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Some Distortion Theorems for Starlike Log-Harmonic Functions

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Abstract

In this paper, we consider univalent log-harmonic mappings of the form $f(z)=zh(z)\overline{g(z)}$ defined on the unit disk $\mathbb{D}$ which are starlike. Some distortion theorems are obtained.

1 Introduction

Let $\mathcal{H}(\mathbb{D})$ be the linear space of all analytic functions defined on the open unit disc $\mathbb{D}=\{z\in\mathbb{C}||z|<1\}$. A log-harmonic mapping is a solution of the non-linear elliptic partial differential equation

$$\overline{f_{\bar{z}}}=wf_{z}\left(\frac{\overline{f}}{f}\right), \tag{1.1}$$

where the second dilatation function $w\in\mathcal{H}(\mathbb{D})$ is such that $|w(z)|<1$ for all $z\in\mathbb{D}$. It has been shown that if f is non-vanishing log-harmonic mapping in $\mathbb{D}$, then f can be expressed as

$$f(z)=h(z)\overline{g(z)}, \tag{1.2}$$

where $h(z)$ and $g(z)$ are analytic in $\mathbb{D}$ with the normalization $h(0)\neq 0$, $g(0)=1$. On the other hand if f vanishes at $z=0$, but not identically zero then f admits the following representation

$$f(z)=z|z|^{2\beta}h(z)\overline{g(z)}, \tag{1.3}$$

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where $\text{Re}\beta > -1/2$, $h(z)$ and $g(z)$ are analytic in $\mathbb{D}$ with the normalization $h(0) \neq 0$, $g(0) = 1$ ([4]). We also note that univalent log-harmonic mappings have been studied extensively in [1], [2], [3], [4], [5], [6] and the class of all univalent log-harmonic mappings is denoted by $\mathcal{S}_{LH}$.

The Jacobian of a logharmonic function of the form $f(z) = zh(z)\overline{g(z)}$ is defined by

$$J_f(z) = |f(z)|^2 \left(\left|\frac{1}{z} + \frac{h'(z)}{h(z)}\right|^2 - \left|\frac{g'(z)}{g(z)}\right|^2 \right) = |f_z(z)|^2 - |f_{\bar{z}}(z)|^2.$$

for all z in $\mathbb{D}$.

Let $f(z) = zh(z)\overline{g(z)}$ be a univalent log-harmonic mapping. We say that f is a starlike log-harmonic mapping if

$$\frac{\partial}{\partial \theta}(\arg f(re^{i\theta})) = \text{Re}\left(\frac{zf_z - \bar{z}f_{\bar{z}}}{f}\right) > 0 \tag{1.4}$$

for every $z \in \mathbb{D}$. The class of all starlike log-harmonic mappings is denoted by $\mathcal{S}^*_{LH}$ ([3]).

Let Ω be the family of functions $\phi(z)$ which are analytic in $\mathbb{D}$ and satisfying the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for all $z \in \mathbb{D}$, and let $s_1(z) = z + a_2z^2 + \cdots$, $s_2(z) = z + b_2z^2 + \cdots$ be analytic functions in $\mathbb{D}$. We say that $s_1(z)$ is subordinate to $s_2(z)$ if there exist $\phi(z) \in \Omega$ such that $s_1(z) = s_2(\phi(z))$ and it is denoted by $s_1(z) \prec s_2(z)$.

Let $\varphi(z)$ be analytic function in $\mathbb{D}$ with the normalization $\varphi(0) = 0$, $\varphi'(0) = 1$. If $\varphi(z)$ satisfies the condition

$$\text{Re}\left(z\frac{\varphi'(z)}{\varphi(z)}\right) > 0 \tag{1.5}$$

for every $z \in \mathbb{D}$, then $\varphi(z)$ is called starlike function. The class of all starlike functions is denoted by $\mathcal{S}^*$.

In our proofs we need following theorems.

Theorem 1.1. *[7] Let $\varphi(z)$ be an element of $\mathcal{S}^*$, then*

$$\frac{1-r}{1+r} \leq \left|z\frac{\varphi'(z)}{\varphi(z)}\right| \leq \frac{1+r}{1-r} \quad (|z| = r < 1). \tag{1.6}$$

Theorem 1.2. *[3] $f(z) = zh(z)\overline{g(z)}$ be a log-harmonic function on $\mathbb{D}$, $0 \notin hg(\mathbb{D})$. Then $f \in \mathcal{S}^*_{LH}$ if and only if $\varphi(z) = \left(z\frac{h(z)}{g(z)}\right) \in \mathcal{S}^*$.*

Theorem 1.3. *[3] Let* $f(z) = zh(z)\overline{g(z)} \in \mathcal{S}^*_{LH}$*, with* $w(0) = 0$*. Then we have*

$$re^{-\frac{4r}{1+r}} \leq |f(z)| \leq re^{\frac{4r}{1-r}} \tag{1.7}$$

for all $|z| = r < 1$*. The equalities occur if and only if* $f(z) = \bar{\zeta} f_0(\zeta z)$, $|\zeta| = 1$*, where*

$$f_0(z) = z\left(\frac{1-\bar{z}}{1-z}\right) e^{Re\frac{4z}{1-z}}.$$

2 Main Results

Lemma 2.1. *Let* $f(z) = zh(z)\overline{g(z)}$ *be an element of* $\mathcal{S}^*_{LH}$*, then*

$$\frac{\varphi'(z)/\varphi(z)}{f_z/f} \prec 1 - z \quad and \quad \frac{\overline{f_{\bar{z}}}/\overline{f}}{\varphi'(z)/\varphi(z)} \prec \frac{z}{1-z} \tag{2.1}$$

where $\varphi(z) = z\frac{h(z)}{g(z)} \in \mathcal{S}^*$ *for all* $z \in \mathbb{D}$*.*

Proof. Since $f(z) = zh(z)\overline{g(z)}$ is the solution of the non-linear elliptic partial differential equation

$$\overline{f_{\bar{z}}} = w(z) f_z \left(\frac{\overline{f}}{f}\right),$$

then we have

$$w(z) = \frac{\overline{f_{\bar{z}}}/\overline{f}}{f_z/f} = \frac{z\frac{g'(z)}{g(z)}}{1 + z\frac{h'(z)}{h(z)}}.$$

Therefore we have $w(0) = 0$. This shows that the second dilatation function satisfies the conditions of Schwarz Lemma and

$$1 - w(z) = \frac{\varphi'(z)/\varphi(z)}{f_z/f}, \quad \frac{w(z)}{1 - w(z)} = \frac{\overline{f_{\bar{z}}}/\overline{f}}{\varphi'(z)/\varphi(z)}. \tag{2.2}$$

Using the subordination principle, the equalities (2.2) can be written in the following form

$$\frac{\varphi'(z)/\varphi(z)}{f_z/f} \prec 1 - z \quad \text{and} \quad \frac{\overline{f_{\bar{z}}}/\overline{f}}{\varphi'(z)/\varphi(z)} \prec \frac{z}{1-z}.$$

$\square$

Theorem 2.2. *Let $f(z) = zh(z)\overline{g(z)} \in S^*_{LH}$, then*

$$e^{-\frac{4r}{1+r}}\frac{1-r}{(1+r)^2} \leq |f_z| \leq e^{\frac{4r}{1-r}}\frac{1+r}{(1-r)^2}, \tag{2.3}$$

$$0 \leq |f_{\bar{z}}| \leq e^{\frac{4r}{1-r}}\frac{r(1+r)}{(1-r)^2} \tag{2.4}$$

for all $|z| = r < 1$.

Proof. Since the transformations $w_1(z) = 1 - z$ and $w_2(z) = \frac{z}{1-z}$ map $|z| = r$ onto the discs with the centers $C_1(r) = (1,0)$, $C_2(r) = \left(\frac{r^2}{1-r^2}, 0\right)$ and radius $\rho_1(r) = r$, $\rho_2(r) = \frac{r}{1-r^2}$ respectively. Using Lemma 2.1 and subordination principle then we can write

$$\left|\frac{\varphi'(z)/\varphi(z)}{f_z/f} - 1\right| \leq r \text{ and } \left|\frac{\overline{f_{\bar{z}}}/\overline{f}}{\varphi'(z)/\varphi(z)} - \frac{r^2}{1-r^2}\right| \leq \frac{r}{1-r^2}. \tag{2.5}$$

Using Theorem 1.1, Theorem 1.2, Theorem 1.3 and inequalities (2.5) and after the straightforward calculations we obtain (2.3) and (2.4). □

As a consequence of Theorem 2.2 we have the following corollary:

Corollary 2.3. *Let $f(z) = zh(z)\overline{g(z)}$ be element of S^*_{LH}, then*

$$e^{-\frac{8r}{1+r}}\frac{(1-r)^2}{(1+r)^4} - e^{\frac{8r^2}{1-r^2}}\frac{r}{(1-r^2)} \leq J_f(z) \leq e^{\frac{8r}{1-r}}\frac{(1+r)^3}{(1-r)^4}.$$

for all $|z| = r < 1$.

Theorem 2.4. *Let $f(z) = zh(z)\overline{g(z)}$ be an element of S^*_{LH}, then*

$$|h(z)| \leq e^{\frac{2}{1-r}}\frac{1}{1-r}, \tag{2.6}$$

$$|g(z)| \leq (1-r)e^{\frac{2}{1-r}}, \tag{2.7}$$

for all $|z| = r < 1$.

Proof. Using standart inequalities for complex numbers, we can write

$$\mathrm{Re}\left(\frac{zf_z}{f}\right) \le \left|\frac{zf_z}{f}\right| \tag{2.8}$$

and

$$\mathrm{Re}\left(\frac{\overline{z}f_{\overline{z}}}{f}\right) \le \left|\frac{\overline{z}f_{\overline{z}}}{f}\right| \tag{2.9}$$

for all $z \in \mathbb{D}$. On the other hand,

$$\mathrm{Re}\left(\frac{zf_z}{f}\right) = \mathrm{Re}\left(1 + z\frac{h'(z)}{h(z)}\right) = 1 + \mathrm{Re}\left(z\frac{h'(z)}{h(z)}\right) = 1 + r\frac{\partial}{\partial r}\log|h(z)| \tag{2.10}$$

and

$$\mathrm{Re}\left(\frac{\overline{z}f_{\overline{z}}}{f}\right) = \mathrm{Re}\left(\overline{z}\frac{\overline{g'(z)}}{\overline{g(z)}}\right) = \mathrm{Re}\left(z\frac{g'(z)}{g(z)}\right) = r\frac{\partial}{\partial r}\log|g(z)| \tag{2.11}$$

for all $z \in \mathbb{D}$.

Using Theorem 2.2 and the inequalities (2.8), (2.9), (2.10) and (2.11), we find

$$\frac{\partial}{\partial r}\log|h(z)| \le \frac{1+r}{r(1-r)^2} - \frac{1}{r} \tag{2.12}$$

and

$$\frac{\partial}{\partial r}\log|g(z)| \le \frac{1+r}{r(1-r)^2}. \tag{2.13}$$

Integrating from zero to r we obtain (2.6) and (2.7). □

Theorem 2.5. *If $f(z) = zh(z)\overline{g(z)}$ is in S^*_{LH} and a is in $\mathbb{D}$, then*

$$\varphi_*(z) = \frac{zg(a)h\left(\frac{z+a}{1+\bar{a}z}\right)}{h(a)(1+\bar{a}z)^2 g\left(\frac{z+a}{1+\bar{a}z}\right)} \quad (z \in \mathbb{D})$$

is likewise in S^*.

Proof. For ρ real, $0 < \rho < 1$, let

$$\varphi_\rho(z) = \frac{zg(\rho a)h\left(\rho\left(\frac{z+a}{1+\bar{a}z}\right)\right)}{h(\rho a)(1+\bar{a}z)^2 g\left(\rho\left(\frac{z+a}{1+\bar{a}z}\right)\right)} \quad (z \in \mathbb{D}),$$

then

$$z\frac{\varphi_\rho'(z)}{\varphi_\rho(z)} = \frac{1-\bar{a}z}{1+\bar{a}z} + (1-|a|^2)\frac{z}{(1+\bar{a}z)(z+a)} \cdot \left[\rho\left(\frac{z+a}{1+\bar{a}z}\right)\frac{h'\left(\rho\left(\frac{z+a}{1+\bar{a}z}\right)\right)}{h\left(\rho\left(\frac{z+a}{1+\bar{a}z}\right)\right)} - \rho\left(\frac{z+a}{1+\bar{a}z}\right)\frac{g'\left(\rho\left(\frac{z+a}{1+\bar{a}z}\right)\right)}{g\left(\rho\left(\frac{z+a}{1+\bar{a}z}\right)\right)}\right]. \tag{2.14}$$

Letting $z = e^{i\theta}$, $a = |a|e^{i\phi}$ and $\nu = \rho\left(\frac{e^{i\theta}+a}{1+\bar{a}e^{i\theta}}\right)$ and after the simple calculations we get

$$z\frac{\varphi_\rho'(z)}{\varphi_\rho(z)} = \frac{1-|a|^2}{|1+ae^{-i\theta}|^2}\left(1+\nu\frac{h'(\nu)}{h(\nu)} - \nu\frac{g'(\nu)}{g(\nu)}\right) + i\frac{2|a|\sin(\phi-\theta)}{|1+ae^{-i\theta}|^2}.$$

Therefore for $|z| = 1$, we have

$$\begin{aligned} \mathrm{Re}\left(z\frac{\varphi_\rho'(z)}{\varphi_\rho(z)}\right) &= \frac{1-|a|^2}{|1+ae^{-i\theta}|^2}\mathrm{Re}\left(1+\nu\frac{h'(\nu)}{h(\nu)} - \nu\frac{g'(\nu)}{g(\nu)}\right) \\ &= \frac{1-|a|^2}{|1+ae^{-i\theta}|^2}\mathrm{Re}\left(\frac{\nu f_\nu - \bar{\nu} f_{\bar{\nu}}}{f}\right) > 0 \end{aligned} \tag{2.15}$$

and we conclude that $\varphi_\rho(z)$ is in $\mathcal{S}^*$ for admissible ρ. From the compactness of $\mathcal{S}^*$ and (2.15) we infer that $\varphi_*(z) = \lim_{\rho\to 1}\varphi_\rho(z)$ is in $\mathcal{S}^*$. □

We also note that if we take $a = v$, $u = \frac{z+a}{1+\bar{a}z} = \frac{z+v}{1+\bar{v}z} \Leftrightarrow z = \frac{u-v}{1-\bar{v}u}$ and using Theorem 2.5 and after simple calculations we obtain the following two point distortion inequalities.

Corollary 2.6. *Let $f(z) = zh(z)\overline{g(z)}$ be an element of S^*_{LH}, then*

$$\begin{aligned} e^{\frac{-4|u-v|}{|1-\bar{v}u|+|u-v|}}\frac{|1-\bar{v}u|(|1-\bar{v}u|-|u-v|)}{(|1-\bar{v}u|+|u-v|)^2} &\le |f_z| \\ &\le e^{\frac{4|u-v|}{|1-\bar{v}u|-|u-v|}}\frac{|1-\bar{v}u|(|1-\bar{v}u|+|u-v|)}{(|1-\bar{v}u|-|u-v|)^2}, \end{aligned}$$

and

$$0 \le |f_{\bar{z}}| \le e^{\frac{4|u-v|}{|1-\bar{v}u|-|u-v|}}\frac{|u-v|(|1-\bar{v}u|+|u-v|)}{(|1-\bar{v}u|-|u-v|)^2},$$

and

$$e^{-\frac{8|u-v|}{|1-\bar{v}u|+|u-v|}}\frac{|1-\bar{v}u|^2(|1-\bar{v}u|-|u-v|)^2}{(|1-\bar{v}u|+|u-v|)^4}$$
$$-e^{\frac{8|u-v|^2}{|1-\bar{v}u|^2-|u-v|^2}}\frac{|1-\bar{v}u||u-v|}{|1-\bar{v}u|^2-|u-v|^2}\leq J_f(z)$$
$$\leq e^{\frac{8|u-v|}{|1-\bar{v}u|-|u-v|}}\frac{|1-\bar{v}u|(|1-\bar{v}u|+|u-v|)^3}{(|1-\bar{v}u|-|u-v|)^4}.$$

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