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ON THE COEFFICIENTS OF THE RIEMANN MAPPING FUNCTION FOR THE COMPLEMENT OF THE MANDELBROT SET

HIROKAZU SHIMAUCHI

ABSTRACT. We denote the Mandelbrot set by $M$, the Riemann sphere by $\hat{\mathbb{C}}$ and the unit disk by $D$. Let $f : D \to \mathbb{C} \setminus \{1/z : z \in M\}$ and $\Psi : \hat{\mathbb{C}} \setminus D \to \hat{\mathbb{C}} \setminus M$ be the Riemann mapping functions and let their expansions be $z + \sum_{m=2}^{\infty} a_m z^m$ and $z + \sum_{m=0}^{\infty} b_m z^{-m}$, respectively. We consider several interesting properties of the coefficients $a_m$ and $b_m$. The detailed studies of these coefficients were given in [1, 3, 4, 5, 8]. This is a partial summary of [11], which contains Zagier's observations (see [1]).

1. INTRODUCTION

For $c \in \mathbb{C}$, let $P_c(z) := z^2 + c$ and $P_c^{\infty}(z) = P_c(P_c(\ldots P_c(z)\ldots))$ be the $n$-th iteration of $P_c(z)$ with $P_c^{0}(z) = z$. In the theory of one-dimensional complex dynamics, there is a detailed study of the dynamics of $P_c(z)$ on the Riemann sphere $\hat{\mathbb{C}}$. For each fixed $c$, the (filled in) Julia set of $P_c(z)$ consists of those values $z$ that remain bounded under iteration. The Mandelbrot set $M$ consists of those parameter values $c$ for which the Julia set is connected. It is known that $M = \{c \in \mathbb{C} : \{P_c^{\infty}(0)\}_{n=0}^{\infty} \text{ is bounded}\}$, compact and is contained in the closed disk of radius 2. Furthermore, $M$ is connected. However, its local connectivity is still unknown, and there is a very important conjecture which states that $M$ is locally connected (see [2]).

Let $G \subset \mathbb{C}$ be a simply connected domain with $w_0 \in G$. Furthermore, let $G' \subset \hat{\mathbb{C}}$ be a simply connected domain with $\infty \in G'$ which has more than one boundary point. Due to the Riemann mapping theorem there exist unique conformal mappings $f : D \to G$ such that $f(0) = 0$ and $f'(0) > 0$ and $g : \mathbb{D}^* \to G'$ such that $g(\infty) = \infty$ and $\lim_{z \to \infty} \frac{g(z)}{z} > 0$ respectively, where $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{D}^* := \hat{\mathbb{C}} \setminus \mathbb{D}$. We call $f$ (and $g$) the Riemann mapping function of $G$ (and $G'$).

Douady and Hubbard demonstrated in [2] the connectedness of the Mandelbrot set by constructing a conformal isomorphism $\Phi : \hat{\mathbb{C}} \setminus M \to \mathbb{D}^*$. Note that $\Psi := \Phi^{-1}$ is the Riemann mapping function of $\hat{\mathbb{C}} \setminus M$. We recall a lemma of Carathéodory.

**Lemma 1** (Carathéodory's Continuity Lemma). Let $G \subset \hat{\mathbb{C}}$ be a simply connected domain and a function $f$ maps $\mathbb{D}$ conformally onto $G$. Then $f$ has a continuous extension to $\hat{\mathbb{D}}$ if and only if the boundary of $G$ is locally connected.

This implies if $\Psi$ can be extended continuously to the unit circle, then the Mandelbrot set is locally connected. This is the motivation of our study.

Jungreis presented an algorithm to compute the coefficients $b_m$ of the Laurent series expansion of $\Psi(z)$ at $\infty$ in [7].

*Key words and phrases.* Mandelbrot set; conformal mapping.
Several detailed studies of \( b_m \) are given in [1, 3, 4, 8] and remarkable empirical observations are mentioned in [1] by Zagier. Especially a formula for \( b_m \) is given in [3]. Many of these coefficients are shown to be zero and infinitely many non-zero coefficients are determined.

In addition, Ewing and Schober [5] studied the coefficients \( a_m \) of the Taylor series expansion of the function \( f(z) := 1/\Psi(1/z) \) at the origin. Note that \( f \) is the Riemann mapping function of the bounded domain \( \mathbb{C} \setminus \{1/z : z \in \mathbb{M}\} \) and \( f \) has a continuous extension to the boundary if and only if the Mandelbrot set is locally connected.

In [12], Komori and Yamashita studied a generalization of \( b_m \). Let \( P_{d,c}(z) = z^d + c \) with an integer \( d \geq 2 \) and let \( \mathbb{M}_d := \{ c \in \mathbb{C} : \{ P_{d,c}^n(0) \}_{n=0}^{\infty} \text{ is bounded} \} \). Constructing the Riemann mapping function \( \Psi_d \) of \( \mathbb{C} \setminus \mathbb{M}_d \), they analyzed the coefficients \( b_{d,m} \) of the Laurent series at \( \infty \).

The author has been studying \( b_{d,m} \) and the coefficients \( a_{d,m} \) of the Taylor series at the origin of the function \( f_d(z) := 1/\Psi_d(1/z) \) in [11].

In [12] and [11], there is a generalization of the results for \( d = 2 \), propositions for \( d > 3 \) and a verification of Zagier's observations.

In this paper, we focus on the case \( d = 2 \). Especially we mention the observations by Zagier and the asymptotic behavior of \( b_m \).

2. Computing the Laurent Series of \( \Psi \)

Now we introduce how to construct \( \Phi \). This is established by Douady and Hubbard (see [1]).

**Theorem 2.** Let \( c \in \mathbb{C} \setminus \mathbb{M} \). Then

\[
\phi_c(z) := z \prod_{k=1}^{\infty} \left( 1 + \frac{c}{P_{c}^{k-1}(z)^2} \right)^{\frac{1}{2}}
\]

is well-defined on some neighborhood of \( \infty \) which includes \( c \). Moreover, \( \Phi(c) := \phi_c(c) \) maps \( \mathbb{C} \setminus \mathbb{M} \) conformally onto \( \mathbb{C} \setminus \overline{D} \), and satisfies \( \Phi(c)/c \to 1 \) as \( c \to \infty \). Thus \( \mathbb{C} \setminus \mathbb{M} \) is simply connected and \( \mathbb{M} \) is connected.

Set \( A_n(c) = P_{c}^{\infty}(c) \) for simplicity. Applying the following proposition, we can calculate the coefficients \( b_m \) of \( \Psi \).

**Proposition 3** (see [1]).

\[
A_n(\Psi(z)) = z^{2^n} + O\left( \frac{1}{z^{2^n-1}} \right).
\]

Jungreis [7] presented an algorithm to compute \( b_m \) and calculated the first 4095 numerical values of \( b_m \). Bielefeld, Fisher and Haeseler calculated the first 8000 terms in [1].

Ewing and Schober [4] computed the first 240000 numerical values of \( b_m \), using an backward recursion formula in the following way.

Let \( n \) be a non-negative integer, and let

\[
A_n(\Psi(z)) = \sum_{m=0}^{\infty} \beta_{n,m} z^{2^n-m} \text{ for } |z| > 1.
\]
Using proposition 3, $\beta_{n,m} = 0$ for $n \geq 1$ and $1 \leq m \leq 2^{n+1} - 2$. Furthermore, $\beta_{n,0} = 1$ for all $n \in \mathbb{N} \cup \{0\}$. Since $P_0(\Psi(z)) = \Psi(z)$, obviously $\beta_{0,m} = b_{m-1}$ for $m \geq 1$. Applying the recursion $A_n(z) = A_{n-1}(z)^2 + z$ to equation (1), we get

$$
\sum_{m=0}^{\infty} \beta_{n,m} z^{2^n-m} = \sum_{m=0}^{\infty} \sum_{k=0}^{m} \beta_{n-1,k} \beta_{n-1,m-k} z^{2^n-m} + \sum_{m=2^n-1}^{\infty} \beta_{0,m-2^n-1} z^{2^n-m}.
$$

For $m \geq 2^n - 1$, we compare the coefficients of the right and left-hand side. Hence

$$
\beta_{n,m} = \sum_{k=0}^{m} \beta_{n-1,k} \beta_{n-1,m-k} + \beta_{0,m-2^n-1}.
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Since $P_0(\Psi(z)) = \Psi(z)$, obviously $\beta_{0,m} = b_{m-1}$ for $m \geq 1$. Applying the recursion $A_n(z) = A_{n-1}(z)^2 + z$ to equation (1), we get

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$$
\beta_{n,m} = \sum_{k=0}^{m} \beta_{n-1,k} \beta_{n-1,m-k} + \beta_{0,m-2^n-1}.
$$

This is the forward recursion to determine $\beta_{n,m}$ in terms of $\beta_{j,k}$ with $j < n$. A corresponding backward recursion formula is derived to be

$$
\beta_{n-1,m} = \frac{1}{2} \left( \beta_{n,m} - \sum_{k=2^n-1}^{m-2^n+1} \beta_{n-1,k} \beta_{n-1,m-k} - \beta_{0,m-2^n-1} \right).
$$

The formula gives $\beta_{m,n}$ in terms of $\beta_{j,k}$ with $j > n, k \leq m$. If $n$ is sufficiently large, then $\beta_{n,m} = 0$ for a fixed $m$. Hence, using this backward recursion formula, we can determine $\beta_{j,m}$ for all $j$.

**Example 4.** Considering $b_0 = \beta_{0,1} = (0 - \beta_{0,0})/2 = -1/2$, $b_1 = \beta_{0,2} = (0 - \beta_{0,1}^2 - \beta_{0,1})/2 = 1/8$, ... yields

$$
\Psi(z) = z - \frac{1}{2} - \frac{1}{8z} + \frac{1}{4z^2} + \frac{15}{128z^3} + \frac{0}{z^4} - \frac{47}{1024z^5} - \frac{1}{16z^6} + \frac{987}{32768z^7} + \cdots
$$

One can make a program for this procedure and derive the exacts value of $b_m$, because $b_m$ is a binary rational number.

**Theorem 5** (see [4]). If $n \geq 0$ and $m \geq 1$, then $2^{2m+3-2^{n+2}}\beta_{n,m}$ is an integer. In particular, $2^{2m+1}b_m$ is an integer.

The coefficient $a_m$ is also a binary rational number, since

$$
a_m = -b_{m-2} - \sum_{j=2}^{m-1} a_j b_{m-1-j} \text{ for } m \geq 2.
$$

**Remark 6.** In [1] Zagier made an empirical observation about the growth of the denominator of $b_m$, which we are going to mention in the next section.

Komori and Yamashita computed the exact values for the first 2000 terms in [12]. In [11], the author made a program to compute the exact values of $b_m$ by using C programing language with multiple precision arithmetic library GMP (see [6]), and the first 30000 exact values of $b_m$ were determined.
3. Observations by Zagier

Based on roughly 1000 coefficients, Zagier made several observations. In this paper, two of them are mentioned. We write $m = m_0 2^n$ with $n \geq 0$, $m_0$ is odd.

Observation 7 (see [1]). It is $b_m = 0$, if and only if $m_0 \leq 2^{n+1} - 5$.

One direction of this statement has been proven in [1] and separately from that in [8].

Theorem 8. If $n \geq 2$ and $m_0 \leq 2^{n+1} - 5$, then $b_m = 0$.

It is still unknown whether the converse of this theorem is true. In [4], the only coefficients which have been observed to be zero are those mentioned in this theorem. In this publication Ewing and Schober proved the following theorem about zero-coefficients of $a_m$.

Theorem 9 (see [5]). If $3 \leq m_0 \leq 2^{n+1}$, then $a_m = 0$.

The truth of the converse of this theorem is unknown. They reported that their computation of 1000 terms of $a_m$ has not produced a zero-coefficient besides those indicated in theorem 9.

Now we consider the growth of the power of 2. For every non-zero rational number $x$, there exists a unique integer $v$ such that $x = 2^v p/q$ with some integers $p$ and $q$ indivisible by 2. The 2-adic valuation $\nu_2 : \mathbb{Q} \setminus \{0\} \rightarrow \mathbb{Z}$ is defined as:

$$\nu_2(x) = v.$$ 

We extend $\nu_2$ to the whole rational field $\mathbb{Q}$ as follows,

$$\nu(x) = \begin{cases} \nu_2(x) & \text{for } x \in \mathbb{Q} \setminus \{0\} \\ +\infty & \text{for } x = 0. \end{cases}$$

Due to theorem 5, if $b_m \neq 0$ then $b_m = C/2^{-\nu(b_m)}$, where $C$ is an odd number. Note that $\nu(((2m+2)!) \leq 2m+1$ for a non-negative integer $m$.

Observation 10 (see [1]). It is $-\nu(b_m) \leq \nu((2m+2)!)$ for all $m$. Equality attained exactly when $m$ is odd.

In [12] a theorem for $b_{d,m}$ which includes this observation was presented. However, $d$ has to be prime and not an arbitrary integer as it was originally stated.

Corollary 11. It is $-\nu(b_m) \leq \nu((2m+2)!)$ for all $m$. Equality attained exactly when $m$ is odd.

For $a_m$ we have the following:

Corollary 12. It is $-\nu(a_m) \leq \nu((2m-2)!)$ for all $m$. Equality attained exactly when $m$ is odd.

The generalization of these result is given in [11].
4. Observation for the Asymptotic Behavior of $b_m$

The result which Ewing and Schober obtained shows that the inequality $|b_m| < 1/m$ holds for $0 < m < 240000$. If there exist positive constants $c$ and $K$ such that the inequality $|b_m| < K/m^{1+c}$ holds for any natural number $m$, this would imply its absolute convergence and give that the Mandelbrot set is locally connected. Furthermore, such a bound imply Hölder continuity (see [1]). However it is not valid because of the following claim given in [1].

**Claim 13.** There is no Hölder continuous extension of $\Psi$ to $\overline{D}$.

On the other hand, the coefficients $b_m$ satisfying $|b_m| \geq 1/m$ have not been found yet.

The author focused on the local maximum of $|b_m|$ and considered the period of Jungreis' algorithm. The observation below for the behavior of $b_m$ can be made.

**Observation 14 (see [11]).** For fixed $1 \leq n \leq 7$, the maximum value of $|b_{2^{2n}-2}|$, $|b_{2^{2n}-1}|$, ..., $|b_{2^{2(n+1)}-3}|$ is $|b_{2^{2n}-2}|$. Furthermore, the sequence $|b_{2^2-2}|, |b_{2^4-2}|, |b_{2^8-2}|$, ..., $|b_{2^{2^n}-2}|$, ... is strictly monotonically decreasing.

It is still unknown whether it would be true for every $n$, and the behavior of $\{ |b_{2^{2n}-2}| \}$ is the material of further research.

**References**


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