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Notes on Nunokawa lemmas

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Abstract

For analytic functions $p(z)$ in the open unit disk $\mathbb{U}$ with $p(0)=1$, Nunokawa has given two results which are called Nunokawa lemmas (Proc. Japan Acad. Ser. A Math. Sci. **68**(1992), 152–153; Proc. Japan Acad. Ser. A Math. Sci. **69**(1993), 234–237). But, since Nunokawa lemmas, nobody gives any examples for the lemmas. The object of the present paper is to consider some simple and interesting examples for Nunokawa lemmas.

1 Introduction

Let $\mathcal{N}$ denote the class of functions $p(z)$ of the form

$$p(z)=1+\sum_{k=1}^{\infty}c_kz^k$$

which are analytic in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$. For functions $p(z)\in\mathcal{N}$, Nunokawa [3, 4] has shown the following lemmas.

Lemma 1 *Let $p(z)\in\mathcal{N}$ and suppose that there exists a point $z_0\in\mathbb{U}$ such that $\mathrm{Re}(p(z))>0$ $(|z|<|z_0|)$, $\mathrm{Re}(p(z_0))=0$ and $p(z_0)\neq 0$. Then, we have*

$$\frac{z_0p'(z_0)}{p(z_0)}=ik$$

where k is real and $|k|\geqq 1$.

Lemma 2 *Let $p(z)\in\mathcal{N}$ with $p(z)\neq 0$ in $\mathbb{U}$ and suppose that there exists a point $z_0\in\mathbb{U}$ such that*

$$|\arg(p(z))|<\frac{\pi\alpha}{2}\qquad for\ |z|<|z_0|$$

and

$$|\arg(p(z_0))|=\frac{\pi\alpha}{2}$$

where $\alpha>0$. Then we have

$$\frac{z_0p'(z_0)}{p(z_0)}=ik\alpha$$

where

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$$k \geqq \frac{1}{2}\left(a + \frac{1}{a}\right) \qquad \textit{when} \quad \arg(p(z_0)) = \frac{\pi\alpha}{2}$$

and

$$k \leqq -\frac{1}{2}\left(a + \frac{1}{a}\right) \qquad \textit{when} \quad \arg(p(z_0)) = -\frac{\pi\alpha}{2}$$

where

$$p(z_0)^{\frac{1}{\alpha}} = \pm ia \qquad \textit{with} \quad a > 0.$$

The above lemmas have been called Nunokawa lemmas and applied to obtain a number of interesting results by many mathematicians (see, for example, [1], [5]). But, nobody enumerated concrete functions satisfying these lemmas. In this article, we obtain the simple and interesting examples of Lemma 1 and Lemma 2, respectively.

2 Examples of Lemma 1

At first, we consider the example for Lemma 1.

Example 1 *Let us consider the function $p(z)$ defined by*

$$p(z) = 1 + \frac{z}{1+iz}.$$

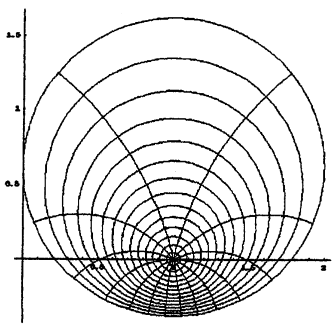
Then, it follows that $p(z) \in \mathcal{N}$, $\mathrm{Re}(p(z)) > 0$ $(|z| < |z_0|)$, $\mathrm{Re}(p(z_0)) = 0$ and $p(z_0) \neq 0$ for a point $z_0 = \dfrac{-2(1-2i)}{5+\sqrt{5}} \in \mathbb{U}$ $\left(|z_0| = \dfrac{\sqrt{5}-1}{2} < 1\right)$. Furthermore, we know that

$$\frac{z_0 p'(z_0)}{p(z_0)} = \frac{5+\sqrt{5}}{2} i \equiv ik$$

and

$$k = \frac{5+\sqrt{5}}{2} = 3.618033\cdots \geqq 1.$$

Thus, $p(z)$ is the function satisfying Lemma 1. Indeed, $p(z)$ maps the circular domain $\{z : |z| < |z_0|\}$ onto the following.



We next discuss an example of Lemma 1 for the case that $p(z)$ maps the circular domain $\{z : |z| \leqq |z_0|\}$ onto the domain which touches the imaginary axis with two points.

Example 2 *Let the function $p(z)$ be given by*

$$p(z) = 1 + m\left(z + \frac{1}{2}z^2\right) \qquad \left(\frac{4}{3} < m < \frac{8}{3}\right).$$

For $z = re^{i\theta}$ $(0 < r < 1,\ \theta \in \mathbb{R})$, we have that

$$\begin{aligned}\mathrm{Re}(p(re^{i\theta})) &= 1 + mr\cos\theta + \frac{1}{2}mr^2\cos 2\theta \\ &= mr^2\cos^2\theta + mr\cos\theta + 1 - \frac{1}{2}mr^2.\end{aligned}$$

Setting $F(t) \equiv mr^2t^2 + mrt + 1 - \frac{1}{2}mr^2$ $(-1 \leqq t = \cos\theta \leqq 1)$ and m is positive, we know that

$$F'(t_0) = mr(2rt_0 + 1) = 0 \quad \text{for} \quad t_0 = -\frac{1}{2r} < 0.$$

(i) For $0 < r \leqq \frac{1}{2}$ (i.e. $t_0 \leqq -1$), since $F'(t) \geqq 0$ in $[-1, 1]$,

$$F(t) \geqq F(-1) = \frac{1}{2}mr^2 - mr + 1 = 0$$

for $r = \dfrac{m - \sqrt{m(m-2)}}{m} \leqq \dfrac{1}{2}$. It follows from $m(m-2) \geqq 0$ and $r \leqq \frac{1}{2}$ that $m \geqq \frac{8}{3}$. Then, we obtain that $p(z_0) = 0$ for $z_0 = -\dfrac{m - \sqrt{m(m-2)}}{m}$. This is unsuitable for the example of the lemma.

(ii) For $\frac{1}{2} < r < 1$ (i.e. $-1 < t_0 < 0$), we derive that

$$F(t) \geqq F(t_0) = -\frac{1}{2}mr^2 + 1 - \frac{1}{4}m$$

and $F(t_0) = 0$ for $r = \sqrt{\dfrac{4-m}{2m}}$ $\left(t_0 = -\sqrt{\dfrac{m}{2(4-m)}}\right)$. Noting that $\frac{1}{2} < r = \sqrt{\dfrac{4-m}{2m}} < 1$, we see that $\frac{4}{3} < m < \frac{8}{3}$. Therefore, it follows that $\mathrm{Re}(p(z_0)) = 0$ $(p(z_0) \neq 0)$ and

$$\mathrm{Re}(p(z)) > 0 \qquad \left(|z| < |z_0| = \sqrt{\frac{4-m}{2m}}\right)$$

for $z_0 = -\dfrac{1}{2} \pm \dfrac{1}{2}\sqrt{\dfrac{8-3m}{m}}i$. Furthermore, simple computations give us that

$$p(z_0) = \pm\frac{\sqrt{m(8-3m)}}{4}i$$

and

$$z_0 p'(z_0) = m z_0(1+z_0) = -\frac{4-m}{2},$$

that is, that

$$\frac{z_0 p'(z_0)}{p(z_0)} = \pm\frac{2(4-m)}{\sqrt{m(8-3m)}} i \equiv ik^{\pm} \qquad \left(|k^{\pm}| = \frac{2(4-m)}{\sqrt{m(8-3m)}} \geqq 1\right).$$

This means that $p(z)$ is an example of Lemma 1. Indeed, taking $m = 2$, we have $p(z) = 1+2z+z^2$ which satisfies

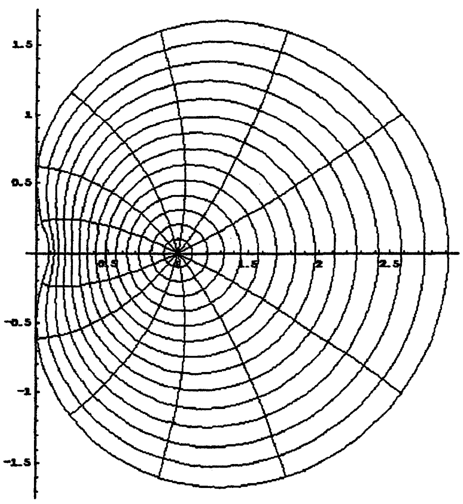
$$p(z_0) = \pm\frac{1}{2}i \neq 0 \qquad (\mathrm{Re}(p(z_0)) = 0),$$

$$\mathrm{Re}(p(z)) > 0 \qquad \left(|z| < |z_0| = \frac{1}{\sqrt{2}}\right)$$

and

$$\frac{z_0 p'(z_0)}{p(z_0)} = \pm 2i \qquad (|k| = 2 \geqq 1)$$

for $z_0 = -\frac{1}{2} \pm \frac{1}{2}i$.



3 Examples of Lemma 2

In this section, we consider a function $p(z)$ satisfying Lemma 2 for every α $(0 < \alpha < 1)$.

Example 3 *A function*

(3.1)
$$p(z) = \frac{1+z}{1-z}$$

is an example of Lemma 2 for every α $(0 < \alpha < 1)$*. Since* $p(z)$ *satisfies*

$$\left| p(z) - \frac{1+r^2}{1-r^2} \right| = \frac{2r}{1-r^2} \qquad (|z| = r < 1)$$

which shows that $p(z)$ *maps the circle* $\{z : |z| = r\}$ *onto the circle of center* $\dfrac{1+r^2}{1-r^2}$ *and radius* $\dfrac{2r}{1-r^2}$*, we know that*

$$\operatorname{Re}(p(z)) > 0 \qquad (z \in \mathbb{U})$$

as $r \longrightarrow 1^-$ *and therefore,* $p(z) \neq 0$ *in* $\mathbb{U}$*. Let* θ *be the angle between the real axis and the tangent line of the above circle passing through the origin, and let* $p(z_0)$ *be the point of contact. Then, we establish*

$$\theta = \pm \sin^{-1}\left(\frac{2r}{1+r^2}\right) \qquad \left(|\theta| = \sin^{-1}\left(\frac{2r}{1+r^2}\right) \equiv \frac{\pi\alpha}{2}\right)$$

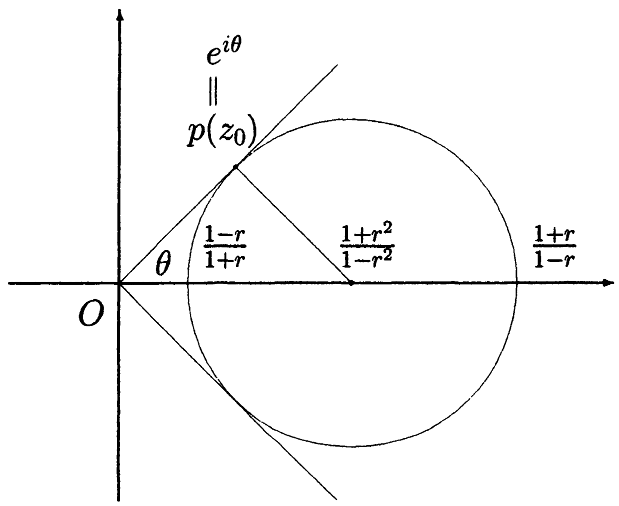
and

$$|p(z_0)| = \sqrt{\left(\frac{1+r^2}{1-r^2}\right)^2 - \left(\frac{2r}{1-r^2}\right)^2} = 1$$

for all r $(0 < r < 1)$*. Namely,* $p(z_0)$ *can be written by*

$$p(z_0) = e^{i\theta} \qquad \left(|\theta| < \frac{\pi}{2}\right).$$

Thus, every point $p(z_0)$ *is on the right-side of the unit circle.*



Since

$$p(z_0) = \frac{1+z_0}{1-z_0} = e^{i\theta} \qquad \left(\theta = \frac{\pi\alpha}{2} \text{ or } \theta = -\frac{\pi\alpha}{2}\right)$$

for some α $(0 < \alpha < 1)$*, we obtain that*

$$z_0 = \frac{-1+e^{i\theta}}{1+e^{i\theta}} = \frac{1-\cos\theta}{\sin\theta} i \quad \text{and} \quad |z_0| = \frac{1-\cos\theta}{|\sin\theta|}.$$

Furthermore, we also derive that

$$\frac{z_0 p'(z_0)}{p(z_0)} = i \sin \theta \equiv ik\alpha \qquad \left(k = \frac{\pi \sin \theta}{2|\theta|}\right)$$

and

$$p(z_0)^{\frac{1}{\alpha}} = e^{i\frac{\pi}{2}\cdot\frac{\theta}{|\theta|}} = \pm i \equiv \pm ia \qquad (a = 1).$$

Then, it follows that

$$k = \frac{\pi \sin \theta}{2|\theta|} \geqq 1 = \frac{1}{2}\left(a + \frac{1}{a}\right) \qquad \left(0 < \theta < \frac{\pi}{2}\right)$$

$$k = \frac{\pi \sin \theta}{2|\theta|} \leqq -1 = -\frac{1}{2}\left(a + \frac{1}{a}\right) \qquad \left(-\frac{\pi}{2} < \theta < 0\right).$$

Therefore, $p(z)$ satisfies Lemma 2. Putting $\alpha = \frac{1}{3}$, we see that

$$\theta = \pm\frac{\pi}{6} = \arg(p(z_0^{\pm})), \quad p(z_0^{\pm}) = \frac{\sqrt{3} \pm i}{2}, \quad z_0^{\pm} = \pm(2 - \sqrt{3})i$$

and

$$\frac{z_0^{\pm} p'(z_0^{\pm})}{p(z_0^{\pm})} = i\left(\pm\frac{3}{2}\right)\frac{1}{3} \equiv ik^{\pm}\alpha \qquad \text{(double sign corresponds)}$$

$$\left(k^{+} = \frac{3}{2}\ \left(\arg(p(z_0^{+})) = \frac{\pi}{6}\right), \quad k^{-} = -\frac{3}{2}\ \left(\arg(p(z_0^{-})) = -\frac{\pi}{6}\right)\right).$$

Finally, we note that

$$p(z_0^{\pm})^{\frac{1}{\alpha}} = \pm i \equiv \pm ia \qquad (a = 1),$$

$$k^{+} = \frac{3}{2} \geqq 1 = \frac{1}{2}\left(a + \frac{1}{a}\right)$$

and

$$k^{-} = -\frac{3}{2} \leqq -1 = -\frac{1}{2}\left(a + \frac{1}{a}\right).$$

4 Appendix

For some real parameters A and B $(-1 \leqq B < A \leqq 1)$, we introduce the following function

$$p(z) = \frac{1 + Az}{1 + Bz} \tag{4.1}$$

which is analytic and univalent in $\mathbb{U}$. This function $p(z)$ has been studied by Janowski [2] as the generalization function of (3.1) and therefore, it is said to be the Janowski function. The Janowski function $p(z)$ given by (4.1) satisfies the following equation

$$\left| p(z) - \frac{1 - ABr^2}{1 - B^2 r^2} \right| = \frac{(A - B)r}{1 - B^2 r^2} \qquad (|z| = r < 1)$$

which implies that $p(z)$ maps the circle $\{z : |z| = r\}$ onto the circle of center $\dfrac{1-ABr^2}{1-B^2r^2}$ and radius $\dfrac{(A-B)r}{1-B^2r^2}$ and

$$\mathrm{Re}(p(z)) > \frac{1-A}{1-B} \geqq 0 \qquad (z \in \mathbb{U}).$$

Thus, we discuss the same things of Example 3 in this section. We first consider the case that $A \neq 0$ and $B \neq 0$.

Let θ be the angle between the real axis and the tangent line of the circle passing through the origin, and let $p(z_0)$ be the point of contact. Then, we see that

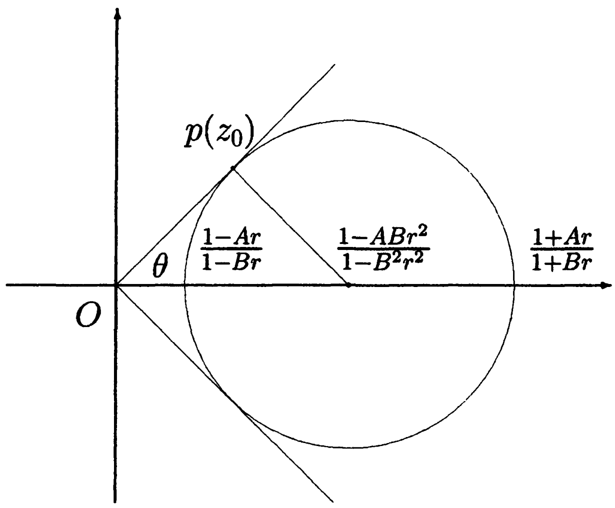
$$\theta = \pm \sin^{-1}\left(\frac{(A-B)r}{1-ABr^2}\right) \qquad \left(0 < |\theta| \equiv \frac{\pi\alpha}{2} < \frac{\pi}{2}\right)$$

which leads us that

$$r = \frac{-(A-B) + \sqrt{(A-B)^2 + 4AB\sin^2\theta}}{2AB\,|\sin\theta|}$$

where r is positive whether or not AB is positive. Furthermore, it follows that

$$p(z_0) = \frac{1+Az_0}{1+Bz_0} = \sqrt{\frac{1-A^2r^2}{1-B^2r^2}}e^{i\theta} \equiv Ce^{i\theta} \qquad \left(z_0 = \frac{-1+Ce^{i\theta}}{A-BCe^{i\theta}}\right).$$



We next need the description of C without r, so that

$$C^2 = \frac{A^2}{B^2} \cdot \frac{-2B\sin^2\theta - (A-B) + \sqrt{(A-B)^2 + 4AB\sin^2\theta}}{2A\sin^2\theta - (A-B) + \sqrt{(A-B)^2 + 4AB\sin^2\theta}}$$

$$= \frac{A^2}{B^2} \cdot \frac{-2B\sin^2\theta - (A-B) + \sqrt{(A-B)^2 + 4AB\sin^2\theta}}{2A\sin^2\theta - (A-B) + \sqrt{(A-B)^2 + 4AB\sin^2\theta}}$$

$$\cdot \frac{2A\sin^2\theta - (A-B) - \sqrt{(A-B)^2 + 4AB\sin^2\theta}}{2A\sin^2\theta - (A-B) - \sqrt{(A-B)^2 + 4AB\sin^2\theta}}$$

$$= \left(\frac{(A+B) - \sqrt{(A-B)^2 + 4AB\sin^2\theta}}{2B\cos\theta} \right)^2$$

which equivalent to

$$C = \frac{\left|(A+B) - \sqrt{(A-B)^2 + 4AB\sin^2\theta}\right|}{2|B|\cos\theta}.$$

If $A + B \leqq 0$, then

$$\left|(A+B) - \sqrt{(A-B)^2 + 4AB\sin^2\theta}\right| = -(A+B) + \sqrt{(A-B)^2 + 4AB\sin^2\theta}.$$

Conversely, because if $A + B > 0$, then

$$(A+B)^2 - \left\{(A-B)^2 + 4AB\sin^2\theta\right\} = 4AB\cos^2\theta,$$

we can deduce that

$$(A+B) - \sqrt{(A-B)^2 + 4AB\sin^2\theta} > 0 \qquad (AB > 0)$$

and

$$-(A+B) + \sqrt{(A-B)^2 + 4AB\sin^2\theta} > 0 \qquad (AB < 0).$$

Although we have to consider three cases (i) $0 < B < A$, (ii) $B < 0 < A$, (iii) $B < A < 0$, by virtue of the above facts, we obtain that

$$C = \frac{(A+B) - \sqrt{(A-B)^2 + 4AB\sin^2\theta}}{2B\cos\theta}$$

in any case. We also derive that

$$\frac{z_0 p'(z_0)}{p(z_0)} = \frac{(A-B)z_0}{(1+Az_0)(1+Bz_0)} = \frac{(-e^{-i\theta} + C)(A - BCe^{i\theta})}{(A-B)C}$$

and put $D \equiv (-e^{-i\theta} + C)(A - BCe^{i\theta})$. Then, we have that

$$\mathrm{Re}(D) = -A\cos\theta + (A+B)C - BC^2\cos\theta$$

$$= -A\cos\theta + \frac{(A+B)^2 - (A+B)\sqrt{(A-B)^2 + 4AB\sin^2\theta}}{2B\cos\theta}$$
$$- \frac{(A+B)^2 + (A-B)^2 + 4AB\sin^2\theta - 2(A+B)\sqrt{(A-B)^2 + 4AB\sin^2\theta}}{4B\cos\theta} = 0$$

and

$$\mathrm{Im}(D) = (A - BC^2)\sin\theta = \frac{4AB\cos^2\theta - (A+B)^2 + (A+B)\sqrt{(A-B)^2 + 4AB\sin^2\theta}}{2B\cos^2\theta}\sin\theta.$$

Since $(A-B)C = \dfrac{(A-B)\left\{(A+B) - \sqrt{(A-B)^2 + 4AB\sin^2\theta}\right\}}{2B\cos\theta}$,

$$\frac{z_0 p'(z_0)}{p(z_0)} = i\left(\frac{4AB\cos^2\theta - (A+B)^2 + (A+B)\sqrt{(A-B)^2 + 4AB\sin^2\theta}}{(A-B)\left\{(A+B) - \sqrt{(A-B)^2 + 4AB\sin^2\theta}\right\}}\tan\theta \right) \equiv ik\alpha$$

$$\left(k = \frac{\pi}{2|\theta|} \cdot \frac{4AB\cos^2\theta - (A+B)^2 + (A+B)\sqrt{(A-B)^2 + 4AB\sin^2\theta}}{(A-B)\left\{(A+B) - \sqrt{(A-B)^2 + 4AB\sin^2\theta}\right\}} \tan\theta \right).$$

Finally, we know that

$$p(z_0)^{\frac{1}{\alpha}} = \pm iC^{\frac{1}{\alpha}} \equiv \pm ia$$

and consequently

$$a = C^{\frac{1}{\alpha}} = \left(\frac{(A+B) - \sqrt{(A-B)^2 + 4AB\sin^2\theta}}{2B\cos\theta} \right)^{\frac{\pi}{2|\theta|}} > 0.$$

Now, it is clear that $p(z)$ satisfies the conditions of Lemma 2. Thus, we expect that

$$k \geqq \frac{1}{2}\left(a + \frac{1}{a}\right) \qquad \left(0 < \theta < \frac{\pi}{2}\right)$$

and

$$k \leqq -\frac{1}{2}\left(a + \frac{1}{a}\right) \qquad \left(-\frac{\pi}{2} < \theta < 0\right).$$

But it is hard that we check it by the manual calculation for the general case.

In the same manner, we derive that

$$\frac{z_0 p'(z_0)}{p(z_0)} = i\tan\theta \equiv ik\alpha \qquad \left(k = \frac{\pi}{2|\theta|}\tan\theta\right)$$

and

$$p(z_0)^{\frac{1}{\alpha}} = \pm i(\cos\theta)^{\frac{\pi}{2|\theta|}} \equiv \pm ia \qquad \left(a = (\cos\theta)^{\frac{\pi}{2|\theta|}}\right)$$

for the case $B = 0$ $(0 < A \leqq 1)$, and

$$\frac{z_0 p'(z_0)}{p(z_0)} = i\tan\theta \equiv ik\alpha \qquad \left(k = \frac{\pi}{2|\theta|}\tan\theta\right)$$

$$p(z_0)^{\frac{1}{\alpha}} = \pm i(\cos\theta)^{-\frac{\pi}{2|\theta|}} \equiv \pm ia \qquad \left(a = (\cos\theta)^{-\frac{\pi}{2|\theta|}}\right)$$

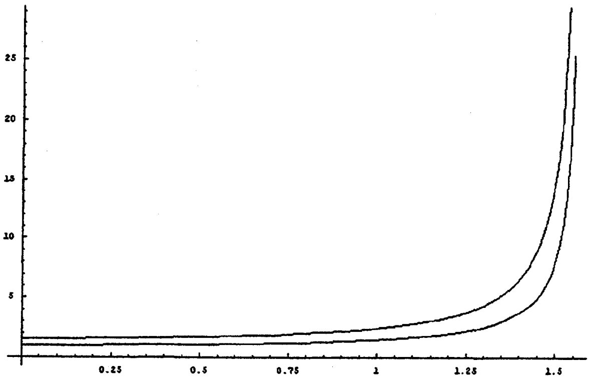
for the case $A = 0$ $(-1 \leqq B < 0)$. For these case, we can prove that

$$k \geqq \frac{1}{2}\left(a + \frac{1}{a}\right) \qquad \left(0 < \theta < \frac{\pi}{2}\right)$$

and

$$k \leqq -\frac{1}{2}\left(a + \frac{1}{a}\right) \qquad \left(-\frac{\pi}{2} < \theta < 0\right).$$

by using Mathematica.



For the particular case $B = -A$ $(0 < A \leqq 1)$, we readily arrive at the same result of Example 3.

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