On the order of strongly starlikeness and order of starlikeness of a certain convex functions

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Let $A$ denote the set of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are analytic in $D = \{z : |z| < 1\}$. Let $f(z) \in A$ and suppose that for $0 < \alpha < 1$ and $0 < \beta < 1$,

1. \[ \text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \alpha \quad \text{in} \ D, \]
2. \[ 1 + \text{Re} \left( \frac{zf''(z)}{f'(z)} \right) > \alpha \quad \text{in} \ D, \]
3. \[ |\arg \left( \frac{zf'(z)}{f(z)} \right)| < \frac{\pi}{2} \beta \quad \text{in} \ D, \]
4. \[ |\arg \left( 1 + \frac{zf''(z)}{f'(z)} \right)| < \frac{\pi}{2} \beta \quad \text{in} \ D, \]
5. \[ |\arg \left( \frac{zf'(z)}{f(z)} - \alpha \right)| < \frac{\pi}{2} \beta \quad \text{in} \ D, \]
6. \[ |\arg \left( 1 + \frac{zf''(z)}{f'(z)} - \alpha \right)| < \frac{\pi}{2} \beta \quad \text{in} \ D. \]

Then if $f(z)$ satisfies the above conditions (1), (2), (3), (4), (5) and (6), we call $f(z)$ starlike of order $\alpha$, convex of order $\alpha$, strongly starlike of order $\beta$, strongly convex of order $\beta$, strongly starlike of order $\beta$ and starlike of order $\alpha$, and strongly convex of order $\beta$ and convex of order $\alpha$ respectively and let us denote the class of functions which satisfy the conditions (1), (2), (3), (4), (5) and (6) by $S^{*}(\alpha), C(\alpha), SS^{*}(\beta), SC(\beta), S^{*}(\alpha, \beta)$ and $C(\alpha, \beta)$ respectively.

Marx [2] and Strohhäcker [5] showed that

\[ f(z) \in C(0) \quad \text{implies} \quad f(z) \in S^{*} \left( \frac{1}{2} \right) \]
and MacGregor [1] and Wilken and Feng [6] obtained more general result that
\[ f(z) \in C(\alpha) \text{ implies } f(z) \in S^*(\beta(\alpha)) \]
where \(0 \leq \alpha < 1\) and
\[
\beta(\alpha) = \begin{cases} 
\frac{1 - 2\alpha}{2^{2-2\alpha}[1 - 2^{2\alpha-1}]} & \text{if } \alpha \neq \frac{1}{2} \\
\frac{1}{2 \log 2} & \text{if } \alpha = \frac{1}{2}.
\end{cases}
\]

Mocanu [3] showed that
\[ f(z) \in SC(\gamma) \text{ implies } f(z) \in SS^*(\beta) \]
where
\[ \tan \frac{\pi \gamma}{2} = \tan \frac{\pi \beta}{2} + \frac{\beta}{(1 - \beta) \cos \frac{\pi \beta}{2}} \left( \frac{1 - \beta}{1 + \beta} \right)^{1 + \delta} \]
and \(0 < \beta < 1\).

On the other hand, Nunokawa [4] obtained that
\[ f(z) \in SC(\alpha(\beta)) \text{ implies } f(z) \in SS^*(\beta) \]
where
\[
\alpha(\beta) = \beta + \frac{2}{\pi} \tan^{-1} \frac{\beta q(\beta) \sin \frac{\pi}{2}(1 - \beta)}{p(\beta) + \beta q(\beta) \cos \frac{\pi}{2}(1 - \beta)}
\]
and \(0 < \beta < 1\).

In this paper, we need the following lemma due to Nunokawa [4].

**Lemma 1** Let \(P(z)\) be analytic in \(D\), \(P(0) = 1\), \(P(z) \neq 0\) in \(D\) and suppose that there exists a point \(z_0 \in D\) such that
\[ |\arg(P(z))| < \frac{\pi}{2} \delta \text{ for } |z| < |z_0| \]
and
\[ |\arg(P(z_0))| = \frac{\pi}{2} \delta \]
where \(0 < \delta\). Then we have
\[ \frac{z_0 P'(z_0)}{P(z_0)} = ik\delta \]
where  
\[ k \geq \frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{when} \quad \arg(P(z_0)) = \frac{\pi}{2} \delta \]

and  
\[ k \leq -\frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{when} \quad \arg(P(z_0)) = -\frac{\pi}{2} \delta \]

where  
\[ P(z_0)^{\frac{1}{\delta}} = \pm ia \quad \text{and} \quad 0 < a. \]

**Theorem 1**  
Let \( f(z) \in A \) and suppose that \( \frac{zf'(z)}{f(z)} \neq \beta(\alpha) \) in \( D \) and  
\[ |\arg \left( 1 + \frac{zf''(z)}{f'(z)} - \alpha \right) | < \frac{\pi}{2} \gamma \quad \text{in} \quad D \]

where \( 0 \leq \alpha < 1 \) and \( 0 < \gamma < 1 \). Then we have  
\[ |\arg \left( \frac{zf'(z)}{f(z)} - \beta(\alpha) \right) | < \frac{\pi}{2} \delta \quad \text{in} \quad D \]

where \( \beta(\alpha) \) is defined by (7), \( 0 < \delta < 1 \),  
\[ \gamma = \frac{2}{\pi} \tan^{-1} \delta(1 - \beta(\alpha)) \left( \frac{a_0^{\delta+1} + a_0^{\delta-1}}{\beta(\alpha) + (1 - \beta(\alpha))a_0^\delta} \right) \]

and \( a_0 \) is the positive root of the equation  
\[ (1 - \beta(\alpha))x^\delta(x^2 - 1) = \beta(\alpha) \{(1 - \delta) - (1 + \delta)x^2\}. \]

**Proof.**  
Let us put  
\[ p(z) = \frac{zf'(z)}{f(z)}, \quad p(0) = 1 \quad \text{and} \quad p(z) \neq \beta(\alpha) \quad \text{in} \quad D. \]

Then it follows that  
\[ 1 + \frac{zf''(z)}{f'(z)} = p(z) + \frac{zp'(z)}{p(z)}. \]

If there exists a point \( z_0 \in D \) such that  
\[ |\arg(P(z))| = |\arg(p(z) - \beta(\alpha))| < \frac{\pi}{2} \delta \]

for \( |z| < |z_0| \) and  
\[ |\arg(P(z_0))| = |\arg(p(z_0) - \beta(\alpha))| = \frac{\pi}{2} \delta \]
where \( P(z) = \frac{p(z) - \beta(\alpha)}{1_{j}' - f(\alpha)} \) and \( P(0) = 1 \), then from Lemma 1, we have

\[
\frac{z_{0}P'(z_{0})}{P(z_{0})} = \frac{z_{0}p'(z_{0})}{p(z_{0}) - \beta(\alpha)} = i\delta k.
\]

For the case \( \arg(P(z_{0})) = \arg(p(\infty) - \beta(\alpha)) = \frac{\pi}{2}\delta \), it follows that

\[(8) \quad \arg \left( 1 + \frac{z_{0}f''(z_{0})}{f'(z_{0})} - \alpha \right) \]

\[
= \arg \left( p(z_{0}) - \beta(\alpha) \right) \left( 1 + \frac{z_{0}f'(z_{0})}{p(z_{0}) - \beta(\alpha)} \cdot \frac{1}{p(z_{0})} + \frac{\beta(\alpha) - \alpha}{p(z_{0}) - \beta(\alpha)} \right)
\]

\[
= \frac{\pi\delta}{2} + \arg \left( 1 + \frac{i\delta k}{\beta(\alpha) + (1 - \beta(\alpha))(ia)^{\delta}} + \frac{\beta(\alpha) - \alpha}{(1 - \beta(\alpha))(ia)^{\delta}} \right)
\]

\[
> \frac{\pi\delta}{2} + \arg \left( e^{-i\frac{\pi}{2}\delta} + \frac{i\delta k}{\beta(\alpha) + (1 - \beta(\alpha))(ia)^{\delta}} + \frac{\beta(\alpha) - \alpha}{(1 - \beta(\alpha))(ia)^{\delta}} \right)
\]

\[
= \frac{\pi\delta}{2} + \arg \left( e^{i\frac{\pi}{2}\delta} + \frac{i\delta k}{\beta(\alpha) + (1 - \beta(\alpha))(ia)^{\delta}} + \frac{\beta(\alpha) - \alpha}{(1 - \beta(\alpha))(ia)^{\delta}} \right)
\]

\[
\geq \arg \left( e^{i\frac{\pi}{2}\delta} + \frac{1}{2} \left( \frac{i\delta a^{+1}}{\beta(\alpha) + (1 - \beta(\alpha))a^{\delta}} + \frac{1}{1 - \beta(\alpha)} \right) \right)
\]

since we have \( 0 < \beta(\alpha) - \alpha \leq \frac{1}{2} \) and Lemma 1. Let us put

\[(9) \quad \varphi(x) = \frac{x^{\delta}(x^{-1} + x)}{\beta(\alpha) + (1 - \beta(\alpha))x^{\delta}} = \frac{x + x^{-1}}{1 - \beta(\alpha) + \beta(\alpha)x^{-\delta}}
\]

for \( 0 < x \). Then it follows that

\[
\varphi'(x) = \frac{1}{x^{2}(1 - \beta(\alpha) + \beta(\alpha)x^{-\delta})^{2}} \left[ (1 - \beta(\alpha))(x^{2} - 1) + \beta(\alpha)x^{-\delta} \left\{(1 + \delta)x^{2} - (1 - \delta) \right\} \right].
\]

Putting \( a_{0} \) be the positive root of the equation \( \varphi'(x) = 0 \) or

\[
x^{\delta}(x^{2} - 1) = \beta(\alpha) \left\{(1 - \delta) - (1 + \delta)x^{2} \right\},
\]

then \( \varphi(x) \) takes its minimum value at \( x = a_{0} \). Therefore, from (8) and (9), we have

\[
\arg \left( 1 + \frac{z_{0}f''(z_{0})}{f'(z_{0})} - \alpha \right) > \arg \left\{ e^{i\frac{\pi}{2}\delta} + \frac{1}{2} \left( \frac{1}{(1 - \beta(\alpha))a_{0}^{\delta}} + \frac{i\delta a_{0}^{+1}}{\beta(\alpha) + (1 - \beta(\alpha))a_{0}^{\delta}} \right) \right\}
\]

\[
\geq \arg \left( \frac{1}{(1 - \beta(\alpha))a_{0}^{\delta}} + i\frac{\delta a_{0}^{+1}}{\beta(\alpha) + (1 - \beta(\alpha))a_{0}^{\delta}} \right)
\]

\[
= \tan^{-1} \delta(1 - \beta(\alpha)) \left( \frac{a_{0}^{\delta+1} + a_{0}^{\delta-1}}{\beta(\alpha) + (1 - \beta(\alpha))a_{0}^{\delta}} \right).
\]
This contradicts hypothesis of Theorem 1.

For the case \( \arg(P(z_0)) = \arg(p(z_0) - \beta(\alpha)) = -\frac{\pi}{2} \delta \), applying the same method as the above and Lemma 1, we have

\[
\arg\left(1 + \frac{z_0 f''(z_0)}{f'(z_0)} - \alpha\right) < -\tan^{-1} \delta(1 - \beta(\alpha)) \left(\frac{a_0^{\delta+1} + a_0^{\delta-1}}{\beta(\alpha) + (1 - \beta(\alpha))a_0^\delta}\right).
\]

This is also a contradiction and therefore it completes the proof of Theorem 1.

\[\square\]

**Remark** Theorem 1 shows that

\[ f(z) \in \mathcal{SC}(\alpha, \gamma) \text{ implies } f(z) \in \mathcal{SS}^*(\beta(\alpha), \delta) \]

where

\[ \gamma = \frac{2}{\pi} \tan^{-1} \delta(1 - \beta(\alpha)) \left(\frac{a_0^{\delta+1} + a_0^{\delta-1}}{\beta(\alpha) + (1 - \beta(\alpha))a_0^\delta}\right) \]

but Theorem 1 is not a sharp result and so, the authors expect that Theorem 1 will be improved by someone in future.

**References**


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