<table>
<thead>
<tr>
<th>Title</th>
<th>ON UNIVALENT FUNCTIONS WITH HALF-INTEGER COEFFICIENTS (Conditions for Univalency of Functions and Applications)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>HIRANUMA, NAOKI</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2011), 1772: 18-20</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2011-12</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/171693">http://hdl.handle.net/2433/171693</a></td>
</tr>
<tr>
<td>Right</td>
<td></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
<tr>
<td>Textversion</td>
<td></td>
</tr>
<tr>
<td>Textversion</td>
<td></td>
</tr>
<tr>
<td>Textversion</td>
<td></td>
</tr>
<tr>
<td>Textversion</td>
<td></td>
</tr>
<tr>
<td>Textversion</td>
<td></td>
</tr>
<tr>
<td>Textversion</td>
<td></td>
</tr>
<tr>
<td>Textversion</td>
<td></td>
</tr>
</tbody>
</table>
ON UNIVALENT FUNCTIONS WITH HALF-INTEGER COEFFICIENTS

NAOKI HIRANUMA

ABSTRACT. Let $S$ be the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_{n} z^{n}$ which are analytic and univalent in the unit disk $D = \{z \in \mathbb{C}: |z| < 1\}$. The subclasses of $S$ whose coefficients $a_{n}$ belong to a quadratic field have been studied by Friedman [3] and Bernardi [1]. Linis [?] gave a short proof of Friedman's theorem which states that if all the $a_{n}$ are "rational integers" then $f$ is rational and has nine forms. In this paper, we consider what will happen if all the $a_{n}$ are "half-integers"; that is, $2a_{n} \in \mathbb{Z}$.

1. PRELIMINARIES

1.1. Notation and Definitions. A domain is an open connected set in the complex plane $\mathbb{C}$. The unit disk $D$ consists of all points $z \in \mathbb{C}$ of modulus $|z| < 1$. A single-valued function $f$ is said to be univalent in a domain $D \subset \mathbb{C}$ if it is injective; that is, if $f(z_{1}) \neq f(z_{2})$ for all points $z_{1}$ and $z_{2}$ in $D$ with $z_{1} \neq z_{2}$. The function $f$ is said to be locally univalent at a point $z_{0} \in D$ if it is univalent in some neighborhood of $z_{0}$. For analytic functions $f$, the condition $f'(z_{0}) \neq 0$ is equivalent to local univalence at $z_{0}$.

We shall be concerned primarily with the class $S$ of functions $f$ analytic and univalent in $D$, normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Thus each $f \in S$ has a Taylor series expansion of the form

$$f(z) = z + a_{2}z^{2} + a_{3}z^{3} + \cdots, \quad |z| < 1.$$  

The important example of a function in the class $S$ is the Koebe function

$$k(z) = \frac{z}{(1-z)^{2}} = z + 2z^{2} + 3z^{3} + \cdots.$$  

1.2. Bieberbach's Conjecture. In 1916, Bieberbach estimated the second coefficient $a_{2}$ of a function in the class $S$. (See [2, p. 30].)

Theorem 1. If $f \in S$ then $|a_{2}| \leq 2$. Equality occurs if and only if $f$ is the Koebe function or one of its rotations.

This suggests the general problem to find

$$A_{n} := \sup_{f \in S} |a_{n}|, \quad n = 2, 3, \ldots.$$  

In a footnote, he wrote "Vielleicht ist überhaupt $A_{n} = n$ (Perhaps it is generally $A_{n} = n$)." Since the Koebe function plays the extremal role in so many problems for the class $S$, it is natural to suspect that it maximizes $|a_{n}|$ for all $n$. This is the famous conjecture of Bieberbach, first proposed in 1916.

Many partial results were obtained in the intervening years, including results for special subclasses of $S$ and for particular coefficients, as well as asymptotic estimates and estimates for general $n$. Finally, de Branges [4] gave a remarkable proof in 1985. (See [6].)
Theorem 2. If $f \in S$ then

$$|a_{n}| \leq n, \quad n = 2, 3, \ldots \quad (1)$$

Equality occurs if and only if $f$ is the Koebe function or one of its rotations.

1.3. Prawitz’ Inequality. Let $f \in S$. Set $F(z) = z/f(z) = \sum_{n=0}^{\infty} b_{n} z^{n}$, then 

$$F(z) = 1 - a_{2} z + (a_{2}^{2} - a_{3}) z^{2} + \cdots .$$

Hence, we have $b_{0} = 1$, $b_{1} = -a_{2}$, $b_{2} = a_{2}^{2} - a_{3}$, $\ldots$. The coefficient $b_{n}$ ($n \geq 1$) can be computed by the relation

$$b_{n} = (-1)^{n} \left| \begin{array}{cccc}
a_{2} & 1 & \cdots & 0 \\
a_{3} & a_{2} & 1 & \cdots & 0 \\
a_{n+1} & a_{n} & \cdots & a_{2} \\
\end{array} \right| .$$

Prawitz [8] discovered an estimate for the coefficient $b_{n}$. It is a generalization of the Gronwall area theorem (see [2, p. 29]) and may be formulated as follows:

Theorem 3. Let $f \in S$ and $[z/f(z)]^{\alpha/2} = \sum_{n=0}^{\infty} \beta_{n} z^{n}$. Then

$$\sum_{n=0}^{\infty} \frac{(2n-\alpha)}{\alpha} |\beta_{n}|^{2} \leq 1$$

for all real $\alpha$.

In particular, for $\alpha = 2$ we have the following

Corollary 1. Let $f \in S$ and $z/f(z) = \sum_{n=0}^{\infty} b_{n} z^{n}$. Then

$$\sum_{n=1}^{\infty} (n-1) |b_{n}|^{2} \leq 1. \quad (2)$$

This corollary is essentially equivalent to the Gronwall area theorem.

2. Motivation

2.1. Friedman’s Theorem. Friedman [3] proved the following theorem which is a part of Salem’s theorem on univalent functions [10]:

Theorem 4. Let $f \in S$. If all the coefficients $a_{n}$ are rational integers then $f(z)$ is one of the following nine functions:

$$z, \quad \frac{z}{1 \pm z}, \quad \frac{z}{1 \pm z^{2}}, \quad \frac{z}{(1 \pm z)^{2}}, \quad \frac{z}{1 \pm z \pm z^{2}} .$$

Proof. Set $F(z) = z/f(z) = 1 + \sum_{n=1}^{\infty} b_{n} z^{n}$, then the coefficients $b_{n}$ are rational integers. Since $b_{1} = -a_{2}$ and $|a_{2}| \leq 2$, it follows that $|b_{1}| \leq 2$. Applying the inequality (2), we have $|b_{2}| \leq 1$ and $b_{n} = 0$ for $n \geq 3$. Therefore, the possible values for $b_{n}$ are:

$$b_{1} = 0, \pm 1, \pm 2; \quad b_{2} = 0, \pm 1; \quad b_{n} = 0 \text{ for } n \geq 3 .$$

From the combination of these values we obtain 15 functions. However, the following six functions must be rejected as having zeros in $\mathbb{D}$:

$$1 \pm 2z, \quad 1 \pm 2z - z^{2}, \quad 1 \pm z - z^{2} .$$

The remaining nine functions prove the theorem. □
2.2. Extensions of Friedman’s Theorem. The method of the proof of Friedman’s theorem in the previous section was given by Linis [7]. He also proved the following

**Theorem 5.** Let \( f \in S \). If all the coefficients \( a_n \) are Gaussian integers then \( f \) has 15 forms. Here, a Gaussian integer is a complex number whose real and imaginary part are both rational integers.

Royster [9] extended the method of the proof given by Linis to quadratic fields with negative discriminant as follows:

**Theorem 6.** Let \( f \in S \). If all the coefficients \( a_n \) are algebraic integers in the quadratic field \( \mathbb{Q}(\sqrt{d}) \) for some square-free rational negative integer \( d \), then \( f \) has 36 forms.

As mentioned above, they have obtained new results by replacing the condition “rational integers” with other conditions.

3. **Main Result**

3.1. Subclass of \( S \) Having Half-integer Coefficients. Now, we shall consider what will happen if all the coefficients \( a_n \) are half-integers. Here, \( a_n \) is said to be a half-integer if \( 2a_n \) is a rational integer.

In a similar way used in the proof of Friedman’s theorem in the second chapter, we set \( F(z) = z/f(z) = 1 + \sum_{n=1}^{\infty} b_n z^n \), then the coefficients \( b_n \) are rational numbers. In the case when the \( a_n \) are rational integers, we could obtain all the possible values for the \( b_n \). But, in this case we cannot obtain them immediately. However, using the inequalities (1) and (2), we can examine the possibilities of coefficients one by one, and obtain the following

**Theorem 7.** Let \( f \in S \). If all the coefficients \( a_n \) are half-integers then \( f(z) \) is one of the following 13 functions:

\[
\begin{align*}
\frac{z}{1\pm z}, & \quad \frac{z}{1\pm z^2}, & \quad \frac{z}{1\pm z^4}, & \quad \frac{z}{(1\pm z)^2}, & \quad \frac{z}{1\pm z + z^2}, & \quad \frac{z(2\pm z)}{2(1\pm z)^2}.
\end{align*}
\]

The detailed proof of this theorem is given in [5].

**References**


**Division of Mathematics**

**Graduate School of Information Sciences**

**Tohoku University**

6-3-09 ARAMAKI-AZA-AOBA, AOBA-KU, SENDAI 980-8579, JAPAN

**E-mail address:** hiramunanaoki@gmail.com