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Jaynes-Cummings model の exactly solvable な拡張について

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Abstract

As an application of noncommutative characteristic equations for matrices with noncommutative entries, we study the exact-solvability of Jaynes-Cummings model (JCM) and its extened models in the view of "operator-valued eigenvalues". Then we find a new method for obtaining invariant subspaces of given models and obtain the evolution operator of an extended JCM Hamiltonian by using the NC spectral decomposition method. Next, we define a class of exactly-solvable matrix Hamiltonian and obtain an infinite number of finite-dimensional invariant subspaces.

1 Introduction

The original Jaynes-Cummings model (JCM) is defined by the Hamiltonian [1]

$$H = \begin{pmatrix} \hbar\omega a^\dagger a + \frac{\epsilon}{2} & \rho a \\ \rho a^\dagger & \hbar\omega a^\dagger a - \frac{\epsilon}{2} \end{pmatrix} \tag{1}$$

where ρ is a real parameter. Note here that the Hamiltonian H is hermitian.

In previous paper [1], they considered an extension of JCM Hamiltonian in the form

$$H = \begin{pmatrix} \hbar\omega a^\dagger a + P(a^\dagger a) + \frac{\epsilon}{2} & \rho a^k \\ \phi\rho (a^\dagger)^k & \hbar\omega a^\dagger a + P(a^\dagger a) - \frac{\epsilon}{2} \end{pmatrix} \tag{2}$$

where $k \in \mathbf{N}$, $\phi = \pm 1$ and $P(a^\dagger a)$ denotes a polynomial of degree $d \geq 2$. If $\phi = 1$, H is hermitian and $\phi = -1$, H is nonhermitian.

In [1], they found H preserved an infinite number of finite-dimensional subspaces

$$\mathcal{V}_n = \mathrm{span}\left\{ \begin{pmatrix} |n> \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ |n+k> \end{pmatrix} \right\}, \quad \forall n \in \mathbf{N} \cup \{0\}. \tag{3}$$

Therefore H is exactly solvable.

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Since

$$H\begin{pmatrix}|n>\\0\end{pmatrix}=\begin{pmatrix}(\hbar\omega a^\dagger a+P(a^\dagger a)+\frac{\epsilon}{2})|n>\\ \phi\rho(a^\dagger)^k|n>\end{pmatrix}=\begin{pmatrix}(\hbar\omega n+P(n)+\frac{\epsilon}{2})|n>\\ \phi\rho\sqrt{(n+1)(n+2)\cdots(n+k)}|n+k>\end{pmatrix},$$

$$H\begin{pmatrix}0\\|n+k>\end{pmatrix}=\begin{pmatrix}\rho a^k|n+k>\\(\hbar\omega a^\dagger a+P(a^\dagger a)-\frac{\epsilon}{2})|n+k>\end{pmatrix}=\begin{pmatrix}\rho\sqrt{(n+k)(n+k-1)\cdots(n+1)}|n>\\(\hbar\omega(n+k)+P(n+k)-\frac{\epsilon}{2})|n+k>\end{pmatrix},$$

the Hamiltonian matrix is

$$H_{n+k}:=\begin{pmatrix}\hbar\omega n+P(n)+\frac{\epsilon}{2} & \rho\sqrt{(n+k)(n+k-1)\cdots(n+1)}\\ \phi\rho\sqrt{(n+1)(n+2)\cdots(n+k)} & \hbar\omega(n+k)+P(n+k)-\frac{\epsilon}{2}\end{pmatrix}. \tag{4}$$

For simplicity, they imposed $P(n)=P(n+k)=0$. Then the eigenvalues of H_{n+k} are

$$\lambda^I_{n+k} = \frac{\hbar\omega(2n+k)+\sqrt{(\hbar\omega k-\epsilon)^2+4\phi\rho^2(n+1)\cdots(n+k)}}{2}, \tag{5}$$

$$\lambda^{II}_{n+k} = \frac{\hbar\omega(2n+k)-\sqrt{(\hbar\omega k-\epsilon)^2+4\phi\rho^2(n+1)\cdots(n+k)}}{2} \tag{6}$$

(equation (34) in [1]).

2 Review of NC Spectral Decomposition

2.1 NC Version of the Characteristic Polynomial

First, we review the noncommutative (NC) version of the characteristic polynomial. For $A=\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix}$, we put $A^n=\begin{pmatrix}a^{(n)}_{11}&a^{(n)}_{12}\\a^{(n)}_{21}&a^{(n)}_{22}\end{pmatrix}$ $(n=0,1,2)$. Then, we denote $\Phi_1(\lambda),\Phi_2(\lambda)$ as two polynomials given by

$$\begin{aligned}\Phi_i(\lambda) &= \begin{vmatrix}a^{(2)}_{i1} & a^{(2)}_{i2} & \boxed{\lambda^2}\\ a^{(1)}_{i1} & a^{(1)}_{i2} & \lambda\\ a^{(0)}_{i1} & a^{(0)}_{i2} & 1\end{vmatrix}\\ &= \lambda^2-(a^{(2)}_{i1},a^{(2)}_{i2})\begin{pmatrix}a^{(1)}_{i1}&a^{(1)}_{i2}\\a^{(0)}_{i1}&a^{(0)}_{i2}\end{pmatrix}^{-1}\begin{pmatrix}\lambda\\1\end{pmatrix}.\end{aligned}$$

More explicitly,

$$\begin{aligned}\Phi_1(\lambda) &= \lambda^2-(a_{11}+a_{12}a_{22}a_{12}^{-1})\lambda+(a_{12}a_{22}a_{12}^{-1}a_{11}-a_{12}a_{21}),\\ \Phi_2(\lambda) &= \lambda^2-(a_{22}+a_{21}a_{11}a_{21}^{-1})\lambda+(a_{21}a_{11}a_{21}^{-1}a_{22}-a_{21}a_{12}).\end{aligned}$$

We remark that the NC Cayley-Hamilton's theorem holds ;

$$A^2-\begin{pmatrix}a_{11}+a_{12}a_{22}a_{12}^{-1} & 0\\ 0 & a_{22}+a_{21}a_{11}a_{21}^{-1}\end{pmatrix}A+\begin{pmatrix}a_{12}a_{22}a_{12}^{-1}a_{11}-a_{12}a_{21} & 0\\ 0 & a_{21}a_{11}a_{21}^{-1}a_{22}-a_{21}a_{12}\end{pmatrix}=O.$$

Proposition 1 (A sufficient condition for no-existence of inverse elements).
For $i = 1, 2$, let $F_i(x)$ be entire functions of x. If relations

$$a_{12}a_{22} = F_1(a_{22})a_{12}, \quad a_{21}a_{11} = F_2(a_{11})a_{21} \tag{7}$$

hold, the noncommutative characteristic polynomials $\Phi_1(\lambda), \Phi_2(\lambda)$ are given by

$$\begin{aligned} \Phi_1(\lambda) &= \lambda^2 - (a_{11} + F_1(a_{22}))\lambda + (F_1(a_{22})a_{11} - a_{12}a_{21}), \\ \Phi_2(\lambda) &= \lambda^2 - (a_{22} + F_2(a_{11}))\lambda + (F_2(a_{11})a_{22} - a_{21}a_{12}). \end{aligned}$$

For example, $a_{12} = a,\ a_{22} = N = a^\dagger a$ and by using the relation $[a, a^\dagger] = 1$, then

$$a_{12}a_{22} = aN = aa^\dagger a = (N+1)a = (a_{22}+1)a_{12}.$$

2.2 NC Spectral Decomposition

Let α_1, α_2 be two solutions of the NC characteristic equation $\Phi_1(\lambda) = 0$ and β_1, β_2 two solutions of $\Phi_2(\lambda) = 0$ respectively. We put

$$x_1 = \begin{pmatrix} \alpha_1 & \\ & \beta_1 \end{pmatrix}, \quad x_2 = \begin{pmatrix} \alpha_2 & \\ & \beta_2 \end{pmatrix} \tag{8}$$

and

$$P_1 = (x_1 - x_2)^{-1}(A - x_2), \tag{9}$$
$$P_2 = (x_2 - x_1)^{-1}(A - x_1), \tag{10}$$

then we have the NC spectral decomposition [3]

$$A^n = x_1^n P_1 + x_2^n P_2 \quad (n = 0, 1, 2, \cdots). \tag{11}$$

3 Operator Method for the Eigenvalue Problem of the Exactly Solvable Hamiltonian (2)

3.1 Operator-valued Eigenvalues

For the eigenvalue problem of the extended JCM Hamiltonian (2)

$$H = \begin{pmatrix} \hbar\omega a^\dagger a + P(a^\dagger a) + \frac{\epsilon}{2} & \rho a^k \\ \phi\rho(a^\dagger)^k & \hbar\omega a^\dagger a + P(a^\dagger a) - \frac{\epsilon}{2} \end{pmatrix},$$

we study another approach. First, we consider a Hamiltonian

$$H = \begin{pmatrix} f_1(N) & c_{12}a^k \\ c_{21}(a^\dagger)^k & f_2(N) \end{pmatrix} \tag{12}$$

where $N = a^\dagger a$, $\quad c_{ij} \in \mathbf{R}$ and $f_i(N)$ $(i = 1, 2)$ are some functions of N. Then we have a proposition as follows;

Proposition 2. *NC characteristic equations of*

$$H = \begin{pmatrix} f_1(N) & c_{12}a^k \\ c_{21}(a^\dagger)^k & f_2(N) \end{pmatrix} \tag{13}$$

are

$$\Phi_1(\lambda) = \lambda^2 - (f_1(N) + f_2(N+k))\lambda + f_1(N)f_2(N+k) - c_{12}c_{21}a^k(a^\dagger)^k = 0, \tag{14}$$

$$\Phi_2(\lambda) = \lambda^2 - (f_1(N-k) + f_2(N))\lambda + f_1(N-k)f_2(N) - c_{12}c_{21}(a^\dagger)^k a^k = 0. \tag{15}$$

The operator solutions α_+, α_- of (14) and β_+, β_- of (15) are

$$\alpha_\pm = \frac{1}{2}\left\{f_1(N) + f_2(N+k) \pm \sqrt{(f_1(N) - f_2(N+k))^2 + 4c_{12}c_{21}a^k(a^\dagger)^k}\right\}, \tag{16}$$

$$\beta_\pm = \frac{1}{2}\left\{f_1(N-k) + f_2(N) \pm \sqrt{(f_1(N-k) - f_2(N))^2 + 4c_{12}c_{21}(a^\dagger)^k a^k}\right\}. \tag{17}$$

Remark 1. *These α_+, α_- and β_+, β_- are considered as "operator-valued eigenvalues".*

3.2 A New Method for Obtaining Invariant Subspaces

Since $a^k(a^\dagger)^k = (N+1)(N+2)\cdots(N+k)$ and $(a^\dagger)^k a^k = N(N-1)\cdots(N-k+1)$, we note that $\alpha_\pm = \alpha_\pm(N)$, $\beta_\pm = \beta_\pm(N)$ and

$$\beta_\pm(N+k) = \alpha_\pm(N). \tag{18}$$

If we put $f_1(N) = \hbar\omega N + P(N) + \frac{\epsilon}{2}$, $f_2(N) = \hbar\omega N + P(N) - \frac{\epsilon}{2}$, $c_{12} = \rho$, $c_{21} = \phi\rho$, we have

$$\alpha_\pm(N) = \frac{1}{2}\left\{\hbar\omega(2N+k) \pm \sqrt{(\hbar\omega k - \epsilon)^2 + 4\phi\rho^2 a^k(a^\dagger)^k}\right\} \tag{19}$$

(here we omitted $P(N)$ and $P(N+k)$ for simplicity). Therefore

$$\begin{aligned} \alpha_\pm(N)|n> &= \frac{1}{2}\left\{\hbar\omega(2N+k) \pm \sqrt{(\hbar\omega k - \epsilon)^2 + 4\phi\rho^2 a^k(a^\dagger)^k}\right\}|n> & (20)\\ &= \frac{1}{2}\left\{\hbar\omega(2n+k) \pm \sqrt{(\hbar\omega k - \epsilon)^2 + 4\phi\rho^2 (n+1)\cdots(n+k)}\right\}|n> . & (21)\end{aligned}$$

These values are nothing but λ^I_{n+k}, λ^{II}_{n+k}. By using $\beta_\pm(N+k) = \alpha_\pm(N)$, we have

$$\beta_\pm(N)|n+k>= \beta_\pm(n+k)|n+k>= \alpha_\pm(n)|n+k>= \lambda^{I,\,II}_{n+k}|n+k> \tag{22}$$

Since $\alpha_\pm$ is derived from $\Phi_1(\lambda) = 0$ (1st row) and $\beta_\pm$ is derived from $\Phi_2(\lambda) = 0$ (2nd row), we can find a basis

$$\left\{\begin{pmatrix}|n>\\0\end{pmatrix}, \begin{pmatrix}0\\|n+k>\end{pmatrix}\right\}$$

of an invariant subspace and eigenvalues λ^I_{n+k}, λ^{II}_{n+k}.

3.3 Evolution Operator and NC Spectral Decomposition Method

Next, we calculate the time evolution operator of H. By the NC spectral decomposition method, the projection operators of H are

$$P_1 = \begin{pmatrix} (\alpha_+ - \alpha_-)^{-1} & \\ & (\beta_+ - \beta_-)^{-1} \end{pmatrix} \begin{pmatrix} f_1(N) - \alpha_- & c_{12}a^k \\ c_{21}(a^\dagger)^k & f_2(N) - \beta_- \end{pmatrix}, \tag{23}$$

$$P_2 = \begin{pmatrix} (\alpha_- - \alpha_+)^{-1} & \\ & (\beta_- - \beta_+)^{-1} \end{pmatrix} \begin{pmatrix} f_1(N) - \alpha_+ & c_{12}a^k \\ c_{21}(a^\dagger)^k & f_2(N) - \beta_+ \end{pmatrix}. \tag{24}$$

Here we put

$$\beta_+(N) - \beta_-(N) = \sqrt{(f_1(N-k) - f_2(N))^2 + 4c_{12}c_{21}(a^\dagger)^k a^k} =: \sqrt{D(N)} \tag{25}$$

and

$$\lambda_\pm(N) := \beta_\pm(N), \tag{26}$$

then

$$x_1 = \begin{pmatrix} \alpha_+(N) & \\ & \beta_+(N) \end{pmatrix} = \begin{pmatrix} \lambda_+(N+k) & \\ & \lambda_+(N) \end{pmatrix}, \quad x_2 = \begin{pmatrix} \lambda_-(N+k) & \\ & \lambda_-(N) \end{pmatrix}, \tag{27}$$

$$P_1 = \begin{pmatrix} (\sqrt{D(N+k)})^{-1} & \\ & (\sqrt{D(N)})^{-1} \end{pmatrix} \begin{pmatrix} f_1(N) - \lambda_-(N+k) & c_{12}a^k \\ c_{21}(a^\dagger)^k & f_2(N) - \lambda_-(N) \end{pmatrix}, \tag{28}$$

$$P_2 = \begin{pmatrix} (-\sqrt{D(N+k)})^{-1} & \\ & (-\sqrt{D(N)})^{-1} \end{pmatrix} \begin{pmatrix} f_1(N) - \lambda_+(N+k) & c_{12}a^k \\ c_{21}(a^\dagger)^k & f_2(N) - \lambda_+(N) \end{pmatrix}. \tag{29}$$

Therefore $H = x_1P_1 + x_2P_2$ implies

$$\begin{aligned} e^{itH} &= e^{itx_1}P_1 + e^{itx_2}P_2 \\ &= \begin{pmatrix} e^{it\lambda_+(N+k)} & \\ & e^{it\lambda_+(N)} \end{pmatrix} \begin{pmatrix} (\sqrt{D(N+k)})^{-1} & \\ & (\sqrt{D(N)})^{-1} \end{pmatrix} \begin{pmatrix} f_1(N)-\lambda_-(N+k) & c_{12}a^k \\ c_{21}(a^\dagger)^k & f_2(N)-\lambda_-(N) \end{pmatrix} \\ &+ \begin{pmatrix} e^{it\lambda_-(N+k)} & \\ & e^{it\lambda_-(N)} \end{pmatrix} \begin{pmatrix} (-\sqrt{D(N+k)})^{-1} & \\ & (-\sqrt{D(N)})^{-1} \end{pmatrix} \begin{pmatrix} f_1(N)-\lambda_+(N+k) & c_{12}a^k \\ c_{21}(a^\dagger)^k & f_2(N)-\lambda_+(N) \end{pmatrix} \\ &= \begin{pmatrix} (1,1) & (1,2) \\ (2,1) & (2,2) \end{pmatrix}, \end{aligned} \tag{30}$$

where

$$(1,1) = e^{\frac{it}{2}\{f_1(N)+f_2(N+k)\}}\left\{\cos\tfrac{t}{2}\sqrt{D(N+k)}+i(f_1(N)-f_2(N+k))\frac{\sin\frac{t}{2}\sqrt{D(N+k)}}{\sqrt{D(N+k)}}\right\} \tag{31}$$

$$(2,2) = e^{\frac{it}{2}\{f_1(N-k)+f_2(N)\}}\left\{\cos\tfrac{t}{2}\sqrt{D(N)}-i(f_1(N-k)-f_2(N))\frac{\sin\frac{t}{2}\sqrt{D(N)}}{\sqrt{D(N)}}\right\} \tag{32}$$

$$(1,2) = 2i\ c_{12}\, e^{\frac{it}{2}\{f_1(N)+f_2(N+k)\}}\frac{\sin\frac{t}{2}\sqrt{D(N+k)}}{\sqrt{D(N+k)}}a^k \tag{33}$$

$$(2,1) = 2i\ c_{21}\, e^{\frac{it}{2}\{f_1(N-k)+f_2(N)\}}\frac{\sin\frac{t}{2}\sqrt{D(N)}}{\sqrt{D(N)}}(a^\dagger)^k. \tag{34}$$

For example, we consider the JCM Hamiltonian (from Wikipedia notation)

$$H_{JC} = \hbar \begin{pmatrix} \nu a^\dagger a + \frac{\omega}{2} & \frac{\Omega}{2} a \\ \frac{\Omega}{2} a^\dagger & \nu a^\dagger a - \frac{\omega}{2} \end{pmatrix}. \tag{35}$$

If we put

$$f_1(N) = \nu N + \frac{\omega}{2}, \quad f_2(N) = \nu N - \frac{\omega}{2}, \quad c_{12} = c_{21} = \frac{\Omega}{2} =: g, \quad k = 1, \quad t \mapsto -t$$

in (30), since

$$f_1(N) + f_2(N+1) = 2\nu\left(N + \frac{1}{2}\right), \quad f_1(N) - f_2(N+1) = \omega - \nu =: \delta, \tag{36}$$

$$D(N+1) = \delta^2 + 4g^2(N+1) = 4\left(\frac{\delta^2}{4} + g^2 N + g^2\right) =: 4(\varphi + g^2), \tag{37}$$

we obtain the famous result

$$e^{-itH_{JC}/\hbar} = \begin{pmatrix} e^{-i\nu t(N+\frac{1}{2})}\left(\cos t\sqrt{\varphi+g^2} - \frac{i\delta}{2}\frac{\sin t\sqrt{\varphi+g^2}}{\sqrt{\varphi+g^2}}\right) & -ige^{-i\nu t(N+\frac{1}{2})}\frac{\sin t\sqrt{\varphi+g^2}}{\sqrt{\varphi+g^2}}a \\ -ige^{-i\nu t(N-\frac{1}{2})}\frac{\sin t\sqrt{\varphi}}{\sqrt{\varphi}}a^\dagger & e^{-i\nu t(N-\frac{1}{2})}\left(\cos t\sqrt{\varphi} + \frac{i\delta}{2}\frac{\sin t\sqrt{\varphi}}{\sqrt{\varphi}}\right) \end{pmatrix}. \tag{38}$$

4 A Generalization of JCM Hamiltonian to $M \times M$ Matrix

We can observe that the extended JCM Hamiltonian

$$H = \begin{pmatrix} f_1(N) & c_{12}a^k \\ c_{21}(a^\dagger)^k & f_2(N) \end{pmatrix}$$

has an infinite number of finite-dimensional invariant subspaces

$$\mathcal{V}_n = \text{span}\left\{ \begin{pmatrix} |n> \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ |n+k> \end{pmatrix} \right\}, \quad \forall n \in \mathbf{N} \cup \{0\}.$$

If we consider a 3×3 matrix Hamiltonian

$$H = \begin{pmatrix} f_1(N) & c_{12}a^{k_1} & c_{13}a^{k_1+k_2} \\ c_{21}(a^\dagger)^{k_1} & f_2(N) & c_{23}a^{k_2} \\ c_{31}(a^\dagger)^{k_1+k_2} & c_{32}(a^\dagger)^{k_2} & f_3(N) \end{pmatrix}, \tag{39}$$

we can find an infinite number of finite-dimensional invariant subspaces

$$\mathcal{V}_{n;k_1,k_2} = \text{span}\left\{ \begin{pmatrix} |n> \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ |n+k_1> \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ |n+k_1+k_2> \end{pmatrix} \right\}, \quad \forall n \in \mathbf{N} \cup \{0\}. \tag{40}$$

We can generalize these matrix Hamiltonians in this way;

Theorem 3. *A $M \times M$ matrix Hamiltonian*

$$H = \begin{pmatrix} f_1(a^\dagger a) & & c_{ij}a^{\sum_{l=i}^{j-1} k_l} \\ & \ddots & \\ c_{ij}(a^\dagger)^{\sum_{l=j}^{i-1} k_l} & & f_M(a^\dagger a) \end{pmatrix} \tag{41}$$

preserves an infinite number of finite-dimensional subspaces

$$\mathcal{V}_{n;k_1,\cdots,k_{M-1}} := span_{1 \le j \le M} \left\{ |n + \sum_{l=1}^{j-1} k_l > \otimes \mathbf{e}_j \right\}, \quad \forall n \in \mathbf{N} \cup \{0\}. \tag{42}$$

Therefore, H is exactly solvable.

The corresponding Hamiltonian matrix restricted to $\mathcal{V}_{n;k_1,\cdots,k_{M-1}}$ is
$H_{n;k_1,\cdots,k_{M-1}} :=$

$$\begin{pmatrix} f_1(n) & & c_{ij}\sqrt{(n+\sum_{l=1}^{j-1} k_l)\cdots(n+\sum_{l=1}^{i-1} k_l+1)} \\ & f_i(n+\sum_{l=1}^{i-1} k_l) & \\ c_{ij}\sqrt{(n+\sum_{l=1}^{j-1} k_l+1)\cdots(n+\sum_{l=1}^{i-1} k_l)} & & f_M(n+\sum_{l=1}^{M-1} k_l) \end{pmatrix} \tag{43}$$

Proof. By using

$$a^k|n+k'>=\sqrt{(n+k')\cdots(n+k'-k+1)}|n+k'-k>, \tag{44}$$

$$(a^\dagger)^k|n+k'>=\sqrt{(n+k'+1)\cdots(n+k'+k)}|n+k'+k>, \tag{45}$$

we have

$$\begin{aligned}
&H|n+\sum_{l=1}^{j-1}k_l>\otimes\mathbf{e}_j\\
&=\begin{pmatrix} c_{ij}a^{\sum_{l=i}^{j-1}k_l}|n+\sum_{l=1}^{j-1}k_l> & (\text{if } i<j)\\ f_i(a^\dagger a)|n+\sum_{l=1}^{j-1}k_l> & (\text{if } i=j)\\ c_{ij}(a^\dagger)^{\sum_{l=j}^{i-1}k_l}|n+\sum_{l=1}^{j-1}k_l> & (\text{if } i>j)\end{pmatrix}\\
&=\begin{pmatrix} c_{ij}\sqrt{(n+\sum_{l=1}^{j-1}k_l)\cdots(n+\sum_{l=1}^{i-1}k_l+1)}|n+\sum_{l=1}^{i-1}k_l>\\ f_i(n+\sum_{l=1}^{i-1}k_l)|n+\sum_{l=1}^{i-1}k_l>\\ c_{ij}\sqrt{(n+\sum_{l=1}^{j-1}k_l+1)\cdots(n+\sum_{l=1}^{i-1}k_l)}|n+\sum_{l=1}^{i-1}k_l>\end{pmatrix}
\end{aligned} \tag{46}$$

□

5 Discussion

We studied the exact-solvability of Jaynes-Cummings model and its extened models in the view of "operator-valued eigenvalues". Analysing relations among "eigenvalues of each row", we found a new method for obtaining invariant subspaces of given models. Then we obtained the evolution operator of an extended JCM Hamiltonian by using the NC spectral decomposition method. Next, we defined a class of exactly-solvable matrix Hamiltonian and obtained an infinite number of finite-dimensional invariant subspaces.

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