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Solutions to The Nonhomogeneous Chebyshev's Equation by Means of N- Fractional Calculus Operator

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Abstract

In this article, solutions to the nonhomogeneous Chebyshev's equations

$$L[\varphi;z;\nu]=\varphi_2\cdot(z^2-1)+\varphi_1\cdot z-\varphi\cdot\nu^2=f \quad (z^2-1\neq 0,\ f\neq 0)$$

$$(\varphi_\alpha=d^\alpha\varphi/dz^\alpha \text{ for } \alpha>0,\ \varphi_0=\varphi=\varphi(z),\ f=f(z))$$

are discussed by means of N-fractional calculus operator (NFCO- Method).

By our method, some particular solutions to the above equations are given as below for example, in fractional differintegrated forms.

Group I.

(i) $\varphi=(G(\nu)\cdot H(\nu))_{-(1+\nu)}\equiv\varphi^*_{[1](z,\nu)}$ (denote)

(ii) $\varphi=(H(\nu)\cdot G(\nu))_{-(1+\nu)}\equiv\varphi^*_{[2](z,\nu)}$

(iII) $\varphi=(G(-\nu)\cdot H(-\nu))_{\nu-1}\equiv\varphi^*_{[3](z,\nu)}$

(iv) $\varphi=(H(-\nu)\cdot G(-\nu))_{\nu-1}\equiv\varphi^*_{[4](z,\nu)}$

where

$$G(\nu)=(f_\nu\cdot(z^2-1)^{\nu-1/2})_{-1},\qquad H(\nu)=(z^2-1)^{-(\nu+1/2)}\quad .$$

§ 0. Introduction (Definition of Fractional Calculus)

and

§ 1. Preliminary

are ommited, then refer to the previous paper for the Homogeneous Chebyshev's equation.

§ 2. Solutions to The Nonhomogeneous Chebyshev's Equations by Means of N-Fractional Calculus Operator

Theorem 1. *Let* $\varphi = \varphi(z) \in F$ and $f = f(z) \in F$, *then the nonhomogeneous Chebyshev's equation*

$$L[\varphi\,;z\,;v] = \varphi_2 \cdot (z^2 - 1) + \varphi_1 \cdot z - \varphi \cdot v^2 = f\,, \qquad (v \in R\,,\; z^2 - 1 \neq 0) \qquad (1)$$

has particular solutions of the forms (fractional differintegrated form);

Group I.

(i) $$\varphi(z) = (G(v) \cdot H(v))_{-(1+v)} \equiv \varphi^*_{[1](z,v)} \quad \text{(denote)} \qquad (2)$$

(ii) $$\varphi(z) = (H(v) \cdot G(v))_{-(1+v)} \equiv \varphi^*_{[2](z,v)} \qquad (3)$$

(iii) $$\varphi(z) = (G(-v) \cdot H(-v))_{v-1} \equiv \varphi^*_{[3](z,v)} \qquad (4)$$

(iv) $$\varphi(z) = (H(-v) \cdot G(-v))_{v-1} \equiv \varphi^*_{[4](z,v)} \qquad (5)$$

where

$$G(v) = (f_v \cdot (z^2 - 1)^{v-1/2})_{-1}\,, \qquad H(v) = (z^2 - 1)^{-(v+1/2)}\,. \qquad (6)$$

Group II.

(i) $$\varphi(z) = (z^2 - 1)^{1/2} (P(v) \cdot H(v))_{-v} \equiv \varphi^*_{[5](z,v)} \qquad (7)$$

(ii) $$\varphi(z) = (z^2 - 1)^{1/2} (H(v) \cdot P(v))_{-v} \equiv \varphi^*_{[6](z,v)} \qquad (8)$$

(iii) $$\varphi(z) = (z^2 - 1)^{1/2} (P(-v) \cdot H(-v))_{v} \equiv \varphi^*_{[7](z,v)} \qquad (9)$$

(iv) $$\varphi(z) = (z^2 - 1)^{1/2} (H(-v) \cdot P(-v))_{v} \equiv \varphi^*_{[8](z,v)} \qquad (10)$$

where

$$P(v) = ((f\cdot(z^2-1)^{-1/2})_{v-1}\cdot(z^2-1)^{v-1/2})_{-1}, \qquad H(v) = (z^2-1)^{-(v+1/2)} \quad .. \ (11)$$

Group I I I .

(i) $$\varphi(z) = (z-1)^{1/2}(Q(v)\cdot S(v))_{-(v+1/2)} \equiv \varphi^*_{[9](z,v)} \qquad (12)$$

(i i) $$\varphi(z) = (z-1)^{1/2}(S(v)\cdot Q(v))_{-(v+1/2)} \equiv \varphi^*_{[10](z,v)} \qquad (13)$$

(i i i) $$\varphi(z) = (z-1)^{1/2}(Q(-v)\cdot S(-v))_{v-1/2} \equiv \varphi^*_{[11](z,v)} \qquad (14)$$

(i v) $$\varphi(z) = (z-1)^{1/2}(S(-v)\cdot Q(-v))_{v-1/2} \equiv \varphi^*_{[12](z,v)} \qquad (15)$$

where

$$Q(v) = ((f\cdot(z-1)^{-1/2})_{v-1/2}\cdot(z-1)^{v}(z+1)^{v-1})_{-1}, \quad S(v) = ((z-1)^{-(v+1)}(z+1)^{-v}) \ . (16)$$

Group I V .

(i) $$\varphi(z) = (z+1)^{1/2}(T(v)\cdot Y(v))_{-(v+1/2)} \equiv \varphi^*_{[13](z,v)} \qquad (17)$$

(i i) $$\varphi(z) = (z+1)^{1/2}(Y(v)\cdot T(v))_{-(v+1/2)} \equiv \varphi^*_{[14](z,v)} \qquad (18)$$

(i i i) $$\varphi(z) = (z+1)^{1/2}(T(-v)\cdot Y(-v))_{v-1/2} \equiv \varphi^*_{[15](z,v)} \qquad (19)$$

(i v) $$\varphi(z) = (z+1)^{1/2}(Y(-v)\cdot T(-v))_{v-1/2} \equiv \varphi^*_{[16](z,v)} \qquad (20)$$

where

$$T(v) = ((f\cdot(z+1)^{-1/2})_{v-1/2}\cdot(z-1)^{v-1}(z+1)^{v})_{-1}, \quad Y(v) = ((z-1)^{-v}(z+1)^{-(v+1)}) \ . (21)$$

Proof of Group I ;

Operate N^{α} to the both sides of (1), we have then

$$\varphi_{2+\alpha}\cdot(z^2-1) + \varphi_{1+\alpha}\cdot z(2\alpha+1) + \varphi_{\alpha}\cdot(\alpha^2-v^2) = f_{\alpha}, \quad (f_{\alpha} \neq 0). \qquad (22)$$

(Refer to the proof of **Group I** in § **2.** in the previous paper for Homogeneous one.)

(I) Case $\alpha = v$;

In this case we obtain

$$\varphi_{2+v}\cdot(z^2-1) + \varphi_{1+v}\cdot z(2v+1) = f_v, \qquad (23)$$

from (22), then letting

$$\varphi_{1+v} = \psi = \psi(z) \qquad (\varphi = \psi_{-(1+v)}), \qquad (24)$$

yields

$$\psi_1\cdot(z^2-1) + \psi\cdot z(2v+1) = f_v \ , \qquad (25)$$

from (24). A particular solution to this linear first order equation is given by

$$\psi = (f_v\cdot(z^2-1)^{v-1/2})_{-1}\cdot(z^2-1)^{-(v+1/2)} \ . \qquad (26)$$

Therefore, we obtain

$$\varphi = \psi_{-(1+\nu)} = ((f_\nu \cdot (z^2-1)^{\nu-12})_{-1} \cdot (z^2-1)^{-(\nu+1/2)})_{-(1+\nu)} \quad (27)$$

$$= (G(\nu)\cdot H(\nu))_{-(1+\nu)} \equiv \varphi^*_{[1](z,\nu)} \quad (2)$$

from (24) and (26).

Inversely, we have

$$\varphi_{2+\nu} = \psi_1 = (f_\nu \cdot (z^2-1)^{\nu-\frac{1}{2}}) \cdot (z^2-1)^{-(\nu+1/2)}$$
$$- (f_\nu \cdot (z^2-1)^{\nu-1/2})_{-1} \cdot (\nu+\tfrac{1}{2}) \cdot 2z(z^2-1)^{-(\nu+1/2+1)} \quad (28)$$

and

$$\varphi_{1+\nu} = \psi = (f_\nu \cdot (z^2-1)^{\nu-1/2})_{-1} \cdot (z^2-1)^{-(\nu+1/2)} \quad (29)$$

from (2) respectively. Then we obtain

$$\text{LHS of (23)} = \{(f_\nu \cdot (z^2-1)^{\nu-\frac{1}{2}}) \cdot (z^2-1)^{-(\nu+\frac{1}{2})}$$
$$- (f_\nu \cdot (z^2-1)^{\nu-\frac{1}{2}})_{-1} \cdot (\nu+\tfrac{1}{2}) \cdot 2z(z^2-1)^{-(\nu+\frac{1}{2}+1)}\}(z^2-1)$$
$$+ z(2\nu+1)\ (f_\nu \cdot (z^2-1)^{\nu-\frac{1}{2}})_{-1} \cdot (z^2-1)^{-(\nu+\frac{1}{2})} = f_\nu \quad (30)$$

applying (28) and (29).

That is, the function shown by (2) satisfies equation (23).

Moreover, we have (1) operating $N^{-\nu}$ to the both sides of (23).

Therefore, the function given by (2) satisfies equation (1).

Next changing the order $G(\nu)$ and $H(\nu)$ in parenthesis $(\quad\cdot\quad)_{-(1+\nu)}$ in (2) we obtain

$$\varphi = (H(\nu)\cdot G(\nu))_{-(1+\nu)} \equiv \varphi^*_{[2](z,\nu)} \quad (3)$$

where

$$\varphi^*_{[1](z,\nu)} \neq \varphi^*_{[2](z,\nu)} \quad (\text{for } -(1+\nu) \notin \mathbf{Z}^+_0) \quad (31)$$

(II) Case $\alpha = -\nu$;

In the same way as **(I)** above, setting $-\nu$ instead of ν in the solutions (2) and (3), we obtain

$$\varphi = (G(-\nu)\cdot H(-\nu))_{\nu-1} \equiv \varphi^*_{[3](z,\nu)} \quad (4)$$

$$\varphi = (H(-\nu)\cdot G(-\nu))_{\nu-1} \equiv \varphi^*_{[4](z,\nu)} \quad (5)$$

where

$$\varphi^*_{[3](z,\nu)} \neq \varphi^*_{[4](z,\nu)} \quad (\text{for } (\nu-1) \notin \mathbf{Z}^+_0) \quad (32)$$

And

$$\varphi^*_{[3](z,\nu)} = \varphi^*_{[1](z,-\nu)}, \qquad \varphi^*_{[4](z,\nu)} = \varphi^*_{[2](z,-\nu)} \quad (33)$$

Note 1. When $f_v = 0$, we have

$$\varphi^*_{[1](z,v)} = ((0)_{-1} \cdot (z^2-1)^{-(v+1/2)})_{-(1+v)} = (K \cdot (z^2-1)^{-(v+1/2)})_{-(1+v)} = 0 \ . \quad (34\text{-}1)$$

(for $(1+v) \notin Z_0^+$)

(K; arbitrary constant for integation)

$$\varphi^*_{[2](z,v)} = ((z^2-1)^{-(v+1/2)} \cdot (0)_{-1})_{-(1+v)} = ((z^2-1)^{-(v+1/2)} \cdot K)_{-(1+v)}$$

$$= ((z^2-1)^{-(v+1/2)})_{-(1+v)} K \quad (34\text{-}2)$$

from (.2) and (3) , respectively, by Lemmas (i v) and (i).

And we have

$$((z^2-1)^{-(v+1/2)} K)_{-(1+v)} = (K(z^2-1)^{-(v+1/2)})_{-(1+v)} = K((z^2-1)^{-(v+1/2)})_{-(1+v)} \quad (34\text{-}3)$$

by our definition **§ 0.** (1) , for N- Fractional Calculus.

Proof of Group I I ;

Set

$$\varphi = (z^2-1)^{\lambda} \phi, \qquad \phi = \phi(z), \quad (35)$$

we have then

$$\phi_2 \cdot (z^2-1) + \phi_1 \cdot z(4\lambda+1) + \phi \cdot \{(4\lambda^2 - v^2) + \frac{2\lambda(2\lambda-1)}{z^2-1}\} = f \cdot (z^2-1)^{-\lambda} \quad (36)$$

from (1), applying (35).

(Refer to the proof of **Group II** in § **2.** in the previous paper for Homogeneous one.)

When $\lambda = 0$, (36) is reduced to (1). We have then the same particular solutions as Group I

When $\lambda = 1/2$, we have

$$\phi_2 \cdot (z^2-1) + \phi_1 \cdot 3z + \phi \cdot (1-v^2) = f \cdot (z^2-1)^{-1/2} \quad (37)$$

from (36).

Operate N^{α} to the both sides of (37), then yields

$$\phi_{2+\alpha} \cdot (z^2-1) + \phi_{1+\alpha} \cdot z(2\alpha+3) + \phi_{\alpha} \cdot ((\alpha+1)^2 - v^2) = (f \cdot (z^2-1)^{-1/2})_{\alpha} \ . \quad (38)$$

(I) Case $\alpha = v-1$;

In this case ,letting

$$\phi_v = V = V(z) \qquad (\phi = V_{-v}), \quad (39)$$

we obtain

$$V_1\cdot(z^2-1)+V\cdot z(2\nu+1)=(f\cdot(z^2-1)^{-1/2})_{\nu-1}\ , \qquad (40)$$

from (38). A particular solution to this linear first order equation is given by

$$V=((f\cdot(z^2-1)^{-1/2})_{\nu-1}\cdot(z^2-1)^{\nu-1/2})_{-1}\cdot(z^2-1)^{-(\nu+1/2)}=P(\nu)H(\nu)\ . \qquad (41)$$

Therefore, we obtain

$$\varphi=(z^2-1)^{1/2}\Big(((f\cdot(z^2-1)^{-1/2})_{\nu-1}\cdot(z^2-1)^{\nu-1/2})_{-1}\cdot(z^2-1)^{-(\nu+1/2)}\Big)_{-\nu} \qquad (42)$$

$$=(z^2-1)^{1/2}(P(\nu)\cdot H(\nu))_{-\nu}\ \equiv\varphi^*_{[5](z,\nu)}\ , \qquad (7)$$

from (35), applying (39) and (41), for $\lambda=1/2$.

Next changing the order $P(\nu)$ and $H(\nu)$ in parenthesis $(\quad\cdot\quad)_{-\nu}$ in (7) we obtain

$$\varphi=(z^2-1)^{1/2}(H(\nu)\cdot P(\nu))_{-\nu}\equiv\varphi^*_{[6](z,\nu)}\ . \qquad (8)$$

where

$$\varphi^*_{[5](z,\nu)}\neq\varphi^*_{[6](z,\nu)}\quad(\text{for}\ \ (-\nu)\notin Z_0^+)\ . \qquad (43)$$

(I I) Case $\alpha=-\nu-1$;

In the same way as **(I)** above, setting $-\nu$ instead of ν in the solutions (7) and (8), we obtain

$$\varphi=(z^2-1)^{1/2}(P(-\nu)\cdot H(-\nu))_{\nu}\equiv\varphi^*_{[7](z,\nu)}\ , \qquad (9)$$

$$\varphi=(z^2-1)^{1/2}(H(-\nu)\cdot P(-\nu))_{\nu}\equiv\varphi^*_{[8](z,\nu)}\ . \qquad (10)$$

where

$$\varphi^*_{[7](z,\nu)}\neq\varphi^*_{[8](z,\nu)}\quad(\text{for}\ \ \nu\notin Z_0^+)\ . \qquad (44)$$

And

$$\varphi^*_{[7](z,\nu)}=\varphi^*_{[5](z,-\nu)},\qquad \varphi^*_{[8](z,\nu)}=\varphi^*_{[6](z,-\nu)}\ . \qquad (45)$$

Proof of Group I I I ;

Set

$$\varphi=(z-1)^{\lambda}\phi,\qquad \phi=\phi(z), \qquad (46)$$

we have then

$$\phi_2\cdot(z^2-1)+\phi_1\cdot\{z(2\lambda+1)+2\lambda\}+\phi\cdot\{(\lambda^2-\nu^2)+\frac{\lambda(2\lambda-1)}{z-1}\}=f\cdot(z-1)^{-\lambda} \qquad (47)$$

from (1), applying (46)

(Refer to the proof of **Group III** in § **2.** in the previous paper for Homogeneous one)

When $\lambda=0$, (47) is reduced to (1). We have then the same particular solutions as Group I .

When $\lambda = 1/2$, we have

$$\phi_2 \cdot (z^2 - 1) + \phi_1 \cdot (2z + 1) + \phi \cdot (1/4 - \nu^2) = f \cdot (z-1)^{-1/2} \tag{48}$$

from (47).

Operate N^α to the both sides of (48), then yields

$$\phi_{2+\alpha} \cdot (z^2 - 1) + \phi_{1+\alpha} \cdot \{z(2\alpha + 2) + 1\} + \phi_\alpha \cdot \{(\alpha + 1/2)^2 - \nu^2\} = (f \cdot (z-1)^{-1/2})_\alpha \ . \tag{49}$$

(I) Case $\alpha = \nu - 1/2$;

In this case letting

$$\phi_{\nu+1/2} = V = V(z) \qquad (\phi = V_{-(\nu+1/2)}), \tag{50}$$

we obtain

$$V_1 \cdot (z^2 - 1) + V \cdot \{z(2\nu + 1) + 1\} = (f \cdot (z-1)^{-1/2})_{\nu-1/2} \ , \tag{51}$$

from (49).

A particular solution to this equation is given by

$$V = ((f \cdot (z-1)^{-1/2})_{\nu-1/2} \cdot \frac{z-1}{(z^2-1)^{1-\nu}})_{-1} \cdot \frac{(z^2-1)^{-\nu}}{z-1} = Q(\nu)S(\nu) \ . \tag{52}$$

Therefore, we obtain

$$\varphi = (z-1)^{1/2}\Big(((f \cdot (z-1)^{-1/2})_{\nu-1/2} \cdot (z-1)^{\nu}(z+1)^{\nu-1})_{-1} \cdot ((z-1)^{-(\nu+1)}(z+1)^{-\nu})\Big)_{-(\nu+1/2)} \tag{53}$$

$$= (z-1)^{1/2}(Q(\nu) \cdot S(\nu))_{-(\nu+1/2)} \equiv \varphi^*_{[9](z,\nu)} \ , \tag{12}$$

from (46) and (50), applying (52), for $\lambda = 1/2$.

Next changing the order $Q(\nu)$ and $S(\nu)$ in parenthesis $(\quad \cdot \quad)_{-(\nu+1/2)}$ in (12) we obtain

$$\varphi = (z-1)^{1/2}(S(\nu) \cdot Q(\nu))_{-\nu} \equiv \varphi^*_{[10](z,\nu)} \ . \tag{13}$$

where

$$\varphi^*_{[9](z,\nu)} \neq \varphi^*_{[10](z,\nu)} \quad (\text{for } -(\nu + 1/2) \notin \boldsymbol{Z}_0^+) \ . \tag{54}$$

(II) Case $\alpha = -\nu - 1/2$;

In the same way as **(I)** above, setting $-\nu$ instead of ν in the solutions (12) and (13), we obtain

$$\varphi = (z-1)^{1/2}(Q(-\nu) \cdot S(-\nu))_\nu \equiv \varphi^*_{[11](z,\nu)} \ , \tag{14}$$

$$\varphi = (z-1)^{1/2}(S(-\nu)\cdot Q(-\nu))_\nu \equiv \varphi^*_{[12](z,\nu)} \cdot \qquad (15)$$

where

$$\varphi^*_{[11](z,\nu)} \neq \varphi^*_{[12](z,\nu)} \quad (\text{for } (\nu - 1/2) \notin Z^+_0) \ . \qquad (55)$$

And

$$\varphi^*_{[11](z,\nu)} = \varphi^*_{[9](z,-\nu)}, \qquad \varphi^*_{[12](z,\nu)} = \varphi^*_{[10](z,-\nu)} \ . \qquad (56)$$

Proof of Group I V ;

Set

$$\varphi = (z+1)^\lambda \phi, \qquad \phi = \phi(z), \qquad (57)$$

we have then

$$\phi_2 \cdot (z^2-1) + \phi_1 \cdot \{z(2\lambda+1) - 2\lambda\} + \phi \cdot \{(\lambda^2 - \nu^2) - \frac{\lambda(2\lambda-1)}{z+1}\} = f \cdot (z+1)^{-\lambda} \qquad (58)$$

from (1) applying (57).

(Refer to the proof of **Group IV** in § **2.** in the previous paper for Homogeneous one)

When $\lambda = 0$, (58) is reduced to (1). We have then the same particular solutions as Group I .

When $\lambda = 1/2$, we have

$$\phi_2 \cdot (z^2-1) + \phi_1 \cdot (2z-1) + \phi \cdot (1/4 - \nu^2) = f \cdot (z+1)^{-1/2} \qquad (59)$$

from (58).

Operate N^α to the both sides of (59), then yields

$$\phi_{2+\alpha} \cdot (z^2-1) + \phi_{1+\alpha} \cdot \{z(2\alpha+2) - 1\} + \phi_\alpha \cdot \{(\alpha + 1/2)^2 - \nu^2\} = (f \cdot (z+1)^{-1/2})_\alpha \ . \ (60)$$

(I) Case $\alpha = \nu - 1/2$;

Letting

$$\phi_{\nu+1/2} = V = V(z) \qquad (\phi = V_{-(\nu+1/2)}), \qquad (61)$$

we obtain

$$V_1 \cdot (z^2-1) + V \cdot \{z(2\nu+1) - 1\} = (f \cdot (z+1)^{-1/2})_{\nu-1/2} \ , \qquad (62)$$

from (60).

A particular solution to this equation is given by

$$V = ((f \cdot (z+1)^{-1/2})_{\nu-1/2} \cdot (z-1)^{\nu-1}(z+1)^\nu)_{-1} \cdot (z-1)^{-\nu}(z+1)^{-(\nu+1)} = T(\nu) \cdot Y(\nu) \ . \qquad (63)$$

Therefore, we obtain

$$\varphi = (z+1)^{1/2}\Big(((f \cdot (z+1)^{-1/2})_{\nu-1/2} \cdot (z-1)^{\nu-1}(z+1)^\nu)_{-1} \cdot ((z-1)^{-\nu}(z+1)^{-(\nu+1)})\Big)_{-(\nu+1/2)} \qquad (64)$$

$$= (z+1)^{1/2}(T(\nu)\cdot Y(\nu))_{-(\nu+1/2)} \equiv \varphi^*_{[13](z,\nu)} \quad , \tag{17}$$

from (57) and (61), applying (63), for $\lambda = 1/2$.

Next changing the order $T(\nu)$ and $Y(\nu)$ in parenthesis $(\quad\cdot\quad)_{-(\nu+1/2)}$ in (17) we obtain

$$\varphi = (z+1)^{1/2}(Y(\nu)\cdot T(\nu))_{-(\nu+1/2)} \equiv \varphi^*_{[14](z,\nu)} \quad . \tag{18}$$

where

$$\varphi^*_{[13](z,\nu)} \neq \varphi^*_{[14](z,\nu)} \quad (\text{for } -(\nu+1/2) \notin \mathbf{Z}_0^+) \ . \tag{65}$$

(II) Case $\alpha = -\nu - 1/2$;

In the same way as **(I)** above, setting $-\nu$ instead of ν in the solutions (17) and (18), we obtain

$$\varphi = (z+1)^{1/2}(T(-\nu)\cdot Y(-\nu))_{\nu-1/2} \equiv \varphi^*_{[15](z,\nu)} \quad , \tag{19}$$

$$\varphi = (z+1)^{1/2}(Y(-\nu)\cdot T(-\nu))_{\nu-1/2} \equiv \varphi^*_{[16](z,\nu)} \quad . \tag{20}$$

where

$$\varphi^*_{[15](z,\nu)} \neq \varphi^*_{[16](z,\nu)} \quad (\text{for } (\nu-1/2) \notin \mathbf{Z}_0^+) \ . \tag{66}$$

And

$$\varphi^*_{[15](z,\nu)} = \varphi^*_{[13](z,-\nu)} \ , \qquad \varphi^*_{[16](z,\nu)} = \varphi^*_{[14](z,-\nu)} \quad . \tag{67}$$

§ 3. Some Example

(i) When $\nu = -1$ and $f = (z-1)^{-1}$ we have

$$\varphi_2 \cdot (z^2-1) + \varphi_1 \cdot z - \varphi = (z-1)^{-1} \tag{15}$$

and

$$\varphi = \varphi^*_{[1](z,-1)} = \left((f_{-1}\cdot(z^2-1)^{-3/2})_{-1}(z^2-1)^{1/2}\right)_0 \tag{16}$$

$$= \left(\log(z-1)\cdot(z^2-1)^{-3/2}\right)_{-1}(z^2-1)^{1/2} \tag{17}$$

from § **2.** (1) and § **2.** (2) respectively.

Hence

$$\varphi_1 = \log(z-1)\cdot(z^2-1)^{-1} + \left(\log(z-1)\cdot(z^2-1)^{-3/2}\right)_{-1}(z^2-1)^{-1/2}z \tag{18}$$

and

$$\varphi_2 = (z-1)^{-1}(z^2-1)^{-1} - \log(z-1)\cdot 2z(z^2-1)^{-2} + \log(z-1)\cdot z(z^2-1)^{-2}$$

$$+\left(\log(z-1)\cdot(z^2-1)^{-3/2}\right)_{-1}\{-z^2(z^2-1)^{-3/2} + (z^2-1)^{-1/2}\} \qquad (19)$$

respectively.

Then applying (17), (18) and (19), we obtain

$$\text{LHS of } (15) = (z-1)^{-1} \quad . \qquad (20)$$

The function shown by (17) satisfies equation (15) clearly.

(i i) When $\nu = -1/2$ and $f = (z-1)^{-1/2}$ we have

$$\varphi_2\cdot(z^2-1) + \varphi_1\cdot z - \varphi\cdot(1/4) = (z-1)^{-1/2} \qquad (21)$$

and

$$\varphi = \varphi^*_{[9](z,-1/2)} = (z+1)^{1/2}\left(\log(z-1)\cdot(z-1)^{-1/2}(z+1)^{-3/2}\right)\Big)_{-1} \qquad (22)$$

from §2. (1) and §2. (9) respectively.

Hence we have

$$\varphi_1 = \tfrac{1}{2}(z+1)^{-1/2}(\,W\,)_{-1} + \log(z-1)\cdot(z-1)^{-1/2}(z+1)^{-1} \qquad (23)$$

$$(W = \log(z-1)\cdot(z-1)^{-1/2}(z+1)^{-3/2}\;)$$

and

$$\varphi_2 = -\tfrac{1}{4}(z+1)^{-3/21}(W)_{-1} + (z-1)^{-3/2}(z+1)^{-1} + \{\tfrac{1}{2}(z-1)^{-1/2}(z+1)^{-2}$$

$$-\tfrac{1}{2}(z-1)^{-3/2}(z+1)^{-1} - (z-1)^{-1/2}(z+1)^{-2}\}\log(z-1) \qquad (24)$$

from (22), respectively.

Then applying (22), (23) and (24), we obtain

$$\text{LHS of } (21) = (\,W\,)_{-1}(z+1)^{-1/2}\{-\tfrac{1}{4}(z-1) + \tfrac{1}{2}z - \tfrac{1}{4}(z+1)\} + (z-1)^{-1/2}$$

$$+(z+1)^{-1}(z-1)^{-1/2}\{\tfrac{1}{2}(z-1) - \tfrac{1}{2}(z+1) - (z-1) + z\}\log(z-1)$$

$$= (z-1)^{-1/2} \quad . \qquad (25)$$

The function shown by (22) satisfies equation (21) clearly.

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