - \beta_{0,1})/2 = 1/8$, ... yields

$$\Psi(z) = z - \frac{1}{2} + \frac{1}{8z} - \frac{1}{4z^2} + \frac{15}{128z^3} + \frac{0}{z^4} - \frac{47}{1024z^5} - \frac{1}{16z^6} + \frac{987}{32768z^7} + \cdots$$

One can make a program for this procedure and derive the exacts value of b_m, because b_m is a binary rational number.

Theorem 5 (see [4]). *If $n \geq 0$ and $m \geq 1$, then $2^{2m+3-2^{n+2}}\beta_{n,m}$ is an integer. In particular, $2^{2m+1}b_m$ is an integer.*

The coefficient a_m is also a binary rational number, since

$$a_m = -b_{m-2} - \sum_{j=2}^{m-1} a_j b_{m-1-j} \text{ for } m \geq 2.$$

Remark 6. In [1] Zagier made an empirical observation about the growth of the denominator of b_m, which we are going to mention in the next section.

Komori and Yamashita computed the exact values for the first 2000 terms in [12]. In [11], the autor made a program to compute the exact values of b_m by using C programing language with multiple precision arithmetic library GMP (see [6]), and the first 30000 exact values of b_m were determined.

3. Observations by Zagier

Based on roughly 1000 coefficients, Zagier made several observations. In this paper, two of them are mentioned. We write $m = m_0 2^n$ with $n \geq 0$, m_0 is odd.

Observation 7 (see [1]). It is $b_m = 0$, if and only if $m_0 \leq 2^{n+1} - 5$.

One direction of this statement has been proven in [1] and separately from that in [8].

Theorem 8. *If $n \geq 2$ and $m_0 \leq 2^{n+1} - 5$, then $b_m = 0$.*

It is still unknown whether the converse of this theorem is true. In [4], the only coefficients which have been observed to be zero are those mentioned in this theorem. In this publication Ewing and Schober proved the following theorem about zero-coefficients of a_m.

Theorem 9 (see [5]). *If $3 \leq m_0 \leq 2^{n+1}$, then $a_m = 0$.*

The truth of the converse of this theorem is unknown. They reported that their computation of 1000 terms of a_m has not produced a zero-coefficient besides those indicated in theorem 9.

Now we consider the growth of the power of 2. For every non-zero rational number x, there exists a unique integer v such that $x = 2^v p/q$ with some integers p and q indivisible by 2. The 2-adic valuation $\nu_2 : \mathbb{Q} \setminus \{0\} \to \mathbb{Z}$ is defined as:

$$\nu_2(x) = v.$$

We extend ν_2 to the whole rational field $\mathbb{Q}$ as follows,

$$\nu(x) = \begin{cases} \nu_2(x) & \text{for } x \in \mathbb{Q} \setminus \{0\} \\ +\infty & \text{for } x = 0. \end{cases}$$

Due to theorem 5, if $b_m \neq 0$ then $b_m = C/2^{-\nu(b_m)}$, where C is an odd number. Note that $\nu((2m+2)!) \leq 2m+1$ for a non-negative integer m.

Observation 10 (see [1]). It is $-\nu(b_m) \leq \nu((2m+2)!)$ for all m. Equality attained exactly when m is odd.

In [12] a theorem for $b_{d,m}$ which includes this observation was presented. However, d has to be prime and not an arbitrary integer as it was originally stated.

Corollary 11. *It is $-\nu(b_m) \leq \nu((2m+2)!)$ for all m. Equality attained exactly when m is odd.*

For a_m we have the following:

Corollary 12. *It is $-\nu(a_m) \leq \nu((2m-2)!)$ for all m. Equality attained exactly when m is odd.*

The generalization of these result is given in [11].

4. Observation for the Asymptotic Behavior of b_m

The result which Ewing and Schober obtained shows that the inequality $|b_m| < 1/m$ holds for $0 < m < 240000$. If there exist positive constants ϵ and K such that the inequality $|b_m| < K/m^{1+\epsilon}$ holds for any natural number m, this would imply its absolute convergence and give that the Mandelbrot set is locally connected. Furthermore, such a bound imply Hölder continuity (see [1]). However it is not valid because of the following claim given in [1].

Claim 13. *There is no Hölder continuous extension of Ψ to $\overline{\mathbb{D}}$.*

On the other hand, the coefficients b_m satisfying $|b_m| \geq 1/m$ have not been found yet.

The author focused on the local maximum of $|b_m|$ and considered the period of Jungreis' algorithm. The observation below for the behavior of b_m can be made.

Observation 14 (see [11]). For fixed $1 \leq n \leq 7$, the maximum value of $|b_{2^{2n}-2}|$, $|b_{2^{2n}-1}|, \ldots, |b_{2^{2(n+1)}-3}|$ is $|b_{2^{2n}-2}|$. Furthermore, the sequence $|b_{2^2-2}|, |b_{2^4-2}|, |b_{2^6-2}|, \cdots, |b_{2^{2n}-2}|, \cdots$ is strictly monotonically decreasing.

It is still unknown whether it would be true for every n, and the behavior of $\{|b_{2^{2n}-2}|\}$ is the material of further research.

References

[1] B. Bielefeld , Y. Fisher and F. V. Haeseler, *Computing the Laurent series of the map* Ψ : $C \setminus \overline{D} \to C \setminus M$, Adv. in Appl. Math. **14** (1993), 25-38.

[2] A. Douady and J. H. Hubbard, *Exploring the Mandelbrot set*, The Orsay Notes, (1985).

[3] J. Ewing and G. Schober, *On the coefficients of the mapping to the exterior of the Mandelbrot set*, Michigan Math. J. **37** (1990), 315-320.

[4] J. Ewing and G. Schober, *The area of the Mandelbrot set*, Numerische Mathematik **61** (1992), 59-72.

[5] J. Ewing and G. Schober, *Coefficients associated with the reciprocal of the Mandelbrot set*, J. Math. Anal. Appl. **170** (1992), no. 1, 104-114.

[6] The GNU Multiple Precision Arithmetic Library, http://gmplib.org/.

[7] I. Jungreis, *The uniformization of the complement of the Mandelbrot set*, Duke Math. J. **52** (1985), no. 4, 935-938.

[8] G. M. Levin, *On the arithmetic properties of a certain sequence of polynomials*, Russian Math. Surveys 43 (1988), 245-246.

[9] S. Morosawa, Y. Nishimura, M. Taniguchi, T. Ueda, *Holomorphic Dynamics*, Cambridge University Press, (2000).

[10] Ch. Pommerenke, *Boundary behaviour of conformal maps*, Springer-Verlag, (1992).

[11] H. Shimauchi, *On the coefficients of the Riemann mapping function for the complement of the Mandelbrot set*, in preparation, (2011).

[12] O. Yamashita, *On the coefficients of the mapping to the exterior of the Mandelbrot set*, Master thesis (1998), Graduate School of Information Science, Nagoya University, http://www.math.human.nagoya-u.ac.jp/master.thesis/1997.html.

Graduate School of Information Sciences,
Tohoku University, Sendai, 980-8579, Japan.
E-mail address: `shimauchi@ims.is.tohoku.ac.jp`