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Notes on Nunokawa lemmas

Toshio Hayami, Hitoshi Shiraishi and Shigeyoshi Owa

Abstract

For analytic functions \( p(z) \) in the open unit disk \( U \) with \( p(0) = 1 \), Nunokawa has given two results which are called Nunokawa lemmas (Proc. Japan Acad. Ser. A Math. Sci. 68(1992), 152–153; Proc. Japan Acad. Ser. A Math. Sci. 69(1993), 234–237). But, since Nunokawa lemmas, nobody gives any examples for the lemmas. The object of the present paper is to consider some simple and interesting examples for Nunokawa lemmas.

1 Introduction

Let \( \mathcal{N} \) denote the class of functions \( p(z) \) of the form

\[ p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k \]

which are analytic in the open unit disk \( U = \{ z \in \mathbb{C} : |z| < 1 \} \). For functions \( p(z) \in \mathcal{N} \), Nunokawa [3, 4] has shown the following lemmas.

Lemma 1 Let \( p(z) \in \mathcal{N} \) and suppose that there exists a point \( z_0 \in U \) such that \( \text{Re}(p(z)) > 0 \) \((|z| < |z_0|)\), \( \text{Re}(p(z_0)) = 0 \) and \( p(z_0) \neq 0 \). Then, we have

\[ \frac{z_0 p'(z_0)}{p(z_0)} = ik \]

where \( k \) is real and \( |k| \geq 1 \).

Lemma 2 Let \( p(z) \in \mathcal{N} \) with \( p(z) \neq 0 \) in \( U \) and suppose that there exists a point \( z_0 \in U \) such that

\[ |\text{arg}(p(z))| < \frac{\pi \alpha}{2} \quad \text{for} \quad |z| < |z_0| \]

and

\[ |\text{arg}(p(z_0))| = \frac{\pi \alpha}{2} \]

where \( \alpha > 0 \). Then we have

\[ \frac{z_0 p'(z_0)}{p(z_0)} = ik\alpha \]

where

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\[ k \geq \frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{when} \quad \arg(p(z_0)) = \frac{\pi \alpha}{2} \]

and

\[ k \leq -\frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{when} \quad \arg(p(z_0)) = -\frac{\pi \alpha}{2} \]

where

\[ p(z_0)^{\frac{1}{\alpha}} = \pm ia \quad \text{with} \quad a > 0. \]

The above lemmas have been called Nunokawa lemmas and applied to obtain a number of interesting results by many mathematicians (see, for example, [1], [5]). But, nobody enumerated concrete functions satisfying these lemmas. In this article, we obtain the simple and interesting examples of Lemma 1 and Lemma 2, respectively.

2 Examples of Lemma 1

At first, we consider the example for Lemma 1.

Example 1 Let us consider the function \( p(z) \) defined by

\[ p(z) = 1 + \frac{z}{1+iz}. \]

Then, it follows that \( p(z) \in \mathcal{N} \), \( \text{Re}(p(z)) > 0 \) \((|z| < |z_0|)\), \( \text{Re}(p(z_0)) = 0 \) and \( p(z_0) \neq 0 \) for a point \( z_0 = \frac{-2(1-2i)}{5 + \sqrt{5}} \in \mathbb{U} \left( |z_0| = \frac{\sqrt{5} - 1}{2} < 1 \right) \). Furthermore, we know that

\[ \frac{z_0p'(z_0)}{p(z_0)} = \frac{5 + \sqrt{5}}{2} i \equiv ik \]

and

\[ k = \frac{5 + \sqrt{5}}{2} = 3.618033 \ldots \geq 1. \]

Thus, \( p(z) \) is the function satisfying Lemma 1. Indeed, \( p(z) \) maps the circular domain \( \{ z : |z| < |z_0| \} \) onto the following.
We next discuss an example of Lemma 1 for the case that $p(z)$ maps the circular domain \{ $z: |z| \leq |z_0|$\} onto the domain which touches the imaginary axis with two points.

**Example 2** Let the function $p(z)$ be given by

$$p(z) = 1 + m \left( z + \frac{1}{2}z^2 \right) \quad \left( \frac{4}{3} < m < \frac{8}{3} \right).$$

For $z = re^{i\theta}$ (0 < $r$ < 1, $\theta$ $\in$ $\mathbb{R}$), we have that

\[
\text{Re}(p(re^{i\theta})) = 1 + mr \cos \theta + \frac{1}{2}mr^2 \cos 2\theta = mr^2 \cos^2 \theta + mr \cos \theta + 1 - \frac{1}{2}mr^2.
\]

Setting $F(t) \equiv mr^2t^2 + mrt + 1 - \frac{1}{2}mr^2 \ (-1 \leq t = \cos \theta \leq 1)$ and $m$ is positive, we know that

$$F'(t_0) = mr(2rt_0 + 1) = 0 \quad \text{for} \quad t_0 = -\frac{1}{2r} < 0.$$

(i) For $0 < r \leq \frac{1}{2}$ (i.e. $t_0 \leq -1$), since $F'(t) \geq 0$ in $[-1, 1]$,

$$F(t) \geq F(-1) = \frac{1}{2}mr^2 - mr + 1 = 0$$

for $r = \frac{m - \sqrt{m(m - 2)}}{m} \leq \frac{1}{2}$. It follows from $m(m - 2) \geq 0$ and $r \leq \frac{1}{2}$ that $m \geq \frac{8}{3}$. Then, we obtain that $p(z_0) = 0$ for $z_0 = -\frac{m - \sqrt{m(m - 2)}}{m}$. This is unsuitable for the example of the lemma.

(ii) For $\frac{1}{2} < r < 1$ (i.e. $-1 < t_0 < 0$), we derive that

$$F(t) \geq F(t_0) = -\frac{1}{2}mr^2 + 1 - \frac{1}{4}m$$

and $F(t_0) = 0$ for $r = \sqrt{\frac{4 - m}{2m}} \left( t_0 = -\sqrt{\frac{m}{2(4 - m)}} \right)$. Noting that $\frac{1}{2} < r = \sqrt{\frac{4 - m}{2m}} < 1$, we see that $\frac{4}{3} < m < \frac{8}{3}$. Therefore, it follows that $\text{Re}(p(z_0)) = 0$ ($p(z_0) \neq 0$) and

$$\text{Re}(p(z)) > 0 \quad \left( |z| < |z_0| = \sqrt{\frac{4 - m}{2m}} \right)$$

for $z_0 = -\frac{1}{2} \pm \frac{1}{2} \sqrt{\frac{8 - 3m}{m}}i$. Furthermore, simple computations give us that

$$p(z_0) = \pm \sqrt{\frac{m(8 - 3m)}{4}}i.$$
and

\[ z_0 p'(z_0) = m z_0 (1 + z_0) = -\frac{4 - m}{2}, \]

that is, that

\[
\frac{z_0 p'(z_0)}{p(z_0)} = \pm \frac{2(4 - m)}{\sqrt{m(8 - 3m)}} i \equiv i k^\pm \quad \left( |k^\pm| = \frac{2(4 - m)}{\sqrt{m(8 - 3m)}} \geq 1 \right).
\]

This means that \( p(z) \) is an example of Lemma 1. Indeed, taking \( m = 2 \), we have \( p(z) = 1 + 2z + z^2 \) which satisfies

\[
p(z_0) = \pm \frac{1}{2} i \neq 0 \quad \text{(Re}(p(z_0)) = 0),
\]

\[
\text{Re}(p(z)) > 0 \quad \left( |z| < |z_0| = \frac{1}{\sqrt{2}} \right)
\]

and

\[
\frac{z_0 p'(z_0)}{p(z_0)} = \pm 2i \quad \left( |k| = 2 \geq 1 \right)
\]

for \( z_0 = -\frac{1}{2} \pm \frac{1}{2} i \).

3 Examples of Lemma 2

In this section, we consider a function \( p(z) \) satisfying Lemma 2 for every \( \alpha \) (0 < \( \alpha < 1 \)).

Example 3  A function

\[
p(z) = \frac{1 + z}{1 - z}
\]
is an example of Lemma 2 for every $\alpha$ ($0 < \alpha < 1$). Since $p(z)$ satisfies

$$\left| p(z) - \frac{1 + r^2}{1 - r^2} \right| = \frac{2r}{1 - r^2} \quad (|z| = r < 1)$$

which shows that $p(z)$ maps the circle $\{z : |z| = r\}$ onto the circle of center $\frac{1 + r^2}{1 - r^2}$ and radius $\frac{2r}{1 - r^2}$, we know that

$$\text{Re}(p(z)) > 0 \quad (z \in U)$$

as $r \to 1^-$ and therefore, $p(z) \neq 0$ in $U$. Let $\theta$ be the angle between the real axis and the tangent line of the above circle passing through the origin, and let $p(z_0)$ be the point of contact. Then, we establish

$$\theta = \pm \sin^{-1} \left( \frac{2r}{1 + r^2} \right) \quad \left( |\theta| = \sin^{-1} \left( \frac{2r}{1 + r^2} \right) = \frac{\pi \alpha}{2} \right)$$

and

$$|p(z_0)| = \sqrt{\left( \frac{1 + r^2}{1 - r^2} \right)^2 - \left( \frac{2r}{1 - r^2} \right)^2} = 1$$

for all $r$ ($0 < r < 1$). Namely, $p(z_0)$ can be written by

$$p(z_0) = e^{i\theta} \quad \left( |\theta| \ < \frac{\pi}{2} \right).$$

Thus, every point $p(z_0)$ is on the right-side of the unit circle.

Since

$$p(z_0) = \frac{1 + z_0}{1 - z_0} = e^{i\theta} \quad \left( \theta = \frac{\pi \alpha}{2} \text{ or } \theta = -\frac{\pi \alpha}{2} \right)$$

for some $\alpha$ ($0 < \alpha < 1$), we obtain that

$$z_0 = \frac{-1 + e^{i\theta}}{1 + e^{i\theta}} = \frac{1 - \cos \theta}{\sin \theta}i \quad \text{and} \quad |z_0| = \frac{1 - \cos \theta}{|\sin \theta|}.$$
Furthermore, we also derive that
\[
\frac{z_0 p'(z_0)}{p(z_0)} = i \sin \theta \equiv ik \alpha \quad (k = \frac{\pi \sin \theta}{2|\theta|})
\]
and
\[
p(z_0)^{\frac{1}{\alpha}} = e^{i \frac{\pi}{2}} = \pm i \equiv \pm ia \quad (a = 1).
\]
Then, it follows that
\[
k = \frac{\pi \sin \theta}{2|\theta|} \geq 1 = \frac{1}{2} \left( a + \frac{1}{a} \right) \quad (0 < \theta < \frac{\pi}{2})
\]
\[
k = \frac{\pi \sin \theta}{2|\theta|} \leq -1 = -\frac{1}{2} \left( a + \frac{1}{a} \right) \quad (-\frac{\pi}{2} < \theta < 0).
\]
Therefore, \( p(z) \) satisfies Lemma 2. Putting \( \alpha = \frac{1}{3} \), we see that
\[
\theta = \pm \frac{\pi}{6} = \arg(p(z_{0}^{\pm})), \quad p(z_{0}^{\pm}) = \frac{\sqrt{3} \pm i}{2}, \quad z_{0}^{\pm} = \pm (2 - \sqrt{3})i
\]
and
\[
\frac{z_{0}^{\pm} p'(z_{0}^{\pm})}{p(z_{0}^{\pm})} = i \left( \pm \frac{3}{2} \right) = \pm i k_{0}^{\pm} \alpha \quad (\text{double sign corresponds})
\]
\[
\left( k_{0}^{+} = \frac{3}{2} \left( \arg(p(z_{0}^{+})) = \frac{\pi}{6} \right), \quad k_{0}^{-} = \frac{3}{2} \left( \arg(p(z_{0}^{-})) = -\frac{\pi}{6} \right) \right).
\]
Finally, we note that
\[
p(z_{0}^{\pm})^{\frac{1}{\alpha}} = \pm i \equiv \pm ia \quad (a = 1),
\]
\[
k_{0}^{+} = \frac{3}{2} \geq 1 = \frac{1}{2} \left( a + \frac{1}{a} \right)
\]
and
\[
k_{0}^{-} = -\frac{3}{2} \leq -1 = -\frac{1}{2} \left( a + \frac{1}{a} \right).
\]

4 Appendix

For some real parameters \( A \) and \( B (-1 \leq B < A \leq 1) \), we introduce the following function
\[
(4.1) \quad p(z) = \frac{1 + Az}{1 + Bz}
\]
which is analytic and univalent in \( U \). This function \( p(z) \) has been studied by Janowski [2] as the generalization function of (3.1) and therefore, it is said to be the Janowski function. The Janowski function \( p(z) \) given by (4.1) satisfies the following equation
\[
\left| p(z) - \frac{1 - ABz^2}{1 - B^2z^2} \right| = \frac{(A - B)r}{1 - B^2r^2} \quad (|z| = r < 1)
\]
which implies that $p(z)$ maps the circle $\{z : |z| = r\}$ onto the circle of center $\frac{1 - ABr^2}{1 - B^2r^2}$ and radius $\frac{(A - B)r}{1 - B^2r^2}$ and

$$\text{Re}(p(z)) > \frac{1 - A}{1 - B} \geq 0 \quad (z \in U).$$

Thus, we discuss the same things of Example 3 in this section. We first consider the case that $A \neq 0$ and $B \neq 0$.

Let $\theta$ be the angle between the real axis and the tangent line of the circle passing through the origin, and let $p(z_0)$ be the point of contact. Then, we see that

$$\theta = \pm \sin^{-1} \left( \frac{(A - B)r}{1 - ABr^2} \right) \quad (0 < |\theta| \equiv \frac{\pi \alpha}{2} < \frac{\pi}{2})$$

which leads us that

$$r = \frac{-(A - B) + \sqrt{(A - B)^2 + 4AB \sin^2 \theta}}{2AB |\sin \theta|}$$

where $r$ is positive whether or not $AB$ is positive. Furthermore, it follows that

$$p(z_0) = \frac{1 + Az_0}{1 + Bz_0} = \sqrt{\frac{1 - A^2r^2}{1 - B^2r^2}} e^{i\theta} \equiv Ce^{i\theta} \quad (z_0 = \frac{-1 + Ce^{i\theta}}{A - BCe^{i\theta}}).$$

\[ \begin{array}{c}
\text{We next need the description of } C \text{ without } r, \text{ so that} \\
C^2 = \frac{A^2}{B^2} \cdot \frac{-2B \sin^2 \theta - (A - B) + \sqrt{(A - B)^2 + 4AB \sin^2 \theta}}{2A \sin^2 \theta - (A - B) + \sqrt{(A - B)^2 + 4AB \sin^2 \theta}} \\
C^2 = \frac{A^2}{B^2} \cdot \frac{-2B \sin^2 \theta - (A - B) + \sqrt{(A - B)^2 + 4AB \sin^2 \theta}}{2A \sin^2 \theta - (A - B) + \sqrt{(A - B)^2 + 4AB \sin^2 \theta}} \\
\end{array} \]
\[
\frac{(A + B) - \sqrt{(A - B)^2 + 4AB \sin^2 \theta}}{2B \cos \theta}^2
\]

which equivalent to

\[
C = \frac{(A + B) - \sqrt{(A - B)^2 + 4AB \sin^2 \theta}}{2|B| \cos \theta}
\]

If \( A + B \leq 0 \), then

\[
|A + B| - \sqrt{(A - B)^2 + 4AB \sin^2 \theta|} = -(A + B) + \sqrt{(A - B)^2 + 4AB \sin^2 \theta}.
\]

Conversely, because if \( A + B > 0 \), then

\[
(A + B)^2 - \{(A - B)^2 + 4AB \sin^2 \theta\} = 4AB \cos^2 \theta,
\]

we can deduce that

\[
(A + B) - \sqrt{(A - B)^2 + 4AB \sin^2 \theta} > 0 \quad (AB > 0)
\]

and

\[
-(A + B) + \sqrt{(A - B)^2 + 4AB \sin^2 \theta} > 0 \quad (AB < 0).
\]

Although we have to consider three cases (i) \( 0 < B < A \), (ii) \( B < 0 < A \), (iii) \( B < A < 0 \), by virtue of the above facts, we obtain that

\[
C = \frac{(A + B) - \sqrt{(A - B)^2 + 4AB \sin^2 \theta}}{2B \cos \theta}
\]

in any case. We also derive that

\[
\frac{z_0 p'(z_0)}{p(z_0)} = \frac{(A - B)z_0}{(1 + Az_0)(1 + Bz_0)} = \frac{(-e^{-i\theta} + C)(A - BCe^{i\theta})}{(A - B)C}
\]

and put \( D \equiv (-e^{-i\theta} + C)(A - BCe^{i\theta}) \). Then, we have that

\[
\text{Re}(D) = -A \cos \theta + (A + B)C - BC^2 \cos \theta
\]

\[
= -A \cos \theta + \frac{(A + B)^2 - (A + B)\sqrt{(A - B)^2 + 4AB \sin^2 \theta}}{2B \cos \theta}
\]

\[
= \frac{(A + B)^2 + (A - B)^2 + 4AB \sin^2 \theta - 2(A + B)\sqrt{(A - B)^2 + 4AB \sin^2 \theta}}{4B \cos \theta} = 0
\]

and

\[
\text{Im}(D) = (A - BC^2) \sin \theta = \frac{4AB \cos^2 \theta - (A + B)^2 + (A + B)\sqrt{(A - B)^2 + 4AB \sin^2 \theta}}{2B \cos \theta} \sin \theta.
\]

Since \( (A - B)C = \frac{(A - B)\{ (A + B) - \sqrt{(A - B)^2 + 4AB \sin^2 \theta}\}}{2B \cos \theta} \),

\[
\frac{z_0 p'(z_0)}{p(z_0)} = i \left( \frac{4AB \cos^2 \theta - (A + B)^2 + (A + B)\sqrt{(A - B)^2 + 4AB \sin^2 \theta}}{(A - B)\{ (A + B) - \sqrt{(A - B)^2 + 4AB \sin^2 \theta}\} \tan \theta} \right) \equiv i \kappa \alpha
\]
Finally, we know that

\[ p(z_0)^{\frac{1}{\alpha}} = \pm i C^{\frac{1}{\alpha}} \equiv \pm ia \]

and consequently

\[
a = C^{\frac{1}{\alpha}} = \left( \frac{(A + B) - \sqrt{(A - B)^2 + 4AB \sin^2 \theta}}{2B \cos \theta} \right)^{\frac{n}{2|\theta|}} > 0.
\]

Now, it is clear that \( p(z) \) satisfies the conditions of Lemma 2. Thus, we expect that

\[
k \geq \frac{1}{2} \left( a + \frac{1}{a} \right) \quad \left( 0 < \theta < \frac{\pi}{2} \right)
\]

and

\[
k \leq -\frac{1}{2} \left( a + \frac{1}{a} \right) \quad \left( -\frac{\pi}{2} < \theta < 0 \right).
\]

But it is hard that we check it by the manual calculation for the general case.

In the same manner, we derive that

\[
\frac{z_0 p'(z_0)}{p(z_0)} = i \tan \theta \equiv i k \alpha \quad \left( k = \frac{\pi}{2|\theta|} \tan \theta \right)
\]

and

\[
p(z_0)^{\frac{1}{\alpha}} = \pm i (\cos \theta)^{\frac{\pi}{2|\theta|}} \equiv \pm ia
\]

for the case \( B = 0 \ (0 < A \leq 1) \), and

\[
\frac{z_0 p'(z_0)}{p(z_0)} = i \tan \theta \equiv i k \alpha \quad \left( k = \frac{\pi}{2|\theta|} \tan \theta \right)
\]

\[
p(z_0)^{\frac{1}{\alpha}} = \pm i (\cos \theta)^{-\frac{\pi}{2|\theta|}} \equiv \pm ia
\]

for the case \( A = 0 \ (-1 \leq B < 0) \). For these case, we can prove that

\[
k \geq \frac{1}{2} \left( a + \frac{1}{a} \right) \quad \left( 0 < \theta < \frac{\pi}{2} \right)
\]

and

\[
k \leq -\frac{1}{2} \left( a + \frac{1}{a} \right) \quad \left( -\frac{\pi}{2} < \theta < 0 \right).
\]

by using Mathematica.
For the particular case \( B = -A \) \((0 < A \leq 1)\), we readily arrive at the same result of Example 3.

References


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