ON THE COEFFICIENTS OF THE RIEMANN MAPPING FUNCTION FOR THE COMPLEMENT OF THE MANDELBROT SET (Conditions for Univalency of Functions and Applications)

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ON THE COEFFICIENTS OF THE RIEMANN MAPPING FUNCTION FOR THE COMPLEMENT OF THE MANDELBROT SET

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ABSTRACT. We denote the Mandelbrot set by \( \mathbb{M} \), the Riemann sphere by \( \hat{\mathbb{C}} \) and the unit disk by \( \mathbb{D} \). Let \( f : \mathbb{D} \to \mathbb{C} \setminus \{1/z : z \in \mathbb{M}\} \) and \( \Psi : \hat{\mathbb{C}} \setminus \bar{\mathbb{D}} \to \hat{\mathbb{C}} \setminus \mathbb{M} \) be the Riemann mapping functions and let their expansions be \( z + \sum_{m=2}^{\infty} a_m z^m \) and \( z + \sum_{m=0}^{\infty} b_m z^{-m} \), respectively. We consider several interesting properties of the coefficients \( a_m \) and \( b_m \). The detailed studies of these coefficients were given in [1, 3, 4, 5, 8]. This is a partial summary of [11], which contains Zagier's observations (see [1]).

1. INTRODUCTION

For \( c \in \mathbb{C} \), let \( P_c(z) := z^2 + c \) and \( P_c^n(z) = P_c(P_c(...P_c(z)...)) \) be the \( n \)-th iteration of \( P_c(z) \) with \( P_c^0(z) = z \). In the theory of one-dimensional complex dynamics, there is a detailed study of the dynamics of \( P_c(z) \) on the Riemann sphere \( \hat{\mathbb{C}} \). For each fixed \( c \), the (filled in) Julia set of \( P_c(z) \) consists of those values \( z \) that remain bounded under iteration. The Mandelbrot set \( \mathbb{M} \) consists of those parameter values \( c \) for which the Julia set is connected. It is known that \( \mathbb{M} = \{ c \in \mathbb{C} : \{P_c^n(0)\}_{n=0}^{\infty} \text{ is bounded}\} \), compact and is contained in the closed disk of radius 2. Furthermore, \( \mathbb{M} \) is connected. However, its local connectivity is still unknown, and there is a very important conjecture which states that \( \mathbb{M} \) is locally connected (see [2]).

Let \( G \subseteq \mathbb{C} \) be a simply connected domain with \( w_0 \in G \). Furthermore, let \( G' \subseteq \hat{\mathbb{C}} \) be a simply connected domain with \( \infty \in G' \) which has more than one boundary point. Due to the Riemann mapping theorem, there exist unique conformal mappings \( f : \mathbb{D} \to G \) such that \( f(0) = 0 \) and \( f'(0) > 0 \) and \( g : \mathbb{D}^* \to G' \) such that \( g(\infty) = \infty \) and \( \lim_{z \to \infty} g(z)/z > 0 \) respectively, where \( \mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \} \) and \( \mathbb{D}^* := \hat{\mathbb{C}} \setminus \mathbb{D} \). We call \( f \) (and \( g \)) the Riemann mapping function of \( G \) (and \( G' \)).

Douady and Hubbard demonstrated in [2] the connectedness of the Mandelbrot set by constructing a conformal isomorphism \( \Phi : \hat{\mathbb{C}} \setminus \mathbb{M} \to \mathbb{D}^* \). Note that \( \Psi := \Phi^{-1} \) is the Riemann mapping function of \( \hat{\mathbb{C}} \setminus \mathbb{M} \). We recall a lemma of Carathéodory.

**Lemma 1** (Carathéodory's Continuity Lemma). Let \( G \subseteq \hat{\mathbb{C}} \) be a simply connected domain and a function \( f \) maps \( \mathbb{D} \) conformally onto \( G \). Then \( f \) has a continuous extension to \( \bar{\mathbb{D}} \) if and only if the boundary of \( G \) is locally connected.

This implies if \( \Psi \) can be extended continuously to the unit circle, then the Mandelbrot set is locally connected. This is the motivation of our study.

Jungreis presented an algorithm to compute the coefficients \( b_m \) of the Laurent series expansion of \( \Psi(z) \) at \( \infty \) in [7].

*Key words and phrases.* Mandelbrot set; conformal mapping.
Several detailed studies of $b_m$ are given in [1, 3, 4, 8] and remarkable empirical observations are mentioned in [1] by Zagier. Especially a formula for $b_m$ is given in [3]. Many of these coefficients are shown to be zero and infinitely many non-zero coefficients are determined.

In addition, Ewing and Schober [5] studied the coefficients $a_m$ of the Taylor series expansion of the function $f(z) := 1/\Psi(1/z)$ at the origin. Note that $f$ is the Riemann mapping function of the bounded domain $\mathbb{C} \setminus \{1/z : z \in \mathbb{M}\}$ and $f$ has a continuous extension to the boundary if and only if the Mandelbrot set is locally connected.

In [12], Komori and Yamashita studied a generalization of $b_m$. Let $P_{d,c}(z) = z^d + c$ with an integer $d \geq 2$ and let $\mathbb{M}_d := \{c \in \mathbb{C} : \{P^{n}_{d,c}(0)\}_{n=0}^{\infty} \text{ is bounded }\}$. Constructing the Riemann mapping function $\Psi_d$ of $\hat{\mathbb{C}} \setminus \mathbb{M}_d$, they analyzed the coefficients $b_{d,m}$ of the Laurent series at $\infty$.

The author has been studying $b_{d,m}$ and the coefficients $a_{d,m}$ of the Taylor series at the origin of the function $f_d(z) := 1/\Psi_d(1/z)$ in [11].

In [12] and [11], there is a generalization of the results for $d = 2$, propositions for $d > 3$ and a verification of Zagier’s observations.

In this paper, we focus on the case $d = 2$. Especially we mention the observations by Zagier and the asymptotic behavior of $b_m$.

2. Computing the Laurent Series of $\Psi$

Now we introduce how to construct $\Phi$. This is established by Douady and Hubbard (see [1]).

**Theorem 2.** Let $c \in \hat{\mathbb{C}} \setminus \mathbb{M}$. Then

$$\phi_c(z) := z \prod_{k=1}^{\infty} \left( 1 + \frac{c}{P^{k-1}_{c}(z)^2} \right)^{1/2}$$

is well-defined on some neighborhood of $\infty$ which includes $c$. Moreover, $\Phi(c) := \phi_c(c)$ maps $\hat{\mathbb{C}} \setminus \mathbb{M}$ conformally onto $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, and satisfies $\Phi(c)/c \rightarrow 1$ as $c \rightarrow \infty$. Thus $\hat{\mathbb{C}} \setminus \mathbb{M}$ is simply connected and $\mathbb{M}$ is connected.

Set $A_n(c) = P^n_{c}(c)$ for simplicity. Applying the following proposition, we can calculate the coefficients $b_m$ of $\Psi$.

**Proposition 3** (see [1]).

$$A_n(\Psi(z)) = z^{2^n} + O\left(\frac{1}{z^{2^n-1}}\right).$$

Jungreis [7] presented an algorithm to compute $b_m$ and calculated the first 4095 numerical values of $b_m$. Bielefeld, Fisher and Haeseler calculated the first 8000 terms in [1].

Ewing and Schober [4] computed the first 240000 numerical values of $b_m$, using an backward recursion formula in the following way.

Let $n$ be a non-negative integer, and let

$$A_n(\Psi(z)) = \sum_{m=0}^{\infty} \beta_{n,m} z^{2^n-m} \quad \text{for } |z| > 1.$$
Using proposition 3, $\beta_{n,m} = 0$ for $n \geq 1$ and $1 \leq m \leq 2^n+1 - 2$. Furthermore $\beta_{n,0} = 1$ for all $n \in \mathbb{N} \cup \{0\}$. Since $P_0(\Psi(z)) = \Psi(z)$, obviously $\beta_{0,m} = b_{m-1}$ for $m \geq 1$. Applying the recursion $A_n(z) = A_{n-1}(z)^2 + z$ to equation (1), we get

$$
\sum_{m=0}^{\infty} \beta_{n,m} z^{2^n-m} = \sum_{m=0}^{\infty} \sum_{k=0}^{m} \beta_{n-1,k} \beta_{n-1,m-k} z^{2^n-m} + \sum_{m=2^n-1}^{\infty} \beta_{0,m-2^n-1} z^{2^n-m}.
$$

For $m \geq 2^n - 1$, we compare the coefficients of the right and left-hand side. Hence

$$
\beta_{n,m} = \sum_{k=0}^{m} \beta_{n-1,k} \beta_{n-1,m-k} + \beta_{0,m-2^n-1}.
$$

Since $\beta_{n-1,0} = 1$ and $\beta_{n,m} = 0$ for $n \geq 1$ and $1 \leq m \leq 2^n+1 - 2$, we obtain the following formula:

$$
\beta_{n,m} = 2\beta_{n-1,m} + \sum_{k=2^n-1}^{m-2^n+1} \beta_{n-1,k} \beta_{n-1,m-k} + \beta_{0,m-2^n-1} \text{ for } n \geq 1 \text{ and } m \geq 2^n - 1.
$$

This is the forward recursion to determine $\beta_{n,m}$ in terms of $\beta_{j,k}$ with $j < n$. A corresponding backward recursion formula is derived to be

$$
\beta_{n-1,m} = \frac{1}{2} \left( \beta_{n,m} - \sum_{k=2^n-1}^{m-2^n+1} \beta_{n-1,k} \beta_{n-1,m-k} - \beta_{0,m-2^n-1} \right).
$$

The formula gives $\beta_{m,n}$ in terms of $\beta_{j,k}$ with $j > n, k \leq m$. If $n$ is sufficiently large, then $\beta_{n,m} = 0$ for a fixed $m$. Hence, using this backward recursion formula, we can determine $\beta_{j,m}$ for all $j$.

**Example 4.** Considering $b_0 = \beta_{0,1} = (0 - \beta_{0,0})/2 = -1/2, b_1 = \beta_{0,2} = (0 - \beta_{0,1}^2 - \beta_{0,1})/2 = 1/8, \ldots$ yields

$$
\Psi(z) = z - \frac{1}{2} + \frac{1}{8}z - \frac{1}{4}z^2 + \frac{15}{128}z^3 + \frac{0}{z^4} - \frac{47}{1024}z^5 - \frac{1}{16}z^6 + \frac{987}{32768}z^7 + \cdots
$$

One can make a program for this procedure and derive the exacts value of $b_m$, because $b_m$ is a binary rational number.

**Theorem 5** (see [4]). If $n \geq 0$ and $m \geq 1$, then $2^{2m+3-2^{n+2}} \beta_{n,m}$ is an integer. In particular, $2^{2m+1} b_m$ is an integer.

The coefficient $a_m$ is also a binary rational number, since

$$
a_m = -b_{m-2} - \sum_{j=2}^{m-1} a_j b_{m-1-j} \text{ for } m \geq 2.
$$

**Remark 6.** In [1] Zagier made an empirical observation about the growth of the denominator of $b_m$, which we are going to mention in the next section.

Komori and Yamashita computed the exact values for the first 2000 terms in [12]. In [11], the author made a program to compute the exact values of $b_m$ by using C programing language with multiple precision arithmetic library GMP (see [6]), and the first 30000 exact values of $b_m$ were determined.
3. Observations by Zagier

Based on roughly 1000 coefficients, Zagier made several observations. In this paper, two of them are mentioned. We write \( m = m_02^n \) with \( n \geq 0 \), \( m_0 \) is odd.

Observation 7 (see [1]). It is \( b_m = 0 \), if and only if \( m_0 \leq 2^{n+1} - 5 \).

One direction of this statement has been proven in [1] and separately from that in [8].

Theorem 8. If \( n \geq 2 \) and \( m_0 \leq 2^{n+1} - 5 \), then \( b_m = 0 \).

It is still unknown whether the converse of this theorem is true. In [4], the only coefficients which have been observed to be zero are those mentioned in this theorem. In this publication Ewing and Schober proved the following theorem about zero-coefficients of \( a_m \).

Theorem 9 (see [5]). If \( 3 \leq m_0 \leq 2^{n+1} \), then \( a_m = 0 \).

The truth of the converse of this theorem is unknown. They reported that their computation of 1000 terms of \( a_m \) has not produced a zero-coefficient besides those indicated in theorem 9.

Now we consider the growth of the power of 2. For every non-zero rational number \( x \), there exists a unique integer \( v \) such that \( x = 2^vp/q \) with some integers \( p \) and \( q \) indivisible by 2. The 2-adic valuation \( \nu_2 : \mathbb{Q} \setminus \{0\} \rightarrow \mathbb{Z} \) is defined as:

\[
\nu_2(x) = v.
\]

We extend \( \nu_2 \) to the whole rational field \( \mathbb{Q} \) as follows,

\[
\nu(x) = \begin{cases} 
\nu_2(x) & \text{for } x \in \mathbb{Q} \setminus \{0\} \\
+\infty & \text{for } x = 0.
\end{cases}
\]

Due to theorem 5, if \( b_m \neq 0 \) then \( b_m = C/2^{-\nu(b_m)} \), where \( C \) is an odd number. Note that \( \nu((2m+2)!) \leq 2m+1 \) for a non-negative integer \( m \).

Observation 10 (see [1]). It is \( -\nu(b_m) \leq \nu((2m+2)!) \) for all \( m \). Equality attained exactly when \( m \) is odd.

In [12] a theorem for \( b_{d,m} \) which includes this observation was presented. However, \( d \) has to be prime and not an arbitrary integer as it was originally stated.

Corollary 11. It is \( -\nu(b_m) \leq \nu((2m+2)!) \) for all \( m \). Equality attained exactly when \( m \) is odd.

For \( a_m \) we have the following:

Corollary 12. It is \( -\nu(a_m) \leq \nu((2m-2)!) \) for all \( m \). Equality attained exactly when \( m \) is odd.

The generalization of these result is given in [11].
4. Observation for the Asymptotic Behavior of $b_m$

The result which Ewing and Schober obtained shows that the inequality $|b_m| < 1/m$ holds for $0 < m < 240000$. If there exist positive constants $c$ and $K$ such that the inequality $|b_m| < K/m^{1+c}$ holds for any natural number $m$, this would imply its absolute convergence and give that the Mandelbrot set is locally connected. Furthermore, such a bound imply Hölder continuity (see [1]). However it is not valid because of the following claim given in [1].

Claim 13. There is no Hölder continuous extension of $\Psi$ to $\overline{D}$.

On the other hand, the coefficients $b_m$ satisfying $|b_m| \geq 1/m$ have not been found yet.

The author focused on the local maximum of $|b_m|$ and considered the period of Jungreis' algorithm. The observation below for the behavior of $b_m$ can be made.

Observation 14 (see [11]). For fixed $1 \leq n \leq 7$, the maximum value of $|b_{2^{2n}-2}|$, $|b_{2^{2n-1}}|$, $\ldots$, $|b_{2^{2(n+1)-2}}|$ is $|b_{22n-2}|$. Furthermore, the sequence $|b_{2^2-2}|$, $|b_{2^4-2}|$, $|b_{2^6-2}|$, $\ldots$, $|b_{2^{2n}-2}|$, $\ldots$ is strictly monotonically decreasing.

It is still unknown whether it would be true for every $n$, and the behavior of $\{|b_{2^{2n}-2}|\}$ is the material of further research.

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