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On the order of strongly starlikeness and order of starlikeness of a certain convex functions

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Let $\mathcal{A}$ denote the set of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are analytic in $\mathbb{D} = \{z : |z| < 1\}$. Let $f(z) \in \mathcal{A}$ and suppose that for $0 < \alpha < 1$ and $0 < \beta < 1$,

$$\text{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad \text{in } \mathbb{D}, \tag{1}$$

$$1 + \text{Re}\left(\frac{zf''(z)}{f'(z)}\right) > \alpha \quad \text{in } \mathbb{D}, \tag{2}$$

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\pi}{2}\beta \quad \text{in } \mathbb{D}, \tag{3}$$

$$\left|\arg\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \frac{\pi}{2}\beta \quad \text{in } \mathbb{D}, \tag{4}$$

$$\left|\arg\left(\frac{zf'(z)}{f(z)} - \alpha\right)\right| < \frac{\pi}{2}\beta \quad \text{in } \mathbb{D}, \tag{5}$$

$$\left|\arg\left(1 + \frac{zf''(z)}{f'(z)} - \alpha\right)\right| < \frac{\pi}{2}\beta \quad \text{in } \mathbb{D}. \tag{6}$$

Then if $f(z)$ satisfies the above conditions (1), (2), (3), (4), (5) and (6), we call $f(z)$ starlike of order α, convex of order α, strongly starlike of order β, strongly convex of order β, strongly starlike of order β and starlike of order α, and strongly convex of order β and convex of order α respectively and let us denote the class of functions which satisfy the conditions (1), (2), (3), (4), (5) and (6) by $\mathcal{S}^*(\alpha)$, $\mathcal{C}(\alpha)$, $\mathcal{SS}^*(\beta)$, $\mathcal{SC}(\beta)$, $\mathcal{S}^*(\alpha,\beta)$ and $\mathcal{C}(\alpha,\beta)$ respectively.

Marx [2] and Strohhäcker [5] showed that

$$f(z) \in \mathcal{C}(0) \quad \text{implies} \quad f(z) \in \mathcal{S}^*\left(\frac{1}{2}\right)$$

and MacGregor [1] and Wilken and Feng [6] obtained more general result that

$$f(z) \in \mathcal{C}(\alpha) \quad \text{implies} \quad f(z) \in \mathcal{S}^*(\beta(\alpha))$$

where $0 \leqq \alpha < 1$ and

$$(7) \qquad \beta(\alpha) = \begin{cases} \dfrac{1-2\alpha}{2^{2-2\alpha}\left[1-2^{2\alpha-1}\right]} & \text{if } \alpha \neq \dfrac{1}{2} \\[2ex] \dfrac{1}{2\log 2} & \text{if } \alpha = \dfrac{1}{2}. \end{cases}$$

Mocanu [3] showed that

$$f(z) \in \mathcal{SC}(\gamma) \quad \text{implies} \quad f(z) \in \mathcal{SS}^*(\beta)$$

where

$$\tan\frac{\pi\gamma}{2} = \tan\frac{\pi\beta}{2} + \frac{\beta}{(1-\beta)\cos\frac{\pi\beta}{2}}\left(\frac{1-\beta}{1+\beta}\right)^{\frac{1+\beta}{2}}$$

and $0 < \beta < 1$.

On the other hand, Nunokawa [4] obtained that

$$f(z) \in \mathcal{SC}(\alpha(\beta)) \quad \text{implies} \quad f(z) \in \mathcal{SS}^*(\beta)$$

where

$$\alpha(\beta) = \beta + \frac{2}{\pi}\tan^{-1}\frac{\beta q(\beta)\sin\frac{\pi}{2}(1-\beta)}{p(\beta) + \beta q(\beta)\cos\frac{\pi}{2}(1-\beta)}$$

$$p(\beta) = (1+\beta)^{\frac{1+\beta}{2}}, \quad q(\beta) = (1-\beta)^{\frac{\beta-1}{2}}$$

and $0 < \beta < 1$.

In this paper, we need the following lemma due to Nunokawa [4].

Lemma 1 *Let $P(z)$ be analytic in $\mathbb{D}$, $P(0) = 1$, $P(z) \neq 0$ in $\mathbb{D}$ and suppose that there exists a point $z_0 \in \mathbb{D}$ such that*

$$|\arg(P(z))| < \frac{\pi}{2}\delta \qquad \text{for} \quad |z| < |z_0|$$

and

$$|\arg(P(z_0))| = \frac{\pi}{2}\delta$$

where $0 < \delta$. Then we have

$$\frac{z_0 P'(z_0)}{P(z_0)} = ik\delta$$

where

$$k \geqq \frac{1}{2}\left(a + \frac{1}{a}\right) \qquad \textit{when} \quad \arg(P(z_0)) = \frac{\pi}{2}\delta$$

and

$$k \leqq -\frac{1}{2}\left(a + \frac{1}{a}\right) \qquad \textit{when} \quad \arg(P(z_0)) = -\frac{\pi}{2}\delta$$

where

$$P(z_0)^{\frac{1}{\delta}} = \pm ia \quad \textit{and} \quad 0 < a.$$

Theorem 1 *Let $f(z) \in \mathcal{A}$ and suppose that $\dfrac{zf'(z)}{f(z)} \neq \beta(\alpha)$ in $\mathbb{D}$ and*

$$\left|\arg\left(1 + \frac{zf''(z)}{f'(z)} - \alpha\right)\right| < \frac{\pi}{2}\gamma \quad \textit{in } \mathbb{D}$$

where $0 \leqq \alpha < 1$ and $0 < \gamma < 1$. Then we have

$$\left|\arg\left(\frac{zf'(z)}{f(z)} - \beta(\alpha)\right)\right| < \frac{\pi}{2}\delta \quad \textit{in } \mathbb{D}$$

where $\beta(\alpha)$ is defined by (7), $0 < \delta < 1$,

$$\gamma = \frac{2}{\pi}\tan^{-1}\delta(1 - \beta(\alpha))\left(\frac{a_0^{\delta+1} + a_0^{\delta-1}}{\beta(\alpha) + (1 - \beta(\alpha))a_0^{\delta}}\right)$$

and a_0 is the positive root of the equation

$$(1 - \beta(\alpha))x^{\delta}(x^2 - 1) = \beta(\alpha)\left\{(1 - \delta) - (1 + \delta)x^2\right\}.$$

Proof. Let us put

$$p(z) = \frac{zf'(z)}{f(z)}, \quad p(0) = 1 \quad \text{and} \quad p(z) \neq \beta(\alpha) \quad \text{in } \mathbb{D}.$$

Then it follows that

$$1 + \frac{zf''(z)}{f'(z)} = p(z) + \frac{zp'(z)}{p(z)}.$$

If there exists a point $z_0 \in \mathbb{D}$ such that

$$|\arg(P(z))| = |\arg(p(z) - \beta(\alpha))| < \frac{\pi}{2}\delta$$

for $|z| < |z_0|$ and

$$|\arg(P(z_0))| = |\arg(p(z_0) - \beta(\alpha))| = \frac{\pi}{2}\delta$$

where $P(z) = \dfrac{p(z) - \beta(\alpha)}{1 - \beta(\alpha)}$ and $P(0) = 1$, then from Lemma 1, we have

$$\frac{z_0 P'(z_0)}{P(z_0)} = \frac{z_0 p'(z_0)}{p(z_0) - \beta(\alpha)} = i\delta k.$$

For the case $\arg(P(z_0)) = \arg(p(z_0) - \beta(\alpha)) = \dfrac{\pi}{2}\delta$, it follows that

$$\begin{aligned}
(8) \qquad & \arg\left(1 + \frac{z_0 f''(z_0)}{f'(z_0)} - \alpha\right) \\
= \; & \arg\left\{(p(z_0) - \beta(\alpha))\left(1 + \frac{z_0 p'(z_0)}{p(z_0) - \beta(\alpha)} \cdot \frac{1}{p(z_0)} + \frac{\beta(\alpha) - \alpha}{p(z_0) - \beta(\alpha)}\right)\right\} \\
= \; & \frac{\pi\delta}{2} + \arg\left\{1 + \frac{i\delta k}{\beta(\alpha) + (1 - \beta(\alpha))(ia)^\delta} + \frac{\beta(\alpha) - \alpha}{(1 - \beta(\alpha))(ia)^\delta}\right\} \\
> \; & \frac{\pi\delta}{2} + \arg\left\{1 + \frac{i\delta k}{(\beta(\alpha) + (1 - \beta(\alpha))a^\delta)e^{i\frac{\pi}{2}\delta}} + \frac{\beta(\alpha) - \alpha}{(1 - \beta(\alpha))a^\delta e^{i\frac{\pi}{2}\delta}}\right\} \\
= \; & \frac{\pi\delta}{2} + \arg\left\{e^{-i\frac{\pi}{2}\delta}\left(e^{i\frac{\pi}{2}\delta} + \frac{i\delta k}{\beta(\alpha) + (1 - \beta(\alpha))a^\delta} + \frac{\beta(\alpha) - \alpha}{(1 - \beta(\alpha))a^\delta}\right)\right\} \\
\geqq \; & \arg\left\{e^{i\frac{\pi}{2}\delta} + \frac{1}{2}\left(\frac{i\delta(a + a^{-1})}{\beta(\alpha) + (1 - \beta(\alpha))a^\delta} + \frac{1}{(1 - \beta(\alpha))a^\delta}\right)\right\}
\end{aligned}$$

since we have $0 < \beta(\alpha) - \alpha \leqq \dfrac{1}{2}$ and Lemma 1. Let us put

$$(9) \qquad \varphi(x) = \frac{x^\delta(x + x^{-1})}{\beta(\alpha) + (1 - \beta(\alpha))x^\delta} = \frac{x + x^{-1}}{1 - \beta(\alpha) + \beta(\alpha)x^{-\delta}}$$

for $0 < x$. Then it follows that

$$\varphi'(x) = \frac{1}{x^2\left(1 - \beta(\alpha) + \beta(\alpha)x^{-\delta}\right)^2}\left[(1 - \beta(\alpha))(x^2 - 1) + \beta(\alpha)x^{-\delta}\left\{(1 + \delta)x^2 - (1 - \delta)\right\}\right].$$

Putting a_0 be the positive root of the equation $\varphi'(x) = 0$ or

$$x^\delta(x^2 - 1) = \beta(\alpha)\left\{(1 - \delta) - (1 + \delta)x^2\right\},$$

then $\varphi(x)$ takes its minimum value at $x = a_0$. Therefore, from (8) and (9), we have

$$\begin{aligned}
\arg\left(1 + \frac{z_0 f''(z_0)}{f'(z_0)} - \alpha\right) \; & > \; \arg\left\{e^{i\frac{\pi}{2}\delta} + \frac{1}{2}\left(\frac{1}{(1 - \beta(\alpha))a_0^\delta} + \frac{i\delta(a_0 + a_0^{-1})}{\beta(\alpha) + (1 - \beta(\alpha))a_0^\delta}\right)\right\} \\
& \geqq \; \arg\left(\frac{1}{(1 - \beta(\alpha))a_0^\delta} + i\frac{\delta(a_0 + a_0^{-1})}{\beta(\alpha) + (1 - \beta(\alpha))a_0^\delta}\right) \\
& = \; \tan^{-1}\delta(1 - \beta(\alpha))\left(\frac{a_0^{\delta+1} + a_0^{\delta-1}}{\beta(\alpha) + (1 - \beta(\alpha))a_0^\delta}\right).
\end{aligned}$$

This contradicts hypothesis of Theorem 1.

For the case $\arg(P(z_0)) = \arg(p(z_0) - \beta(\alpha)) = -\frac{\pi}{2}\delta$, applying the same method as the above and Lemma 1, we have

$$\arg\left(1 + \frac{z_0 f''(z_0)}{f'(z_0)} - \alpha\right) < -\tan^{-1}\delta(1-\beta(\alpha))\left(\frac{a_0^{\delta+1} + a_0^{\delta-1}}{\beta(\alpha) + (1-\beta(\alpha))a_0^{\delta}}\right).$$

This is also a contradiction and therefore it completes the proof of Theorem 1. □

Remark Theorem 1 shows that

$$f(z) \in \mathcal{SC}(\alpha,\gamma) \quad \text{implies} \quad f(z) \in \mathcal{SS}^*(\beta(\alpha),\delta)$$

where

$$\gamma = \frac{2}{\pi}\tan^{-1}\delta(1-\beta(\alpha))\left(\frac{a_0^{\delta+1} + a_0^{\delta-1}}{\beta(\alpha) + (1-\beta(\alpha))a_0^{\delta}}\right)$$

but Theorem 1 is not a sharp result and so, the authors expect that Theorem 1 will be improved by someone in future.

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