<table>
<thead>
<tr>
<th>Title</th>
<th>Univalency of analytic functions associated with Schwarzian derivative (Conditions for Univalency of Functions and Applications)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Nunokawa, Mamoru; Uyanik, Neslihan; Owa, Shigeyoshi</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2011), 1772: 104-108</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2011-12</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/171684">http://hdl.handle.net/2433/171684</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Univalency of analytic functions associated with Schwarzian derivative

Mamoru Nunokawa, Neslihan Uyanik and Shigeyoshi Owa

The authors would like to dedicate this paper to the late Professor Shigeo Ozaki

1 Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $U = \{ z \in \mathbb{C} : |z| < 1 \}$. For $f(z) \in \mathcal{A}$, the following differential operator

$$\{f(z), z\} = \left( \frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left( \frac{f''(z)}{f'(z)} \right)^2$$

is said to be the Schwarzian derivative of $f(z)$ or the Schwarzian differential operator of $f(z)$. For the Schwarzian derivative of $f(z) \in \mathcal{A}$, the following results by Nehari [2] are well-known.

**Theorem A** If $f(z) \in \mathcal{A}$ is univalent in $U$, then

$$|\{f(z), z\}| \leq \frac{6}{(1 - |z|^2)^2} \quad (z \in U).$$

The equality is attained by Koebe function $f(z)$ given by

$$f(z) = \frac{z}{(1 - z)^2}$$

and its rotation.

**Theorem B** If $f(z) \in \mathcal{A}$ satisfies

---

2010 *Mathematics Subject Classification*: Primary 30C45

*Key Words and Phrases*: Analytic function, Univalent function, Schwarzian derivative
then $f(z)$ is univalent in $U$.

For Theorem B, Hille [1] has noticed that 2 in (1.5) is the best possible constant.

Let us define the function $g(z)$ by

$$g(z) = \frac{f'(x)(1-|x|^2)}{f\left(\frac{z+x}{1+\overline{x}z}\right) - f(x)}$$

(1.6)  

$$= \frac{1}{z} + \overline{x} - \frac{1}{2}(1-|x|^2)\frac{f''(x)}{f'(x)} - \frac{1}{6}(1-|x|^2)^2\left\{\left(\frac{f''(x)}{f'(x)}\right)' - \frac{1}{2}\left(\frac{f''(x)}{f'(x)}\right)^2\right\}z + \cdots$$

$$= \frac{1}{z} + h(z, x)$$

for $f(z) \in A$ and some complex $x$ such that $|x| < 1$, where

(1.7)  

$$h(z, x) = \overline{x} - \frac{1}{2}(1-|x|^2)\frac{f''(x)}{f'(x)} - \frac{1}{6}(1-|x|^2)^2\left\{\left(\frac{f''(x)}{f'(x)}\right)' - \frac{1}{2}\left(\frac{f''(x)}{f'(x)}\right)^2\right\}z + \cdots$$

Then, it is easy to see that $g(z)$ is univalent in $U$ if and only if $f(z)$ is univalent in $U$.

On the other hand, Ozaki and Nunokawa [3] have given the following result.

**Theorem C**  If $f(z) \in A$ is univalent in $U$, then

(1.8)  

$$|h'(0, x)| \leq \frac{(1-|x|^2)^2}{6} \{f(x), x\} \leq 1 \quad (|x| < 1).$$

If $f(z) \in A$ satisfies

(1.9)  

$$|h'(0, x)| \leq \frac{1}{3} \quad (|x| < 1),$$

then $f(z)$ is univalent in $U$.

To discuss the univalency for our problem, we have to recall here the following result which is called Darboux theorem.

**Lemma 1**  Let $E$ be a domain covered by Jordan curve $C$ and let $w = f(z)$ be analytic in $E$. If a point $z$ moves on $C$ in the positive direction, then $w$ also moves on the Jordan curve $\Gamma = f(C)$ in the positive direction. Let $\Delta$ be the inside of the curve $\Gamma$. Then $w = f(z)$ is univalent in $E$ and maps $E$ onto $\Delta$ conformally.
Proof Let $w_0 \in \Delta$ and $\phi(z) = w - w_0 = f(z) - w_0$. Then $\phi(z)$ is analytic in $E$, $\phi(z) \neq 0$ on $C$, and

\[
\frac{1}{2\pi} \int_C d\arg \phi(z) = \frac{1}{2} \int_{\Gamma} d\arg(w-w_0).
\]

From the argument theorem, the left hand side of (1.10) shows that the number of zeros of $\phi(z)$ in $E$ and the right hand side of (1.10) shows the argument momentum when $w$ moves on $\Gamma$ in the positive direction. Therefore, the right hand side of (1.10) should be just one. This shows us that $\phi(z) = f(z) - w_0$ has one zero in $E$.

Let us put $w_0 = f(z_0)$. Then there exists only one point $z_0 \in E$ for an arbitrary $w_0 \in \Delta$. This means that $f(z)$ is univalent in $E$.

For the case of $w_0 \not\in \Delta$, we obtain that

\[
\int_C d\arg(w-w_0) = 0,
\]

which gives us that $\phi(z) = f(z) - w_0$ has no zero in $E$. This completes the proof of the lemma.

We note that we owe the proof of Lemma 1 by Tsuji [4].

2 Univalency of functions associated with Schwarzian derivative

An application for Lemma 1 derives

Theorem 1 If $f(z) \in A$ satisfies

\[
\text{Re} h'(z,x) > \alpha \quad (z \in U)
\]

for some real $\alpha$ ($\alpha > 1$) and for all $|x| < 1$, then $f(z)$ is univalent in $U$, where $h(z,x)$ is given by (1.7).

Proof Let us put $0 < |z| < 1$ and $|x| < 1$. Then, using $g(z)$ and $h(z,x)$ given by (1.7), we have that

\[
g(z) - \frac{1}{z} = h(z,x)
\]

is analytic in $U$. Note that $f(z)$ is univalent in $U$ if and only if $g(z)$ is univalent in $U$. We know that

\[
\left( g(z_2) - \frac{1}{z_2} \right) - \left( g(z_1) - \frac{1}{z_1} \right) = h(z_2,x) - h(z_1,x) = \int_{z_1}^{z_2} \left( \frac{dh(z,x)}{dz} \right) dz,
\]

where the integral is taken on the line segment $z_1 z_2$ such that $z_1 \neq z_2$ and $0 < |z_1| = |z_2| = r < 1$. Letting

\[
z = z_1 + (z_2 - z_1)t \quad (0 \leq t \leq 1),
\]
we have that

\begin{equation}
\int_{z_1}^{z_2} \left( \frac{dh(z,x)}{dz} \right) dz = (z_2 - z_1) \int_{0}^{1} \left( \frac{dh(z,x)}{dz} \right) dz.
\end{equation}

Therefore, we obtain that

\[ g(z_2) - g(z_1) + \frac{z_2 - z_1}{z_1 z_2} = (z_2 - z_1) \int_{0}^{1} h'(z,x) dt. \]

This gives us that

\begin{equation}
\frac{g(z_2) - g(z_1)}{z_2 - z_1} = \int_{0}^{1} h'(z,x) dt - \frac{1}{z_1 z_2}
\end{equation}

If there exist two points $z_1$ and $z_2$ such that $z_1 \neq z_2$ and $|z_1| = |z_2| = r < 1$ for which $g(z_1) = g(z_2)$, then we have that

\[ 0 = \int_{0}^{1} \text{Re} \left( h'(z,x) - \frac{1}{z_1 z_2} \right) dt > \int_{0}^{1} \left( \alpha - \frac{1}{|z_1 z_2|} \right) dt = \frac{\alpha r^2 - 1}{r^2}. \]

Therefore, letting $r \to 1^-$, we see that

\[ \int_{0}^{1} \text{Re} \left( h'(z,x) - \frac{1}{z_1 z_2} \right) dt > 0. \]

This is the contradiction and shows that there exist no points $z_1$ and $z_2$ such that $z_1 \neq z_2$ and $g(z_1) = g(z_2)$ in $U$. Since $g(z)$ is univalent in $U$, using Lemma 1, we conclude that $f(z)$ is univalent in $U$.

References


Mamoru Nunokawa  
Emeritus Professor of University of Gunma  
Hoshikuki 798-8, Chuou-Ward, Chiba 260-0808, Japan  
E-mail: mamoru.nuno@doctor.nifty.jp

Neslihan Uyanik  
Department of mathematics  
Kazim Karabekir Faculty of Education  
Atatürk University  
25240 Erzurum, Turkey  
E-mail: nesuyan@yahoo.com

Shigeyoshi Owa  
Department of Mathematics  
Kinki University  
Higashi-Osaka, Osaka 577-8502, Japan  
E-mail: owa@math.kindai.ac.jp