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Univalency of analytic functions associated with Schwarzian derivative

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The authors would like to dedicate this paper to the late Professor Shigeo Ozaki

1 Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For $f(z) \in \mathcal{A}$, the following differential operator

$$\begin{aligned}\{f(z), z\} &= \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 \\ &= \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2\end{aligned} \tag{1.2}$$

is said to be the Schwarzian derivative of $f(z)$ or the Schwarzian differential operator of $f(z)$. For the Schwarzian derivative of $f(z) \in \mathcal{A}$, the following results by Nehari [2] are well-known.

Theorem A *If $f(z) \in \mathcal{A}$ is univalent in $\mathbb{U}$, then*

$$|\{f(z), z\}| \leqq \frac{6}{(1-|z|^2)^2} \qquad (z \in \mathbb{U}). \tag{1.3}$$

The equality is attained by Koebe function $f(z)$ given by

$$f(z) = \frac{z}{(1-z)^2} \tag{1.4}$$

and its rotation.

Theorem B *If $f(z) \in \mathcal{A}$ satisfies*

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$$(1.5) \qquad |\{f(z),z\}| \leqq \frac{2}{(1-|z|^2)^2} \qquad (z\in\mathbb{U}),$$

then $f(z)$ is univalent in $\mathbb{U}$.

For Theorem B, Hille [1] has noticed that 2 in (1.5) is the best possible constant.
Let us define the function $g(z)$ by

$$(1.6) \qquad g(z) = \frac{f'(x)(1-|x|^2)}{f\left(\dfrac{z+x}{1+\overline{x}z}\right)-f(x)}$$

$$= \frac{1}{z}+\overline{x}-\frac{1}{2}(1-|x|^2)\frac{f''(x)}{f'(x)}-\frac{1}{6}(1-|x|^2)^2\left\{\left(\frac{f''(x)}{f'(x)}\right)'-\frac{1}{2}\left(\frac{f''(x)}{f'(x)}\right)^2\right\}z+\cdots$$

$$= \frac{1}{z}+h(z,x)$$

for $f(z)\in\mathcal{A}$ and some complex x such that $|x|<1$, where

$$(1.7) \quad h(z,x) = \overline{x}-\frac{1}{2}(1-|x|^2)\frac{f''(x)}{f'(x)}-\frac{1}{6}(1-|x|^2)^2\left\{\left(\frac{f''(x)}{f'(x)}\right)'-\frac{1}{2}\left(\frac{f''(x)}{f'(x)}\right)^2\right\}z+\cdots.$$

Then, it is easy to see that $g(z)$ is univalent in $\mathbb{U}$ if and only if $f(z)$ is univalent in $\mathbb{U}$.
On the other hand, Ozaki and Nunokawa [3] have given the following result.

Theorem C *If $f(z)\in\mathcal{A}$ is univalent in $\mathbb{U}$, then*

$$(1.8) \qquad |h'(0,x)| \leqq \frac{(1-|x|^2)^2}{6}|\{f(x),x\}| \leqq 1 \qquad (|x|<1).$$

If $f(z)\in\mathcal{A}$ satisfies

$$(1.9) \qquad |h'(0,x)| \leqq \frac{1}{3} \qquad (|x|<1),$$

then $f(z)$ is univalent in $\mathbb{U}$.

To discuss the univalency for our problem, we have to recall here the following result which is called Darboux theorem.

Lemma 1 *Let $\mathbb{E}$ be a domain covered by Jordan curve C and let $w=f(z)$ be analytic in $\mathbb{E}$. If a point z moves on C in the positive direction, then w also moves on the Jordan curve $\Gamma=f(C)$ in the positive direction. Let Δ be the inside of the curve Γ. Then $w=f(z)$ is univalent in $\mathbb{E}$ and maps $\mathbb{E}$ onto Δ conformally.*

Proof Let $w_0 \in \Delta$ and $\phi(z) = w - w_0 = f(z) - w_0$. Then $\phi(z)$ is analytic in $\mathbb{E}, \phi(z) \neq 0$ on C, and

$$\frac{1}{2\pi}\int_C d\arg\phi(z) = \frac{1}{2}\int_\Gamma d\arg(w - w_0). \tag{1.10}$$

From the argument theorem, the left hand side of (1.10) shows that the number of zeros of $\phi(z)$ in $\mathbb{E}$ and the right hand side of (1.10) shows the argument momentum when w moves on Γ in the positive direction. Therefore, the right hand side of (1.10) should be just one. This shows us that $\phi(z) = f(z) - w_0$ has one zero in $\mathbb{E}$.

Let us put $w_0 = f(z_0)$. Then there exists only one point $z_0 \in \mathbb{E}$ for an arbitrary $w_0 \in \Delta$. This means that $f(z)$ is univalent in $\mathbb{E}$.

For the case of $w_0 \notin \Delta$, we obtain that

$$\int_C d\arg(w - w_0) = 0, \tag{1.11}$$

which gives us that $\phi(z) = f(z) - w_0$ has no zero in $\mathbb{E}$. This completes the proof of the lemma.

We note that we owe the proof of Lemma 1 by Tsuji [4].

2 Univalency of functions associated with Schwarzian derivative

An application for Lemma 1 derives

Theorem 1 *If $f(z) \in \mathcal{A}$ satisfies*

$$\mathrm{Re}\, h'(z,x) > \alpha \qquad (z \in \mathbb{U}) \tag{2.1}$$

for some real α ($\alpha > 1$) and for all $|x| < 1$, then $f(z)$ is univalent in $\mathbb{U}$, where $h(z,x)$ is given by (1.7).

Proof Let us put $0 < |z| < 1$ and $|x| < 1$. Then, using $g(z)$ and $h(z,x)$ given by (1.7), we have that

$$g(z) - \frac{1}{z} = h(z,x) \tag{2.2}$$

is analytic in $\mathbb{U}$. Note that $f(z)$ is univalent in $\mathbb{U}$ if and only if $g(z)$ is univalent in $\mathbb{U}$. We know that

$$\left(g(z_2) - \frac{1}{z_2}\right) - \left(g(z_1) - \frac{1}{z_1}\right) = h(z_2,x) - h(z_1,x) = \int_{z_1}^{z_2}\left(\frac{dh(z,x)}{dz}\right)dz, \tag{2.3}$$

where the integral is taken on the line segment $z_1 z_2$ such that $z_1 \neq z_2$ and $0 < |z_1| = |z_2| = r < 1$. Letting

$$z = z_1 + (z_2 - z_1)t \qquad (0 \leqq t \leqq 1),$$

we have that

$$\int_{z_1}^{z_2} \left(\frac{dh(z,x)}{dz}\right) dz = (z_2 - z_1) \int_0^1 \left(\frac{dh(z,x)}{dz}\right) dz. \tag{2.4}$$

Therefore, we obtain that

$$g(z_2) - g(z_1) + \frac{z_2 - z_1}{z_1 z_2} = (z_2 - z_1) \int_0^1 h'(z,x)dt.$$

This gives us that

$$\begin{aligned} \frac{g(z_2) - g(z_1)}{z_2 - z_1} &= \int_0^1 h'(z,x)dt - \frac{1}{z_1 z_2} \\ &= \int_0^1 \left(h'(z,x) - \frac{1}{z_1 z_2}\right) dt. \end{aligned} \tag{2.5}$$

If there exist two points z_1 and z_2 such that $z_1 \neq z_2$ and $|z_1| = |z_2| = r < 1$ for which $g(z_1) = g(z_2)$, then we have that

$$0 = \int_0^1 \mathrm{Re}\left(h'(z,x) - \frac{1}{z_1 z_2}\right) dt > \int_0^1 \left(\alpha - \frac{1}{|z_1 z_2|}\right) dt = \frac{\alpha r^2 - 1}{r^2}.$$

Therefore, letting $r \to 1^-$, we see that

$$\int_0^1 \mathrm{Re}\left(h'(z,x) - \frac{1}{z_1 z_2}\right) dt > 0.$$

This is the contradiction and shows that there exist no points z_1 and z_2 such that $z_1 \neq z_2$ and $g(z_1) = g(z_2)$ in $\mathbb{U}$. Since $g(z)$ is univalent in $\mathbb{U}$, using Lemma 1, we conclude that $f(z)$ is univalent in $\mathbb{U}$.

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