Notes on Nunokawa lemmas

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Abstract

For analytic functions $p(z)$ in the open unit disk $U$ with $p(0) = 1$, Nunokawa has given two results which are called Nunokawa lemmas (Proc. Japan Acad. Ser. A Math. Sci. 68(1992), 152–153; Proc. Japan Acad. Ser. A Math. Sci. 69(1993), 234–237). But, since Nunokawa lemmas, nobody gives any examples for the lemmas. The object of the present paper is to consider some simple and interesting examples for Nunokawa lemmas.

1 Introduction

Let $\mathcal{N}$ denote the class of functions $p(z)$ of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. For functions $p(z) \in \mathcal{N}$, Nunokawa [3, 4] has shown the following lemmas.

**Lemma 1** Let $p(z) \in \mathcal{N}$ and suppose that there exists a point $z_0 \in U$ such that $\text{Re}(p(z)) > 0$ $(|z| < |z_0|)$, $\text{Re}(p(z_0)) = 0$ and $p(z_0) \neq 0$. Then, we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik$$

where $k$ is real and $|k| \geq 1$.

**Lemma 2** Let $p(z) \in \mathcal{N}$ with $p(z) \neq 0$ in $U$ and suppose that there exists a point $z_0 \in U$ such that

$$|\arg(p(z))| < \frac{\pi \alpha}{2} \quad \text{for } |z| < |z_0|$$

and

$$|\arg(p(z_0))| = \frac{\pi \alpha}{2}$$

where $\alpha > 0$. Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = i k \alpha$$

where

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\[ k \geq \frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{when} \quad \arg(p(z_0)) = \frac{\pi \alpha}{2} \]

and

\[ k \leq -\frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{when} \quad \arg(p(z_0)) = -\frac{\pi \alpha}{2} \]

where

\[ p(z_0)^{\pm} = \pm ia \quad \text{with} \quad a > 0. \]

The above lemmas have been called Nunokawa lemmas and applied to obtain a number of interesting results by many mathematicians (see, for example, [1], [5]). But, nobody enumerated concrete functions satisfying these lemmas. In this article, we obtain the simple and interesting examples of Lemma 1 and Lemma 2, respectively.

2 Examples of Lemma 1

At first, we consider the example for Lemma 1.

Example 1 Let us consider the function \( p(z) \) defined by

\[ p(z) = 1 + \frac{z}{1 + iz}. \]

Then, it follows that \( p(z) \in N, \text{Re}(p(z)) > 0 (|z| < |z_0|), \text{Re}(p(z_0)) = 0 \) and \( p(z_0) \neq 0 \) for a point \( z_0 = \frac{-2(1 - 2i)}{5 + \sqrt{5}} \in U \left( |z_0| = \frac{\sqrt{5} - 1}{2} < 1 \right) \). Furthermore, we know that

\[ \frac{z_0p'(z_0)}{p(z_0)} = \frac{5 + \sqrt{5}}{2}i \equiv ik \]

and

\[ k = \frac{5 + \sqrt{5}}{2} = 3.618033 \cdots \geq 1. \]

Thus, \( p(z) \) is the function satisfying Lemma 1. Indeed, \( p(z) \) maps the circular domain \( \{ z : |z| < |z_0| \} \) onto the following.
We next discuss an example of Lemma 1 for the case that $p(z)$ maps the circular domain \( \{ z : |z| \leq |z_0| \} \) onto the domain which touches the imaginary axis with two points.

**Example 2** Let the function $p(z)$ be given by

\[
p(z) = 1 + m \left( z + \frac{1}{2} z^2 \right) \quad \left( \frac{4}{3} < m < \frac{8}{3} \right).
\]

For $z = re^{i\theta}$ (0 < $r$ < 1, $\theta \in \mathbb{R}$), we have that

\[
\text{Re}(p(re^{i\theta})) = 1 + mr \cos \theta + \frac{1}{2}mr^2 \cos 2\theta
\]

\[
= mr^2 \cos^2 \theta + mr \cos \theta + 1 - \frac{1}{2}mr^2.
\]

Setting $F(t) \equiv mr^2t^2 + mrt + 1 - \frac{1}{2}mr^2$ \((-1 \leq t = \cos \theta \leq 1)\) and $m$ is positive, we know that

\[
F'(t_0) = mr(2rt_0 + 1) = 0 \quad \text{for} \quad t_0 = -\frac{1}{2r} < 0.
\]

(i) For $0 < r \leq \frac{1}{2}$ (i.e. $t_0 \leq -1$), since $F'(t) \geq 0$ in $[-1, 1]$,

\[
F(t) \geq F(-1) = \frac{1}{2}mr^2 - mr + 1 = 0
\]

for $r = \frac{m - \sqrt{m(m-2)}}{m} \leq \frac{1}{2}$. It follows from $m(m-2) \geq 0$ and $r \leq \frac{1}{2}$ that $m \geq \frac{8}{3}$. Then, we obtain that $p(z_0) = 0$ for $z_0 = -\frac{m - \sqrt{m(m-2)}}{m}$. This is unsuitable for the example of the lemma.

(ii) For $\frac{1}{2} < r < 1$ (i.e. $-1 < t_0 < 0$), we derive that

\[
F(t) \geq F(t_0) = -\frac{1}{2}mr^2 + 1 - \frac{1}{4}m
\]

and $F(t_0) = 0$ for $r = \sqrt{\frac{4-m}{2m}} \left( t_0 = -\sqrt{\frac{m}{2(4-m)}} \right)$. Noting that $\frac{1}{2} < r = \sqrt{\frac{4-m}{2m}} < 1$, we see that $\frac{4}{3} < m < \frac{8}{3}$. Therefore, it follows that $\text{Re}(p(z_0)) = 0$ ($p(z_0) \neq 0$) and

\[
\text{Re}(p(z)) > 0 \quad \left( |z| < |z_0| = \sqrt{\frac{4-m}{2m}} \right)
\]

for $z_0 = -\frac{1}{2} \pm \frac{1}{2} \sqrt{\frac{8-3m}{m}} i$. Furthermore, simple computations give us that

\[
p(z_0) = \pm \sqrt{\frac{m(8-3m)}{4}} i.
\]
and

\[ z_0p'(z_0) = mz_0(1 + z_0) = -\frac{4-m}{2}, \]

that is, that

\[ \frac{z_0p'(z_0)}{p(z_0)} = \pm \frac{2(4-m)}{\sqrt{m(8-3m)}}i \equiv ik^\pm \quad \left( |k^\pm| = \frac{2(4-m)}{\sqrt{m(8-3m)}} \geq 1 \right). \]

This means that \( p(z) \) is an example of Lemma 1. Indeed, taking \( m = 2 \), we have \( p(z) = 1 + 2z + z^2 \) which satisfies

\[ p(z_0) = \pm \frac{1}{2}i \neq 0 \quad (\text{Re}(p(z_0)) = 0), \]

\[ \text{Re}(p(z)) > 0 \quad \left( |z| < |z_0| = \frac{1}{\sqrt{2}} \right) \]

and

\[ \frac{z_0p'(z_0)}{p(z_0)} = \pm 2i \quad (|k| = 2 \geq 1) \]

for \( z_0 = -\frac{1}{2} \pm \frac{1}{2}i. \)

3 Examples of Lemma 2

In this section, we consider a function \( p(z) \) satisfying Lemma 2 for every \( \alpha \) \((0 < \alpha < 1)\).

Example 3 A function

\[ p(z) = \frac{1 + z}{1 - z} \]
is an example of Lemma 2 for every $\alpha (0 < \alpha < 1)$. Since $p(z)$ satisfies

$$\left| p(z) - \frac{1 + r^2}{1 - r^2} \right| = \frac{2r}{1 - r^2} \quad (|z| = r < 1)$$

which shows that $p(z)$ maps the circle $\{z : |z| = r\}$ onto the circle of center $\frac{1 + r^2}{1 - r^2}$ and radius $\frac{2r}{1 - r^2}$, we know that

$$\text{Re}(p(z)) > 0 \quad (z \in U)$$

as $r \to 1^-$ and therefore, $p(z) \neq 0$ in $U$. Let $\theta$ be the angle between the real axis and the tangent line of the above circle passing through the origin, and let $p(z_0)$ be the point of contact. Then, we establish

$$\theta = \pm \sin^{-1} \left( \frac{2r}{1 + r^2} \right) \quad (|\theta| = \sin^{-1} \left( \frac{2r}{1 + r^2} \right) \equiv \frac{\pi \alpha}{2})$$

and

$$|p(z_0)| = \sqrt{\left( \frac{1 + r^2}{1 - r^2} \right)^2 - \left( \frac{2r}{1 - r^2} \right)^2} = 1$$

for all $r \ (0 < r < 1)$. Namely, $p(z_0)$ can be written by

$$p(z_0) = e^{i\theta} \quad (|\theta| < \frac{\pi}{2}).$$

Thus, every point $p(z_0)$ is on the right-side of the unit circle.

Since

$$p(z_0) = \frac{1 + z_0}{1 - z_0} = e^{i\theta} \quad (\theta = \frac{\pi \alpha}{2} \text{ or } \theta = -\frac{\pi \alpha}{2})$$

for some $\alpha \ (0 < \alpha < 1)$, we obtain that

$$z_0 = \frac{-1 + e^{i\theta}}{1 + e^{i\theta}} = \frac{1 - \cos \theta}{\sin \theta}i \quad \text{and} \quad |z_0| = \frac{1 - \cos \theta}{|\sin \theta|}.$$
Furthermore, we also derive that
\[ \frac{z_0 p'(z_0)}{p(z_0)} = i \sin \theta \equiv ik \alpha \quad (k = \frac{\pi \sin \theta}{2|\theta|}) \]
and
\[ p(z_0)^{1/\alpha} = e^{i \frac{\theta}{2}} = \pm i \equiv \pm ia \quad (a = 1). \]
Then, it follows that
\[ k = \frac{\pi \sin \theta}{2|\theta|} \geq 1 = \frac{1}{2} \left( a + \frac{1}{a} \right) \quad (0 < \theta < \frac{\pi}{2}) \]
\[ k = \frac{\pi \sin \theta}{2|\theta|} \leq -1 = -\frac{1}{2} \left( a + \frac{1}{a} \right) \quad (-\frac{\pi}{2} < \theta < 0). \]
Therefore, \( p(z) \) satisfies Lemma 2. Putting \( \alpha = \frac{1}{3} \), we see that
\[ \theta = \pm \frac{\pi}{6} = \arg(p(z_0^\pm)), \quad p(z_0^\pm) = \frac{\sqrt{3} \pm i}{2}, \quad z_0^\pm = \pm (2 - \sqrt{3})i \]
and
\[ \frac{z_0^\pm p'(z_0^\pm)}{p(z_0^\pm)} = i \left( \pm \frac{3}{2} \right) \frac{1}{3} \equiv ik^\pm \alpha \quad \text{(double sign corresponds)} \]
\[ \left( k^+ = \frac{3}{2} \left( \arg(p(z_0^+)) = \frac{\pi}{6} \right), \quad k^- = -\frac{3}{2} \left( \arg(p(z_0^-)) = -\frac{\pi}{6} \right) \right). \]
Finally, we note that
\[ p(z_0^\pm)^{1/\alpha} = \pm i \equiv \pm ia \quad (a = 1), \]
\[ k^+ = \frac{3}{2} \geq 1 = \frac{1}{2} \left( a + \frac{1}{a} \right) \]
and
\[ k^- = -\frac{3}{2} \leq -1 = -\frac{1}{2} \left( a + \frac{1}{a} \right). \]

4 Appendix

For some real parameters \( A \) and \( B (-1 \leq B < A \leq 1) \), we introduce the following function

\[ p(z) = \frac{1 + Az}{1 + Bz} \]

which is analytic and univalent in \( U \). This function \( p(z) \) has been studied by Janowski [2] as the generalization function of (3.1) and therefore, it is said to be the Janowski function. The Janowski function \( p(z) \) given by (4.1) satisfies the following equation

\[ \left| p(z) - \frac{1 - AB^2r^2}{1 - B^2r^2} \right| = \frac{(A - B)r}{1 - B^2r^2} \quad (|z| = r < 1) \]
which implies that $p(z)$ maps the circle $\{z : |z| = r\}$ onto the circle of center $\frac{1 - ABr^2}{1 - B^2r^2}$ and radius $\frac{(A - B)r}{1 - B^2r^2}$ and

$$\text{Re}(p(z)) > \frac{1 - A}{1 - B} \geq 0 \quad (z \in U).$$

Thus, we discuss the same things of Example 3 in this section. We first consider the case that $A \neq 0$ and $B \neq 0$.

Let $\theta$ be the angle between the real axis and the tangent line of the circle passing through the origin, and let $p(z_0)$ be the point of contact. Then, we see that

$$\theta = \pm \sin^{-1} \left( \frac{(A - B)r}{1 - ABr^2} \right) \quad \left( 0 < |\theta| \equiv \frac{\pi \alpha}{2} < \frac{\pi}{2} \right)$$

which leads us that

$$r = \frac{-(A - B) \pm \sqrt{(A - B)^2 + 4AB \sin^2 \theta}}{2AB |\sin \theta|}$$

where $r$ is positive whether or not $AB$ is positive. Furthermore, it follows that

$$p(z_0) = \frac{1 + Az_0}{1 + Bz_0} = \sqrt{\frac{1 - A^2r^2}{1 - B^2r^2}} e^{i\theta} \equiv Ce^{i\theta} \quad \left( z_0 = \frac{-1 + Ce^{i\theta}}{A - BCe^{i\theta}} \right).$$

We next need the description of $C$ without $r$, so that

$$C^2 = \frac{A^2}{B^2} \cdot \frac{-2B \sin^2 \theta - (A - B) + \sqrt{(A - B)^2 + 4AB \sin^2 \theta}}{2A \sin^2 \theta - (A - B) + \sqrt{(A - B)^2 + 4AB \sin^2 \theta}}$$

$$= \frac{A^2}{B^2} \cdot \frac{-2B \sin^2 \theta - (A - B) + \sqrt{(A - B)^2 + 4AB \sin^2 \theta}}{2A \sin^2 \theta - (A - B) + \sqrt{(A - B)^2 + 4AB \sin^2 \theta}}$$

$$= \frac{2A \sin^2 \theta - (A - B) - \sqrt{(A - B)^2 + 4AB \sin^2 \theta}}{2A \sin^2 \theta - (A - B) - \sqrt{(A - B)^2 + 4AB \sin^2 \theta}}.$$
\[
\frac{(A + B) - \sqrt{(A - B)^2 + 4AB \sin^2 \theta}}{2B \cos \theta}
\]

which equivalent to
\[
C = \frac{|(A + B) - \sqrt{(A - B)^2 + 4AB \sin^2 \theta}|}{2|B| \cos \theta}
\]

If \(A + B \leq 0\), then
\[
|(A + B) - \sqrt{(A - B)^2 + 4AB \sin^2 \theta}| = -(A + B) + \sqrt{(A - B)^2 + 4AB \sin^2 \theta}.
\]

Conversely, because if \(A + B > 0\), then
\[
(A + B)^2 - \{(A - B)^2 + 4AB \sin^2 \theta\} = 4AB \cos^2 \theta,
\]
we can deduce that
\[
(A + B) - \sqrt{(A - B)^2 + 4AB \sin^2 \theta} > 0 \quad (AB > 0)
\]
and
\[
-(A + B) + \sqrt{(A - B)^2 + 4AB \sin^2 \theta} > 0 \quad (AB < 0).
\]

Although we have to consider three cases (i) \(0 < B < A\), (ii) \(B < 0 < A\), (iii) \(B < A < 0\), by virtue of the above facts, we obtain that
\[
C = \frac{(A + B) - \sqrt{(A - B)^2 + 4AB \sin^2 \theta}}{2B \cos \theta}
\]
in any case. We also derive that
\[
\frac{z_0 p'(z_0)}{p(z_0)} = \frac{(A - B)z_0}{(1 + Az_0)(1 + Bz_0)} = \frac{(-e^{-i\theta} + C)(A - BCe^{i\theta})}{(A - B)C}
\]
and put \(D \equiv (-e^{-i\theta} + C)(A - BCe^{i\theta})\). Then, we have that
\[
\text{Re}(D) = -A \cos \theta + (A + B)C - BC^2 \cos \theta
\]
and
\[
\text{Im}(D) = (A - BC^2) \sin \theta = \frac{4AB \cos^2 \theta - (A + B)^2 + (A + B) \sqrt{(A - B)^2 + 4AB \sin^2 \theta}}{2B \cos^2 \theta} \sin \theta.
\]
Since \((A - B)C = \frac{(A - B) \left\{ (A + B) - \sqrt{(A - B)^2 + 4AB \sin^2 \theta} \right\}}{2B \cos \theta}\),
\[
\frac{z_0 p'(z_0)}{p(z_0)} = i \left( \frac{4AB \cos^2 \theta - (A + B)^2 + (A + B) \sqrt{(A - B)^2 + 4AB \sin^2 \theta}}{(A - B) \left\{ (A + B) - \sqrt{(A - B)^2 + 4AB \sin^2 \theta} \right\} \tan \theta} \right) = ik\alpha
\]
Finally, we know that
\[ p(z_0)^{\frac{1}{a}} = \pm iC^\frac{1}{\alpha} \equiv \pm ia \]
and consequently
\[ a = C^\frac{1}{\alpha} = \left( \frac{(A + B) - \sqrt{(A - B)^2 + 4AB\sin^2 \theta}}{2B\cos \theta} \right)^{\frac{n}{2|\theta|}} > 0. \]

Now, it is clear that \( p(z) \) satisfies the conditions of Lemma 2. Thus, we expect that
\[ k \geq \frac{1}{2} \left( a + \frac{1}{a} \right) \quad (0 < \theta < \frac{\pi}{2}) \]
and
\[ k \leq -\frac{1}{2} \left( a + \frac{1}{a} \right) \quad (-\frac{\pi}{2} < \theta < 0). \]

But it is hard that we check it by the manual calculation for the general case.

In the same manner, we derive that
\[ \frac{z_0p'(z_0)}{p(z_0)} = i\tan \theta \equiv ik\alpha \quad \left( k = \frac{\pi}{2|\theta|} \tan \theta \right) \]
and
\[ p(z_0)^{\frac{1}{a}} = \pm i\cos \theta \frac{\tan}{\tan} \equiv \pm ia \quad \left( a = \left( \cos \theta \right)^{\frac{\pi}{2|\theta|}} \right) \]
for the case \( B = 0 (0 < A \leq 1) \), and
\[ \frac{z_0p'(z_0)}{p(z_0)} = i\tan \theta \equiv ik\alpha \quad \left( k = \frac{\pi}{2|\theta|} \tan \theta \right) \]
\[ p(z_0)^{\frac{1}{a}} = \pm i\cos \theta \frac{\tan}{\tan} \equiv \pm ia \quad \left( a = \left( \cos \theta \right)^{\frac{\pi}{2|\theta|}} \right) \]
for the case \( A = 0 (-1 \leq B < 0) \). For these case, we can prove that
\[ k \geq \frac{1}{2} \left( a + \frac{1}{a} \right) \quad (0 < \theta < \frac{\pi}{2}) \]
and
\[ k \leq -\frac{1}{2} \left( a + \frac{1}{a} \right) \quad (-\frac{\pi}{2} < \theta < 0). \]
by using Mathematica.
For the particular case $B = -A$ ($0 < A \leq 1$), we readily arrive at the same result of Example 3.

References


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