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Kyoto University
Univalency of analytic functions associated with Schwarzian derivative

Mamoru Nunokawa, Neslihan Uyanik and Shigeyoshi Owa

The authors would like to dedicate this paper to the late Professor Shigeo Ozaki

1 Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. For $f(z) \in \mathcal{A}$, the following differential operator

$$\{f(z), z\} = \left( \frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left( \frac{f''(z)}{f'(z)} \right)^2$$

is said to be the Schwarzian derivative of $f(z)$ or the Schwarzian differential operator of $f(z)$. For the Schwarzian derivative of $f(z) \in \mathcal{A}$, the following results by Nehari [2] are well-known.

**Theorem A** If $f(z) \in \mathcal{A}$ is univalent in $U$, then

$$\left| \{f(z), z\} \right| \leq \frac{6}{(1-|z|^2)^2} \quad (z \in U).$$

The equality is attained by Koebe function $f(z)$ given by

$$f(z) = \frac{z}{(1-z)^2}$$

and its rotation.

**Theorem B** If $f(z) \in \mathcal{A}$ satisfies

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then $f(z)$ is univalent in $U$.

For Theorem B, Hille [1] has noticed that 2 in (1.5) is the best possible constant.

Let us define the function $g(z)$ by

\[
g(z) = \frac{f'(x)(1 - |x|^2)}{f\left(\frac{z + x}{1 + \overline{x}z}\right) - f(x)}
\]

\[
= \frac{1}{z} + \overline{x} - \frac{1}{2}(1 - |x|^2)\frac{f''(x)}{f'(x)} - \frac{1}{6}(1 - |x|^2)^2\left\{\left(\frac{f''(x)}{f'(x)}\right)' - \frac{1}{2}\left(\frac{f''(x)}{f'(x)}\right)^2\right\}z + \cdots
\]

for $f(z) \in A$ and some complex $x$ such that $|x| < 1$, where

\[
h(z, x) = \overline{x} - \frac{1}{2}(1 - |x|^2)\frac{f''(x)}{f'(x)} - \frac{1}{6}(1 - |x|^2)^2\left\{\left(\frac{f''(x)}{f'(x)}\right)' - \frac{1}{2}\left(\frac{f''(x)}{f'(x)}\right)^2\right\}z + \cdots
\]

Then, it is easy to see that $g(z)$ is univalent in $U$ if and only if $f(z)$ is univalent in $U$.

On the other hand, Ozaki and Nunokawa [3] have given the following result.

**Theorem C** If $f(z) \in A$ is univalent in $U$, then

\[
|h'(0, x)| \leq \frac{(1 - |x|^2)^2}{6} |\{f(x), x\}| \leq 1 \quad (|x| < 1).
\]

If $f(z) \in A$ satisfies

\[
|h'(0, x)| \leq \frac{1}{3} \quad (|x| < 1),
\]

then $f(z)$ is univalent in $U$.

To discuss the univalency for our problem, we have to recall here the following result which is called Darboux theorem.

**Lemma 1** Let $E$ be a domain covered by Jordan curve $C$ and let $w = f(z)$ be analytic in $E$. If a point $z$ moves on $C$ in the positive direction, then $w$ also moves on the Jordan curve $\Gamma = f(C)$ in the positive direction. Let $\Delta$ be the inside of the curve $\Gamma$. Then $w = f(z)$ is univalent in $E$ and maps $E$ onto $\Delta$ conformally.
Proof Let $w_0 \in \Delta$ and $\phi(z) = w - w_0 = f(z) - w_0$. Then $\phi(z)$ is analytic in $E$, $\phi(z) \neq 0$ on $C$, and

\[ \frac{1}{2\pi} \int_C d\arg \phi(z) = \frac{1}{2} \int_{\Gamma} d\arg(w - w_0). \]

From the argument theorem, the left hand side of (1.10) shows that the number of zeros of $\phi(z)$ in $E$ and the right hand side of (1.10) shows the argument momentum when $w$ moves on $\Gamma$ in the positive direction. Therefore, the right hand side of (1.10) should be just one. This shows us that $\phi(z) = f(z) - w_0$ has one zero in $E$.

Let us put $w_0 = f(z_0)$. Then there exists only one point $z_0 \in E$ for an arbitrary $w_0 \in \Delta$. This means that $f(z)$ is univalent in $E$.

For the case of $w_0 \notin \Delta$, we obtain that

\[ \int_C d\arg(w - w_0) = 0, \]

which gives us that $\phi(z) = f(z) - w_0$ has no zero in $E$. This completes the proof of the lemma.

We note that we owe the proof of Lemma 1 by Tsuji [4].

2 Univalency of functions associated with Schwarzian derivative

An application for Lemma 1 derives

Theorem 1 If $f(z) \in A$ satisfies

\[ \text{Re} h'(z, x) > \alpha \quad (z \in U) \]

for some real $\alpha$ ($\alpha > 1$) and for all $|x| < 1$, then $f(z)$ is univalent in $U$, where $h(z, x)$ is given by (1.7).

Proof Let us put $0 < |z| < 1$ and $|z| < 1$. Then, using $g(z)$ and $h(z, x)$ given by (1.7), we have that

\[ g(z) - \frac{1}{z} = h(z, x) \]

is analytic in $U$. Note that $f(z)$ is univalent in $U$ if and only if $g(z)$ is univalent in $U$. We know that

\[ \left( g(z_2) - \frac{1}{z_2} \right) - \left( g(z_1) - \frac{1}{z_1} \right) = h(z_2, x) - h(z_1, x) = \int_{z_1}^{z_2} \left( \frac{dh(z, x)}{dz} \right) dz, \]

where the integral is taken on the line segment $z_1z_2$ such that $z_1 \neq z_2$ and $0 < |z_1| = |z_2| = r < 1$. Letting

\[ z = z_1 + (z_2 - z_1)t \quad (0 \leq t \leq 1), \]
we have that

\[(2.4) \quad \int_{z_1}^{z_2} \left(\frac{dh(z,x)}{dz}\right) \, dz = (z_2 - z_1) \int_{0}^{1} \left(\frac{dh(z,x)}{dz}\right) \, dz.\]

Therefore, we obtain that

\[g(z_2) - g(z_1) + \frac{z_2 - z_1}{z_1 z_2} = (z_2 - z_1) \int_{0}^{1} h'(z,x) \, dt.\]

This gives us that

\[(2.5) \quad \frac{g(z_2) - g(z_1)}{z_2 - z_1} = \int_{0}^{1} h'(z,x) \, dt - \frac{1}{z_1 z_2} = \int_{0}^{1} \left(h'(z,x) - \frac{1}{z_1 z_2}\right) \, dt.\]

If there exist two points \(z_1\) and \(z_2\) such that \(z_1 \neq z_2\) and \(|z_1| = |z_2| = r < 1\) for which \(g(z_1) = g(z_2)\), then we have that

\[0 = \int_{0}^{1} \Re \left(h'(z,x) - \frac{1}{z_1 z_2}\right) \, dt > \int_{0}^{1} \left(\alpha - \frac{1}{|z_1 z_2|}\right) \, dt = \frac{\alpha r^2 - 1}{r^2}.\]

Therefore, letting \(r \to 1^-\), we see that

\[\int_{0}^{1} \Re \left(h'(z,x) - \frac{1}{z_1 z_2}\right) \, dt > 0.\]

This is the contradiction and shows that there exist no points \(z_1\) and \(z_2\) such that \(z_1 \neq z_2\) and \(g(z_1) = g(z_2)\) in \(U\). Since \(g(z)\) is univalent in \(U\), using Lemma 1, we conclude that \(f(z)\) is univalent in \(U\).

References


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