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Kyoto University
Solutions to The Nonhomogeneous Chebyshev’s Equation by Means of N-Fractional Calculus Operator

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Abstract

In this article, solutions to the nonhomogeneous Chebyshev’s equations

\[ L[\varphi;z;\nu] = \varphi_2 \cdot (z^2 - 1) + \varphi_1 \cdot z - \varphi \cdot \nu^2 = f \quad (z^2 - 1 \neq 0, f \neq 0) \]

\[ (\varphi_a = d^\alpha \varphi / dz^\alpha \text{ for } \alpha > 0, \varphi_0 = \varphi = \varphi(z), f = f(z)) \]

are discussed by means of N-fractional calculus operator (NFCO-Method).

By our method, some particular solutions to the above equations are given as below for example, in fractional differintegrated forms.

Group I.

(i) \[ \varphi = (G(\nu) \cdot H(\nu))_{-(1+\nu)} = \varphi_{[1](\nu)}^* \] (denote)

(ii) \[ \varphi = (H(\nu) \cdot G(\nu))_{-(1+\nu)} = \varphi_{[2](\nu)}^* \]

(iii) \[ \varphi = (G(-\nu) \cdot H(-\nu))_{-\nu-1} = \varphi_{[3](\nu)}^* \]

(iv) \[ \varphi = (H(-\nu) \cdot G(-\nu))_{-\nu-1} = \varphi_{[4](\nu)}^* \]

where

\[ G(\nu) = (g \cdot (z^2 - 1)^{-\nu-1/2})_{-1}, \quad H(\nu) = (z^2 - 1)^{-\nu-1/2} \]
§ 0. Introduction (Definition of Fractional Calculus) and
§ 1. Preliminary
are omitted, then refer to the previous paper for the Homogeneous Chebyshev's equation.

§ 2. Solutions to The Nonhomogeneous Chebyshev's Equations by Means of N-Fractional Calculus Operator

Theorem 1. Let $\varphi = \varphi(z) \in F$ and $f = f(z) \in F$, then the nonhomogeneous Chebyshev's equation

$$L[\varphi; z; v] = \varphi_2 \cdot (z^2 - 1) + \varphi_1 \cdot z - \varphi \cdot v^2 = f, \quad (v \in R, z^2 - 1 \neq 0)$$

has particular solutions of the forms (fractional differintegrated form);

Group I.

(i) $\varphi(z) = (G(v) \cdot H(v))_{-1-v} \equiv \varphi_{[1]}(z,v)$ (denote) (2)

(ii) $\varphi(z) = (H(v) \cdot G(v))_{-1-v} \equiv \varphi_{[2]}(z,v)$ (3)

(iii) $\varphi(z) = (G(-v) \cdot H(-v))_{v-1} \equiv \varphi_{[3]}(z,v)$ (4)

(iv) $\varphi(z) = (H(-v) \cdot G(-v))_{v-1} \equiv \varphi_{[4]}(z,v)$ (5)

where

$G(v) = (f \cdot (z^2 - 1)^{v-1/2})_{-1}$, \hspace{1cm} $H(v) = (z^2 - 1)^{-(v+1/2)}$. (6)

Group II.

(i) $\varphi(z) = (z^2 - 1)^{1/2} \cdot (P(v) \cdot H(v))_{-v} \equiv \varphi_{[5]}(z,v)$ (7)

(ii) $\varphi(z) = (z^2 - 1)^{1/2} \cdot (H(v) \cdot P(v))_{-v} \equiv \varphi_{[6]}(z,v)$ (8)

(iii) $\varphi(z) = (z^2 - 1)^{1/2} \cdot (P(-v) \cdot H(-v))_{v} \equiv \varphi_{[7]}(z,v)$ (9)

(iv) $\varphi(z) = (z^2 - 1)^{1/2} \cdot (H(-v) \cdot P(-v))_{v} \equiv \varphi_{[8]}(z,v)$ (10)

where
\[ P(v) = ((f \cdot (z^2 - 1)^{-1/2})_{v-1} \cdot (z^2 - 1)^{v-1/2})_{-1}, \quad H(v) = (z^2 - 1)^{-(v+1/2)} \] .. (11)

**Group III.**

(i) \[ \varphi(z) = (z - 1)^{1/2} (Q(v) \cdot S(v))_{-(v+1/2)} = \varphi^{*}_{[9]}(z, v) \] .. (12)

(ii) \[ \varphi(z) = (z - 1)^{1/2} (S(v) \cdot Q(v))_{-(v+1/2)} = \varphi^{*}_{[10]}(z, v) \] .. (13)

(iii) \[ \varphi(z) = (z - 1)^{1/2} (Q(-v) \cdot S(-v))_{v^{-1/2}} = \varphi^{*}_{[11]}(z, v) \] .. (14)

(iv) \[ \varphi(z) = (z - 1)^{1/2} (S(-v) \cdot Q(-v))_{v^{-1/2}} = \varphi^{*}_{[12]}(z, v) \] .. (15)

where

\[ Q(v) = ((f \cdot (z - 1)^{-1/2})_{v-1} \cdot (z - 1)^{v^{-1/2}})_{-1}, \quad S(v) = ((z - 1)^{-v} + (z + 1)^{-v}) \] .. (16)

**Group IV.**

(i) \[ \varphi(z) = (z + 1)^{1/2} (T(v) \cdot Y(v))_{-(v+1/2)} = \varphi^{*}_{[13]}(z, v) \] .. (17)

(ii) \[ \varphi(z) = (z + 1)^{1/2} (Y(v) \cdot T(v))_{-(v+1/2)} = \varphi^{*}_{[14]}(z, v) \] .. (18)

(iii) \[ \varphi(z) = (z + 1)^{1/2} (T(-v) \cdot Y(-v))_{v^{-1/2}} = \varphi^{*}_{[15]}(z, v) \] .. (19)

(iv) \[ \varphi(z) = (z + 1)^{1/2} (Y(-v) \cdot T(-v))_{v^{-1/2}} = \varphi^{*}_{[16]}(z, v) \] .. (20)

where

\[ T(v) = ((f \cdot (z + 1)^{-1/2})_{v^{-1/2}} \cdot (z - 1)^{-v} (z + 1)^{v})_{-1}, \quad Y(v) = ((z - 1)^{-v} (z + 1)^{-v}) \] .. (21)

**Proof of Group I:**

Operate \( N^{\alpha} \) to the both sides of (1), we have then

\[ \varphi_{2+\alpha} \cdot (z^2 - 1) + \alpha_1 + \varphi_{\alpha} \cdot (z(2\alpha + 1)) + \varphi_{\alpha} \cdot (\alpha^2 - v^2) = f_{\alpha}, \quad (f_{\alpha} \neq 0) \] .. (22)

(Refer to the proof of Group I in § 2 in the previous paper for Homogeneous one.)

**Case** \( \alpha = v \);

In this case we obtain

\[ \varphi_{2+v} \cdot (z^2 - 1) + \varphi_{1+v} \cdot z(2v + 1) = f_{v}, \] .. (23)

from (22), then letting

\[ \varphi_{1+v} = \psi = \psi(z) \quad (\varphi = \psi_{-a+v}) \] .. (24)

yields

\[ \psi \cdot (z^2 - 1) + \psi \cdot z(2v + 1) = f_{v} \] .. (25)

from (24). A particular solution to this linear first order equation is given by

\[ \psi = (f_{v} \cdot (z^2 - 1)^{-1/2})_{-1} \cdot (z^2 - 1)^{-(v+1/2)} \] .. (26)
Therefore, we obtain
\[ \varphi = \psi_{-(1+v)} = ((f_v \cdot (z^2 - 1)^{v-12})_{-1} \cdot (z^2 - 1)^{-(v+1/2)})_{-(1+v)} = \varphi_{[1](z,v)}^* \]  
(27)

from (24) and (26).

Inversely, we have
\[ \varphi_{2+v} = \psi = (f_v \cdot (z^2 - 1)^{v+1/2})_{-1} \cdot (z^2 - 1)^{-(v+1/2)} \]
(28)

and
\[ \varphi_{1+v} = \psi = (f_v \cdot (z^2 - 1)^{v+1/2})_{-1} \cdot (z^2 - 1)^{-(v+1/2)} \]
(29)

from (2) respectively. Then we obtain
\[ \text{LHS of (23)} = \{(f_v \cdot (z^2 - 1)^{v-1/2}) \cdot (z^2 - 1)^{-(v+1/2)} \}
- (f_v \cdot (z^2 - 1)^{v-1/2})_{-1} \cdot (v + \frac{1}{2}) \cdot 2z(z^2 - 1)^{-(v+1/2+1)} \]
(30)

applying (28) and (29).

That is, the function shown by (2) satisfies equation (23).

Moreover, we have (1) operating \( N^{-v} \) to the both sides of (23).

Therefore, the function given by (2) satisfies equation (1).

Next changing the order \( G(v) \) and \( H(v) \) in parenthesis ( ) in (2) we obtain
\[ \varphi = (H(v) \cdot G(v))_{-(1+v)} \equiv \varphi_{[2](z,v)}^* \]
(3)

where
\[ \varphi_{[1](z,v)}^* \neq \varphi_{[2](z,v)}^* \]  
(31)

(11) Case \( \alpha = -v \);

In the same way as (1) above, setting \( -v \) instead of \( v \) in the solutions (2) and (3), we obtain
\[ \varphi = (G(-v) \cdot H(-v))_{-1} \equiv \varphi_{[3](z,v)}^* \]  
(4)

where
\[ \varphi_{[3](z,v)}^* \neq \varphi_{[4](z,v)}^* \]  
(32)

And
\[ \varphi_{[3](z,v)}^* = \varphi_{[1](z,-v)}^* \]
(33)
Note 1. When \( f_v = 0 \), we have
\[
\varphi^*_{[1]}(z,v) = ((0)_{-1} \cdot (z^2 - 1)^{-(v+1/2)})_{-(1+v)} = (K \cdot (z^2 - 1)^{-(v+1/2)})_{-(1+v)} = 0.
\] (34 - 1)
(for \( 1 + v \notin \mathbb{Z}_0^+ \))

\[
\varphi^*_{[2]}(z,v) = ((z^2 - 1)^{-(v+1/2)} \cdot (0)_{-1})_{-(1+v)} = ((z^2 - 1)^{-(v+1/2)} \cdot K)_{-(1+v)} = ((z^2 - 1)^{-(v+1/2)})_{-(1+v)} K.
\] (34 - 2)

from (2) and (3), respectively, by Lemmas (i v) and (i).

And we have

\[
((z^2 - 1)^{-(v+1/2)} K)_{-(1+v)} = (K(z^2 - 1)^{-(v+1/2)})_{-(1+v)} = (K(z^2 - 1)^{-(v+1/2)})_{-(1+v)} (34 - 3)
\]
by our definition \( \S 0 \), (1), for N-Fractional Calculus.

**Proof of Group II**;

Set
\[
\varphi = (z^2 - 1)\lambda \phi, \quad \phi = \phi(z),
\] (35)
we have then
\[
\phi_2 \cdot (z^2 - 1) + \phi \cdot z(4\lambda + 1) + \lambda \cdot ((4\lambda^2 - \nu^2) + \frac{2\lambda(2\lambda - 1)}{z^2 - 1}) = f \cdot (z^2 - 1)^{-\lambda}
\] (36)
from (1), applying (35).

(Refer to the proof of Group II in \( \S 2 \), in the previous paper for Homogeneous one.)

When \( \lambda = 0 \), (36) is reduced to (1). We have then the same particular solutions as Group I

When \( \lambda = 1/2 \), we have
\[
\phi_2 \cdot (z^2 - 1) + \phi_1 \cdot 3z + \phi \cdot (1 - \nu^2) = f \cdot (z^2 - 1)^{-1/2}
\] (37)
from (36).

Operate \( N^\alpha \) to the both sides of (37), then yields
\[
\phi_{2+\alpha} \cdot (z^2 - 1) + \phi_{1+\alpha} \cdot z(2\alpha + 3) + \phi_\alpha \cdot ((\alpha + 1)^2 - \nu^2) = (f \cdot (z^2 - 1)^{-1/2})_{\alpha}.
\] (38)

(1) Case \( \alpha = \nu - 1 \);

In this case, letting
\[
\phi = V = V(z) \quad (\phi = V,)
\] (39)
we obtain
from (38). A particular solution to this linear first order equation is given by

$$V = ((f \cdot (z^2 - 1)^{-1/2})_{v-1} \cdot (z - 1)^{v-1/2})_{-1} \cdot (z^2 - 1)^{-1(v+1/2)} = P(v) H(v) \quad (41)$$

Therefore, we obtain

$$\varphi = (z^2 - 1)^{1/2}((f \cdot (z^2 - 1)^{-1/2})_{v-1} \cdot (z^2 - 1)^{v-1/2})_{-1} \cdot (z^2 - 1)^{-1(v+1/2)} = \varphi^{*}_{[5](z,v)} \quad (42)$$

from (35), applying (39) and (41), for $\lambda = 1/2$.

Next changing the order $P(v)$ and $H(v)$ in parenthesis $(\cdot)_v$ in (7) we obtain

$$\varphi = (z^2 - 1)^{1/2}(P(v) \cdot H(v))_{-v} = \varphi^{*}_{[6](z,v)} \quad (8)$$

where

$$\varphi^{*}_{[5](z,v)} \neq \varphi^{*}_{[6](z,v)} \quad (43)$$

(11) Case $\alpha = -v - 1$;

In the same way as (1) above, setting $-v$ instead of $v$ in the solutions (7) and (8), we obtain

$$\varphi = (z^2 - 1)^{1/2}(P(-v) \cdot H(-v))_{-v} = \varphi^{*}_{[7](z,v)} \quad (9)$$

$$\varphi = (z^2 - 1)^{1/2}(H(-v) \cdot P(-v))_{-v} = \varphi^{*}_{[8](z,v)} \quad (10)$$

where

$$\varphi^{*}_{[7](z,v)} \neq \varphi^{*}_{[8](z,v)} \quad (for \quad v \notin \mathbb{Z}_{0}^{+}) \quad (44)$$

And

$$\varphi^{*}_{[7](z,v)} = \varphi^{*}_{[5](z,-v)} \quad \varphi^{*}_{[8](z,v)} = \varphi^{*}_{[6](z,-v)} \quad (45)$$

**Proof of Group III**

Set

$$\varphi = (z - 1)^{\lambda} \phi, \quad \phi = \phi(z) \quad (46)$$

we have then

$$\phi_{2} \cdot (z^2 - 1) + \phi_{1} \cdot (2\lambda + 1 + 2\lambda) + \phi \cdot ((\lambda^2 - v^2) + \frac{\lambda^{2}(\lambda - 1)}{z - 1}) = f \cdot (z - 1)^{-\lambda} \quad (47)$$

from (1), applying (46).

(Refer to the proof of Group III in §2, in the previous paper for Homogeneous one)

When $\lambda = 0$, (47) is reduced to (1). We have then the same particular solutions as Group I.
When $\lambda = 1/2$, we have
\[\phi_2 \cdot (z^2 - 1) + \phi_1 \cdot (2z + 1) + \phi \cdot (1/4 - v^2) = f \cdot (z - 1)^{-1/2}\] (48)
from (47).

Operate $N^\alpha$ to the both sides of (48), then yields
\[\phi_{2+\alpha} \cdot (z^2 - 1) + \phi_{1+\alpha} \cdot (2z + 1) + \phi_{\alpha} \cdot ((\alpha + 1/2)^2 - v^2) = (f \cdot (z - 1)^{-1/2})_\alpha\] (49)

(I) Case $\alpha = v - 1/2$;

In this case letting
\[\phi_{v+1/2} = V = V(z), \quad (\phi = V_{-(v+1/2)})\]
we obtain
\[V_1 \cdot (z^2 - 1) + V \cdot (2z + 1) = (f \cdot (z - 1)^{-1/2})_{v-1/2}\] (50)
from (49).

A particular solution to this equation is given by
\[V = ((f \cdot (z - 1)^{-1/2})_{v-1/2} \cdot (z-1)^v \cdot (z+1)^v_{-1} \cdot ((z-1)^{-(v+1)}(z+1)^{-v}))_{-(v+1/2)}\]
\[= (z-1)^{1/2}((Q(v) \cdot S(v))_{-(v+1/2)}\equiv \phi_{(I)(z,v)}^{*}\]
(12)

Next changing the order $Q(v)$ and $S(v)$ in parenthesis (·)_{-(v+1/2)} in (12) we obtain
\[\varphi = (z-1)^{1/2}((Q(v) \cdot S(v))_{-(v+1/2)})_{-(v+1/2)}\]
where
\[\varphi_{(I)(z,v)}^{*} = \varphi_{(10)(z,v)}^{*} \quad (for \quad -(v+1/2) \notin Z_0^*)\]
(54)

(II) Case $\alpha = -v-1/2$;

In the same way as (I) above, setting $-v$ instead of $v$ in the solutions (12) and (13), we obtain
\[\varphi = (z-1)^{1/2}(Q(-v) \cdot S(-v))_{-v} \equiv \varphi_{(II)(z,v)}^{*}\]
(14)
\begin{equation}
\varphi = (z - 1)^{1/2} (S(-\nu) \cdot Q(-\nu)) \cdot \varphi^*_{[12](\xi, \nu)} \end{equation}

where
\begin{equation}
\varphi^*_{[11](\xi, \nu)} \neq \varphi^*_{[12](\xi, \nu)} \quad \text{(for } \nu - 1/2 \notin \mathbb{Z}^*_0 \text{)}.
\end{equation}

And
\begin{equation}
\varphi^*_{[11](\xi, \nu)} = \varphi^*_{[9](\xi, -\nu)}, \quad \varphi^*_{[12](\xi, \nu)} = \varphi^*_{[10](\xi, -\nu)}.
\end{equation}

**Proof of Group I V:**

Set
\begin{equation}
\varphi = (z + 1)^{1/2}\phi, \quad \phi = \phi(z),
\end{equation}

we have then
\begin{equation}
\phi_2 \cdot (z^2 - 1) + \phi_1 \cdot (2\lambda + 1 - 2\lambda) + \phi \cdot \left(\lambda^2 - \nu^2 - \frac{\lambda(2\lambda - 1)}{z + 1}\right) = f \cdot (z + 1)^{-\lambda}
\end{equation}

from (1) applying (57).

(Refer to the proof of Group IV in § 2. in the previous paper for Homogeneous one)

When \(\lambda = 0\), (58) is reduced to (1). We have then the same particular solutions as Group I.

When \(\lambda = 1/2\), we have
\begin{equation}
\phi_2 \cdot (z^2 - 1) + \phi_1 \cdot (2z - 1) + \phi \cdot \left(1/4 - \nu^2\right) = f \cdot (z + 1)^{-1/2}
\end{equation}

from (58).

Operate \(N^\alpha\) to the both sides of (59), then yields
\begin{equation}
\phi_2 + \alpha \cdot (z^2 - 1) + \phi_1 \cdot (2\alpha + 1 - 1) + \phi_\alpha \cdot \left((\alpha + 1/2)^2 - \nu^2\right) = (f \cdot (z + 1)^{-1/2})_{\alpha}.
\end{equation}

(1) Case \(\alpha = \nu - 1/2\);

Letting
\begin{equation}
\phi_{\nu + 1/2} = V = V(z) \quad (\phi = V_{-(\nu + 1/2)}),
\end{equation}

we obtain
\begin{equation}
V_1 \cdot (z^2 - 1) + V \cdot (2\nu + 1 - 1) = (f \cdot (z + 1)^{-1/2})_{\nu - 1/2}
\end{equation}

from (60).

A particular solution to this equation is given by
\begin{equation}
V = ((f \cdot (z + 1)^{-1/2})_{\nu - 1/2} \cdot (z - 1)^{\nu - 1}(z + 1)^\nu)_{\nu - 1} \cdot (z - 1)^{\nu - (\nu + 1)}(z + 1)^{(\nu + 1)} = T(\nu) \cdot Y(\nu).
\end{equation}

Therefore, we obtain
\begin{equation}
\varphi = (z + 1)^{1/2} \left(((f \cdot (z + 1)^{-1/2})_{\nu - 1/2} \cdot (z - 1)^{\nu - 1}(z + 1)^\nu)_{\nu - 1} \cdot ((z - 1)^{-\nu}(z + 1)^{(\nu + 1)})_{\nu - (\nu + 1/2)}\right)
\end{equation}
\[
(z+1)^{1/2} (T(v) \cdot Y(v))_{-(v+1/2)} \equiv \varphi^*_{[13](z,v)} ,
\]
from (57) and (61), applying (63), for \( \lambda = 1/2 \).

Next changing the order \( T(v) \) and \( Y(v) \) in parenthesis \((\cdot)_{-(v+1/2)}\) in (17) we obtain
\[
\varphi = (z+1)^{1/2} (Y(v) \cdot T(v))_{-(v+1/2)} \equiv \varphi^*_{[14](z,v)} ,
\]
where
\[
\varphi^*_{[13](z,v)} \neq \varphi^*_{[14](z,v)} \quad \text{for} \quad -(v + 1/2) \notin \mathbb{Z}^+_0 .
\]

(11) Case \( \alpha = -v - 1/2 \);

In the same way as (1) above, setting \(-v\) instead of \(v\) in the solutions (17) and (18), we obtain
\[
\varphi = (z+1)^{1/2} (T(-v) \cdot Y(-v))_{v-1/2} \equiv \varphi^*_{[15](z,v)} ,
\]
\[
\varphi = (z+1)^{1/2} (Y(-v) \cdot T(-v))_{v-1/2} \equiv \varphi^*_{[16](z,v)} ,
\]
where
\[
\varphi^*_{[15](z,v)} = \varphi^*_{[16](z,v)} \quad \text{(for} \quad (v-1/2) \notin \mathbb{Z}^+_0 \text{)} .
\]

\[
\varphi^*_{[15](z,v)} = \varphi^*_{[13](z,-v)} , \quad \varphi^*_{[16](z,v)} = \varphi^*_{[14](z,-v)} .
\]

§3. Some Example

(i) When \( v = -1 \) and \( f = (z-1)^{-1} \) we have
\[
\varphi_2 \cdot (z^2 - 1) + \varphi_1 \cdot z - \varphi = (z - 1)^{-1}
\]
and
\[
\varphi = \varphi^*_{[1](z,-1)} = \left( (f_{-1} \cdot (z^2 - 1)^{-3/2})_{-1} (z^2 - 1)^{1/2} \right)_0
\]
\[
= \left( \log(z-1) \cdot (z^2 - 1)^{-3/2} \right)_{-1} (z^2 - 1)^{1/2}
\]
from §2. (1) and §2. (2) respectively.

Hence
\[
\varphi_1 = \log(z-1) \cdot (z^2 - 1)^{-1} + \left( \log(z-1) \cdot (z^2 - 1)^{-3/2} \right)_{-1} (z^2 - 1)^{-1/2} z
\]
and
\[
\varphi_2 = (z - 1)^{-1}(z^2 - 1)^{-1} - \log(z - 1) \cdot 2z(z^2 - 1)^{-2} + \log(z - 1) \cdot z(z^2 - 1)^{-2} \\
+ \left(\log(z - 1) \cdot (z^2 - 1)^{-3/2}\right)_{-1} \{-z^2(z^2 - 1)^{-3/2} + (z^2 - 1)^{-1/2}\} \\
(19)
\]
respectively.

Then applying (17), (18) and (19), we obtain

\[
\text{LHS of (15) } = (z - 1)^{-1} . \\
(20)
\]

The function shown by (17) satisfies equation (15) clearly.

(i) When \( v = -1/2 \) and \( f = (z - 1)^{-1/2} \) we have

\[
\varphi_2 \cdot (z^2 - 1) + \varphi_1 \cdot z - \varphi \cdot (1/4) = (z - 1)^{-1/2} \\
(21)
\]
and

\[
\varphi = \varphi^{*}_{(1/4, -1/2)} = (z + 1)^{1/2}\left[\log(z - 1) \cdot (z - 1)^{-1/2}(z + 1)^{-3/2}\right]_{-1} \\
(22)
\]
from \( \S 2. (1) \) and \( \S 2. (9) \) respectively.

Hence we have

\[
\varphi_1 = \frac{1}{2}(z + 1)^{-1/2}(W)_{-1} + \log(z - 1) \cdot (z - 1)^{-1/2}(z + 1)^{-1} \\
(23)
\]
and

\[
\varphi_2 = -\frac{1}{4}(z + 1)^{-3/2}(W)_{-1} + (z - 1)^{-3/2}(z + 1)^{-1} + \left\{\frac{1}{2}(z - 1)^{-1/2}(z + 1)^{-2} \\
- \frac{1}{2}(z - 1)^{-3/2}(z + 1)^{-1} - (z - 1)^{-1/2}(z + 1)^{-2}\right\}\log(z - 1) \\
(24)
\]
from (22), respectively.

Then applying (22), (23) and (24), we obtain

\[
\text{LHS of (21) } = (W)_{-1}(z + 1)^{1/2}\{-\frac{1}{4}(z - 1)^{-1} + \frac{1}{2}z - \frac{1}{4}(z + 1)\} + (z - 1)^{-1/2} \\
+ (z + 1)^{-1}(z - 1)^{-1/2}\left\{\frac{1}{2}(z - 1) - \frac{1}{2}(z + 1) - (z - 1) + z\right\}\log(z - 1) \\
= (z - 1)^{-1/2} . \\
(25)
\]

The function shown by (22) satisfies equation (21) clearly.
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