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**Note:** The Kazoja's page appears to be in Japanese, and the text seems to be a translation or an abstract of the cited publication. The page number is not visible in the image provided.
On weakly $\Phi$-like of order $\alpha$
with respect to certain analytic functions

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Abstract

For analytic functions $f(z)$ in the open unit disk $E$, weakly $\Phi$-like of order $\alpha$ with respect to a function $g(z)$ is introduced. The purpose of the present paper is to drive univalency for weakly $\Phi$-like of order $\alpha$ with respect to $g(z)$.

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1. Introduction

Let $A$ be the class of functions of form

\begin{equation}
 f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\end{equation}
which are analytic in the unit disk $E = \{ z : |z| < 1 \}$. A function $f(z) \in A$ is called starlike if $f(z)$ satisfies the condition

\[
\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > 0 \quad (z \in E).
\]  

For $f(z)$ given by (1.1) and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, let $\Phi(f(z), g(z))$ be analytic on $(f(E), g(E)) \in \mathbb{C}^2$ with $\Phi(f(0), g(0)) = 0$, $\Phi(f(z), g(z)) \neq 0$ and $f(z) \neq 0$ in $0 < |z| < 1$, and for arbitrary $\omega \in f(E)$, $\Phi(\omega, g(re^{i\theta}))$ $(0 < r < 1$ and $0 \leq \theta \leq 2\pi)$ satisfy

\[
\frac{d}{d\theta} \arg \Phi(\omega, g(re^{i\theta})) < \frac{1}{2}(3 - \alpha) \quad (z \in E)
\]

where $1 < \alpha < 2$.

A function is said to be weakly $\Phi$-like of order $\alpha$ with respect to a function $g(z)$ which satisfies the above condition of upper order $\frac{1}{2}(3 - \alpha)$ if it satisfies

\[
| \arg \frac{zf'(z)}{\Phi(f(z), g(z))} | < \frac{\pi}{2} \alpha \quad (z \in E)
\]

for $1 < \alpha < 2$ (cf. [1] and [2]).

2. Main result

**Theorem 1.** If $f(z)$ is a weakly $\Phi$-like of order $\alpha$ with respect to a function $g(z)$ of upper order $\frac{1}{2}(3 - \alpha)$ for $1 < \alpha < 2$, then $f(z)$ is univalent in $E$.

**Proof.** We will prove it by reductive absurdity. Let us suppose that there exists a positive real number $r$ $(0 < r < 1)$ for which $f(z)$ is univalent in $|z| < r$, but $f(z)$ is not univalent on $|z| = r$.

![Figure 1](image-url)
In view of Figure 1, we know that there are two points $z_1$ and $z_2$ ($z_1 \neq z_2$) such that

$$|z_1| = |z_2| = r, \ z_1 = re^{i\theta_1}, \ z_2 = re^{i\theta_2}, \ 0 < \beta = \theta_2 - \theta_1,$$

for which $f(z_1) = f(z_2)$. Let us put $C = \{z : z = re^{i\theta}, \theta_1 < \theta \leq \theta_2\}$ and $C_{f(z)} = \{f(z) : z \in C\}$. On the other hand, from the assumption of the theorem, we have

$$\int_{|z|=r} d\arg \frac{zf'(z)}{\phi(f(z),g(z))}$$

$$= \int_{|z|=r} d\arg z + \sum_{|z|=r} d\arg df(z) - \int_{|z|=r} d\arg dz - \int_{|z|=r} d\arg \phi(f(z),g(z))$$

$$= 2\pi + \sum_{|z|=r} d\arg df(z) - 2\pi - \int_{|z|=r} d\arg \phi(f(z),g(z))$$

and

$$\pi \alpha > \sum_{|z|=r} d\arg df(z) - \int_{|z|=r} d\arg \phi(f(z),g(z)) > -\pi \alpha.$$

Now then, it is trivial that

$$\int_{|z|=r} d\arg \phi(f(z),g(z)) = 2\pi.$$

This shows that

$$4\pi > \pi \alpha + 2\pi > \int_{|z|=r} d\arg df(z) > 2\pi - \pi \alpha > 0$$

and therefore it must be

$$\int_{|z|=r} d\arg df(z) = 2\pi.$$

Now, we have

$$\int_{C_{f(z)}} d\arg df(z) = \int_{C} d\arg f'(z)dz = -\pi.$$

Putting $L = \{z : |z| = r\}$, then from the assumption of the theorem, we have

$$\pi \alpha > \int_{L-C} d\arg \frac{zf'(z)}{\Phi(f(z),g(z))} > -\pi \alpha$$

and so, we have

$$\pi \alpha > \int_{L-C} d\arg df(z) - \int_{L-C} d\arg \Phi(f(z),g(z)) > -\pi \alpha.$$
It follows that

$$\int_{L-C} d\arg df(z)$$

$$< \pi \alpha + \arg \Phi(f(z_1), g(re^{i(\theta_1+2\pi)})) - \arg \Phi(f(z_2), g(re^{i\theta_2}))$$

$$= \pi \alpha + \arg \Phi(f(z_2), g(re^{i(\theta_1+2\pi)})) - \arg \Phi(f(z_2), g(re^{i\theta_2}))$$

$$= \pi \alpha + \int_{\theta_2}^{\theta_1+2\pi} \frac{d}{d\theta} \arg \Phi(f(z_2), g(re^{i\theta}))d\theta$$

$$< \pi \alpha + \int_{\theta_2}^{\theta_1+2\pi} \frac{1}{2}(3 - \alpha)d\theta$$

$$= \pi \alpha + \frac{1}{2}(3 - \alpha)(2\pi - \beta)$$

$$< \pi \alpha + (3 - \alpha)\pi = 3\pi$$

This shows that

$$\int_{|z|=r} d\arg df(z) - \oint_{C_{f(z)}} d\arg df(z) < 3\pi.$$ 

From (2.1), (2.2) and (2.3), we have a contradiction. This completes the proof of the theorem.

**Remark.** When $f(z)$ satisfies the hypotheses of Theorem 1, the real part of the function $zf'/\Phi(f(z), g(z))$ can be negative.

**Theorem 2.** Let $\Phi(f(z), g(z))$ be analytic on $(f(E), g(E))$ with $\Phi(f(0), g(0)) = 0$, $\Phi(f(z), g(z)) \neq 0$ and $f(z) \neq 0$ in $0 < |z| < 1$ and for arbitrary $w \in f(E)$, $\Phi(w, g(re^{i\theta}))$ satisfies the following condition

$$\frac{d}{d\theta} \arg \Phi(w, g(re^{i\theta})) > -\frac{1}{2} \ (z \in E)$$

where $0 < r < 1$ and $0 \leq \theta \leq 2\pi$. Then, if $f(z)$ satisfies the following condition

$$\text{Re} \frac{z^2(f'(z))^2}{\Phi(f(z), g(z))} > 0 \ (z \in E)$$

then $f(z)$ is univalent in $E$.

**Proof.** Applying the same method as the proof of Theorem 1, let us suppose that there exists a positive real number $r$ ($0 < r < 1$) for which $f(z)$ is univalent in $|z| < r$, but $f(z)$ is not univalent on $|z| = r$. Also, in view of Figure 1, we know that there are two points $z_1$ and $z_2$ ($z_1 \neq z_2$) such that $|z_1| = |z_2| = r$, $z_1 = re^{i\theta_1}$, $z_2 = re^{i\theta_2}$, $0 < \theta_2 - \theta_1$, for which $f(z_1) = f(z_2)$. From the assumption of the theorem, we have
\[
\pi > \int_{C} d \arg \frac{z^{2}(f'(z))^{2}}{\Phi(f(z), g(z))}
\]
\[
= \int_{C} d \arg z^{2} + \int_{C} d \arg (f'(z))^{2} - \int_{C} d \arg \Phi(f(z), g(z))
\]
\[
= 2 \int_{C} d \arg z + 2 \int_{C} d \arg f'(z) - \int_{C} d \arg \Phi(f(z), g(z))
\]
\[
= 2 \int_{C} d \arg df(z) - (\arg \Phi(f(z_{2}), g(re^{i\theta_{2}})) - \arg \Phi(f(z_{1}), g(re^{i\theta_{1}})))
\]
\[
> -\pi.
\]

It follow that
\[
2 \int_{C} d \arg df(z)
\]
\[
> (\arg \Phi(f(z_{1}), g(re^{i\theta_{2}})) - \arg \Phi(f(z_{1}), g(re^{i\theta_{1}}))) - \pi
\]
\[
= \int_{\theta_{1}}^{\theta_{2}} \frac{d}{d\theta} \arg \Phi(f(z_{1}), g(re^{i\theta})) d\theta - \pi
\]
\[
> -\frac{1}{2} \int_{\theta_{1}}^{\theta_{2}} d\theta - \pi
\]
\[
> -\pi - \pi = -2\pi
\]

and therefore, we have
\[
\int_{C} d \arg df(z) > -\pi,
\]

but from the assumption, we have
\[
\int_{C} d \arg df(z) = -\pi.
\]

This is a contradiction and it completes the proof.

Applying the proof of Theorem 2, we have following corollary:

Corollary 1. Let \( f(z) = z + \sum_{n=2}^{\infty} a_{n} z^{n} \) and \( g(z) = z + \sum_{n=2}^{\infty} b_{n} z^{n} \) be analytic in \( E \) and suppose that
\[
\operatorname{Re} \frac{z^{2}(f'(z))^{2}}{f(z)^{2-\beta} g(z)^{\beta}} > 0 \quad (z \in E)
\]
where $\beta > 0$ and

$$
\text{Re} \frac{zg'(z)}{g(z)} > -\frac{1}{2\beta} \quad \text{in } E,
$$

then $f(z)$ is univalent in $E$.

**Remark.** If $g(z)$ satisfies the condition (2.4), then $g(z)$ is not necessarily a starlike function.

**References**
