### Title
THE STUDY OF MATHEMATICAL HISTORY ON THE EQUATIONS OF NAVIER-STOKES AND BOLTZMANN AS THE MICROSCOPICALLY-DESCRIPTIVE HYDRODYNAMIC EQUATIONS (Mathematical analysis of the Euler equation: 100 years of the Karman vortex street and unsteady vortex motion)

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THE STUDY OF MATHEMATICAL HISTORY ON THE EQUATIONS OF NAVIER-STOKES
AND BOLTZMANN AS THE MICROSCOPICALLY-DESCRITIVE HYDRODYNAMIC
EQUATIONS

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ABSTRACT. The microscopically-description of hydromechanics equations are followed by the description
equations of gas theory by Maxwell, Kirchoff and Boltzmann. Above all, in 1872, Boltzmann
formulated the Boltzmann equations, expressed by the following today’s formulation:
\[ \partial_t f + v \cdot \nabla_x f = Q(f, g), \quad t > 0, \quad x, v \in \mathbb{R}^n(n \geq 3), \quad x = (x, y, z), \quad v = (v, \eta, \zeta), \]
\[ Q(f, g)(t, x, v) = \int_{\mathbb{R}^3} \int_{S^2} B(v - v_*, \sigma)[g(v_*')f(v') - g(v_*)f(v)]d\sigma dv_*, \quad g(v_*) = g(t, x, v_*), \text{ etc.} \]

These equations are able to be reduced for the general form of the hydrodynamic equations, after
the formulations by Maxwell and Kirchoff, and from which the Euler equations and the Navier-Stokes
equations are reduced as a special case.

After Stokes’ linear equations, the equations of gas theories were deduced by Maxwell in 1865,
Kirchoff in 1868 and Boltzmann in 1872. They contributed to formulate the fluid equations and to fix
the Navier-Stokes equations, when Prandtl stated the today’s formulation in using the nomenclature as
the “so-called Navier-Stokes equations” in 1934, in which Prandl included the three terms of nonlinear
and two linear terms with the ratio of two coefficients as $3:1$, which arose Poisson in 1831, Saint-Venant
in 1843, and Stokes in 1845.

Mathematics Subject Classification 2010 : 76-03, 01.

1. Introduction

We have studied the original microscopically-descriptive Navier-Stokes (MDNS) equations as the
progenitors, Navier, Cauchy, Poisson, Saint-Venant and Stokes, and endeavor to ascertain their aims and
conceptual thoughts in formulations these new equation. “The two-constant theory” was introduced
first introduced in 1805 by Laplace in regard to capillary action with constants denoted by $H$ and $K$.

Thereafter, various pairs of constants have been proposed by their progenitors in formulating MDNS
equations or equations describing equilibrium or capillary situations. It is commonly accepted that this
theory describes isotropic, linear elasticity. We can find the “two-constant” in the equations of gas
theories by Maxwell, Kirchoff and Boltzmann, which were fixed into the common linear terms, and
which originally takes its rise in Poisson and Stokes.

The gas theorists studied also the general equations of hydromechanics, which have the same proportion
of coefficients as the equations deduced by Poisson and Stokes with only the linear term and the ratio of
the coefficient of Laplacian to that of gradient of divergence term is $3:1$. (cf Table 1.)

Date: 2011/09/04.

1\(^{(a)}\) Throughout this paper, in citation of bibliographical sources, we show our own paragraph or sentences of commentaries
by surrounding between \((\_\_\_\_\_\_)\) and \((\_\_\_\_\_\_)\). \((\_\_\_\_\_\_)\) is used only when not following to next section, \(_\_\_\_\_\_\_\_. \text{ And by } =^*, \text{ we detail
the statement by original authors, because we would like to discriminate and to avoid confusion from the descriptions by original authors. The mark } \Rightarrow \text{ means transformation of the statements in brevity by ours. And all the frames surrounding the statements are inserted for important remark of ours. Of course, when the descriptions are explicitly distinct without
these marks, these are not the descriptions in citation of bibliographical sources.}

2\(^{(a)}\) Of capillary action, Laplace[9, V.4, Supplement p.2] acknowledges Clailaut [6, p.22], and Clailaut cites Maupertuis.

3\(^{(a)}\) Darrigol [7, p.121].
TABLE 1. The kinetic equations of the hydrodynamics until the "Navier-Stokes equations" was fixed.

<table>
<thead>
<tr>
<th>no</th>
<th>name/prob</th>
<th>the kinetic equations</th>
<th>$\Delta$</th>
<th>$gr$</th>
<th>$dv$</th>
<th>$E$</th>
<th>$F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Euler</td>
<td>$X - \frac{1}{\beta} \Delta X = u \frac{dx}{dt} + v \frac{dy}{dt} + w \frac{dz}{dt}$, $Y - \frac{1}{\beta} \Delta Y = u \frac{dx}{dt} + v \frac{dy}{dt} + w \frac{dz}{dt}$, $Z - \frac{1}{\beta} \Delta Z = u \frac{dx}{dt} + v \frac{dy}{dt} + w \frac{dz}{dt}$</td>
<td>$\varepsilon$</td>
<td>$2\varepsilon^2$</td>
<td>$\frac{1}{\beta}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Navier</td>
<td>(6.1) $\Pi$</td>
<td>$2R$</td>
<td>$G$</td>
<td>$\Pi &gt; 0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(1827) [14]</td>
<td>$\frac{\partial X}{\partial t} + \frac{\partial}{\partial x} (\rho u X) + \frac{\partial}{\partial y} (\rho u Y) + \frac{\partial}{\partial z} (\rho u Z) = \rho \frac{\partial}{\partial t} (\rho u X) + \rho \frac{\partial}{\partial x} (\rho u u X) + \rho \frac{\partial}{\partial y} (\rho u u Y) + \rho \frac{\partial}{\partial z} (\rho u u Z)$</td>
<td>$\varepsilon$</td>
<td>$2\varepsilon^2$</td>
<td>$\frac{1}{\beta}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Navier</td>
<td>(1827) [15]</td>
<td>$\frac{\partial X}{\partial t} + \frac{\partial}{\partial x} (\rho u X) + \frac{\partial}{\partial y} (\rho u Y) + \frac{\partial}{\partial z} (\rho u Z) = \rho \frac{\partial}{\partial t} (\rho u X) + \rho \frac{\partial}{\partial x} (\rho u u X) + \rho \frac{\partial}{\partial y} (\rho u u Y) + \rho \frac{\partial}{\partial z} (\rho u u Z)$</td>
<td>$\varepsilon$</td>
<td>$2\varepsilon^2$</td>
<td>$\frac{1}{\beta}$</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>Cauchy</td>
<td>(1828) [5]</td>
<td>$\frac{\partial X}{\partial t} + \frac{\partial}{\partial x} (\rho u X) + \frac{\partial}{\partial y} (\rho u Y) + \frac{\partial}{\partial z} (\rho u Z) = \rho \frac{\partial}{\partial t} (\rho u X) + \rho \frac{\partial}{\partial x} (\rho u u X) + \rho \frac{\partial}{\partial y} (\rho u u Y) + \rho \frac{\partial}{\partial z} (\rho u u Z)$</td>
<td>$\varepsilon$</td>
<td>$2\varepsilon^2$</td>
<td>$\frac{1}{\beta}$</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>Poisson</td>
<td>(1831) [17]</td>
<td>$\frac{\partial X}{\partial t} + \frac{\partial}{\partial x} (\rho u X) + \frac{\partial}{\partial y} (\rho u Y) + \frac{\partial}{\partial z} (\rho u Z) = \rho \frac{\partial}{\partial t} (\rho u X) + \rho \frac{\partial}{\partial x} (\rho u u X) + \rho \frac{\partial}{\partial y} (\rho u u Y) + \rho \frac{\partial}{\partial z} (\rho u u Z)$</td>
<td>$\varepsilon$</td>
<td>$2\varepsilon^2$</td>
<td>$\frac{1}{\beta}$</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>Poisson</td>
<td>(1831) [17]</td>
<td>$\frac{\partial X}{\partial t} + \frac{\partial}{\partial x} (\rho u X) + \frac{\partial}{\partial y} (\rho u Y) + \frac{\partial}{\partial z} (\rho u Z) = \rho \frac{\partial}{\partial t} (\rho u X) + \rho \frac{\partial}{\partial x} (\rho u u X) + \rho \frac{\partial}{\partial y} (\rho u u Y) + \rho \frac{\partial}{\partial z} (\rho u u Z)$</td>
<td>$\varepsilon$</td>
<td>$2\varepsilon^2$</td>
<td>$\frac{1}{\beta}$</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>Saint-Venant</td>
<td>(1843) [21]</td>
<td>$\frac{\partial X}{\partial t} + \frac{\partial}{\partial x} (\rho u X) + \frac{\partial}{\partial y} (\rho u Y) + \frac{\partial}{\partial z} (\rho u Z) = \rho \frac{\partial}{\partial t} (\rho u X) + \rho \frac{\partial}{\partial x} (\rho u u X) + \rho \frac{\partial}{\partial y} (\rho u u Y) + \rho \frac{\partial}{\partial z} (\rho u u Z)$</td>
<td>$\varepsilon$</td>
<td>$2\varepsilon^2$</td>
<td>$\frac{1}{\beta}$</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>Stokes</td>
<td>(1849) [22]</td>
<td>$\frac{\partial X}{\partial t} + \frac{\partial}{\partial x} (\rho u X) + \frac{\partial}{\partial y} (\rho u Y) + \frac{\partial}{\partial z} (\rho u Z) = \rho \frac{\partial}{\partial t} (\rho u X) + \rho \frac{\partial}{\partial x} (\rho u u X) + \rho \frac{\partial}{\partial y} (\rho u u Y) + \rho \frac{\partial}{\partial z} (\rho u u Z)$</td>
<td>$\varepsilon$</td>
<td>$2\varepsilon^2$</td>
<td>$\frac{1}{\beta}$</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>Maxwell</td>
<td>(1855-56) [12]</td>
<td>$\frac{\partial X}{\partial t} + \frac{\partial}{\partial x} (\rho u X) + \frac{\partial}{\partial y} (\rho u Y) + \frac{\partial}{\partial z} (\rho u Z) = \rho \frac{\partial}{\partial t} (\rho u X) + \rho \frac{\partial}{\partial x} (\rho u u X) + \rho \frac{\partial}{\partial y} (\rho u u Y) + \rho \frac{\partial}{\partial z} (\rho u u Z)$</td>
<td>$\varepsilon$</td>
<td>$2\varepsilon^2$</td>
<td>$\frac{1}{\beta}$</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>Kirchhoff</td>
<td>(1870) [8]</td>
<td>$\frac{\partial X}{\partial t} + \frac{\partial}{\partial x} (\rho u X) + \frac{\partial}{\partial y} (\rho u Y) + \frac{\partial}{\partial z} (\rho u Z) = \rho \frac{\partial}{\partial t} (\rho u X) + \rho \frac{\partial}{\partial x} (\rho u u X) + \rho \frac{\partial}{\partial y} (\rho u u Y) + \rho \frac{\partial}{\partial z} (\rho u u Z)$</td>
<td>$\varepsilon$</td>
<td>$2\varepsilon^2$</td>
<td>$\frac{1}{\beta}$</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>Rayleigh</td>
<td>(1883) [20]</td>
<td>$\frac{\partial X}{\partial t} + \frac{\partial}{\partial x} (\rho u X) + \frac{\partial}{\partial y} (\rho u Y) + \frac{\partial}{\partial z} (\rho u Z) = \rho \frac{\partial}{\partial t} (\rho u X) + \rho \frac{\partial}{\partial x} (\rho u u X) + \rho \frac{\partial}{\partial y} (\rho u u Y) + \rho \frac{\partial}{\partial z} (\rho u u Z)$</td>
<td>$\varepsilon$</td>
<td>$2\varepsilon^2$</td>
<td>$\frac{1}{\beta}$</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>Boltzmann</td>
<td>(1898)[1]</td>
<td>$\frac{\partial X}{\partial t} + \frac{\partial}{\partial x} (\rho u X) + \frac{\partial}{\partial y} (\rho u Y) + \frac{\partial}{\partial z} (\rho u Z) = \rho \frac{\partial}{\partial t} (\rho u X) + \rho \frac{\partial}{\partial x} (\rho u u X) + \rho \frac{\partial}{\partial y} (\rho u u Y) + \rho \frac{\partial}{\partial z} (\rho u u Z)$</td>
<td>$\varepsilon$</td>
<td>$2\varepsilon^2$</td>
<td>$\frac{1}{\beta}$</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>Prandtl</td>
<td>(1904)[19]</td>
<td>$\frac{\partial X}{\partial t} + \frac{\partial}{\partial x} (\rho u X) + \frac{\partial}{\partial y} (\rho u Y) + \frac{\partial}{\partial z} (\rho u Z) = \rho \frac{\partial}{\partial t} (\rho u X) + \rho \frac{\partial}{\partial x} (\rho u u X) + \rho \frac{\partial}{\partial y} (\rho u u Y) + \rho \frac{\partial}{\partial z} (\rho u u Z)$</td>
<td>$\varepsilon$</td>
<td>$2\varepsilon^2$</td>
<td>$\frac{1}{\beta}$</td>
<td></td>
</tr>
</tbody>
</table>

FOR INCOMPRESSIBLE, IT IS SIMPLIFIED $\text{div} \ w = 0$, $\frac{\partial}{\partial t} = g - \frac{1}{\beta} \text{grad} \ p + \nu \Delta w$
<table>
<thead>
<tr>
<th>No</th>
<th>Name</th>
<th>Tensor</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Növian fluid</td>
<td>[ (p - \varepsilon u_{k} u_{i}) \delta_{ij} - \varepsilon (u_{i} u_{j} + u_{j} u_{i}) ]</td>
</tr>
<tr>
<td>2</td>
<td>Cauchy system</td>
<td>[ (\eta \chi_{k} - \eta_{i} i_{j} + \mu (v_{i} v_{j} + v_{j} v_{i}) ]</td>
</tr>
<tr>
<td>3</td>
<td>Poisson fluid</td>
<td>[ \beta (\frac{dx}{dz} + \frac{dy}{dz}) \beta (\frac{dy}{dz} + \frac{dz}{dx}) \pi + 2 \beta \frac{dx}{dz} ]</td>
</tr>
<tr>
<td>4</td>
<td>Saint-Venant fluid</td>
<td>[ \pi = -p - 2 \beta \frac{dx}{dz} ]</td>
</tr>
<tr>
<td>5</td>
<td>Stokes fluid</td>
<td>[ \pi = -p - \mu (\frac{dx}{dz} + \frac{dy}{dz}) ]</td>
</tr>
<tr>
<td>6</td>
<td>Maxwell fluid</td>
<td>[ \pi = -p - \mu (\frac{dx}{dz} + \frac{dy}{dz}) ]</td>
</tr>
<tr>
<td>7</td>
<td>Kirchhoff fluid</td>
<td>[ \pi = -p - 2 \beta \frac{dx}{dz} ]</td>
</tr>
<tr>
<td>8</td>
<td>Boltzmann fluid</td>
<td>[ \pi = -p - 2 \beta \frac{dx}{dz} ]</td>
</tr>
</tbody>
</table>

2. The succession of the linear equations from Poisson to Stokes

4 We discuss the linear fluid equations. Poisson’s tensor of the pressures in fluid reads as follows:

\[
\begin{bmatrix}
U_1 & U_2 & U_3 \\
V_1 & V_2 & V_3 \\
W_1 & W_2 & W_3
\end{bmatrix}
= \begin{bmatrix}
\beta \left( \frac{dx}{dz} + \frac{dy}{dz} \right) & \beta \left( \frac{dy}{dz} + \frac{dz}{dx} \right) & p - \alpha \frac{dx}{dt} + \frac{\pi}{x_t} + 2\beta \frac{dx}{dz} \\
\beta \left( \frac{dy}{dz} + \frac{dz}{dx} \right) & p - \alpha \frac{dx}{dt} + \frac{\pi}{x_t} + 2\beta \frac{dy}{dz} \\
p - \alpha \frac{dx}{dt} + \frac{\pi}{x_t} + 2\beta \frac{dx}{dz} & \beta \left( \frac{dy}{dz} + \frac{dz}{dx} \right) & \beta \left( \frac{dx}{dz} + \frac{dy}{dz} \right)
\end{bmatrix},
\]

\[\text{(7-7)}\]

\[\text{In Poisson [17], the title of the chapter 7 is “Calcul des Pressions dans les Fluides en mouvement ; équations differentielles de ce mouvement.”}\]
where \( \chi t \) is the density of the fluid around the point \( M \), and \( \psi t \) is the pressure. We put \( \omega \) as following:

\[
\omega = p - \alpha \frac{d\psi t}{dt} - \frac{\beta + \beta'}{\chi t} \frac{d\chi t}{dt},
\]

then we get the linear equation by Poisson as following:

\[
\begin{align*}
\rho(X - \frac{d^2}{dx^2}) &= \frac{du}{dx} + \beta (\frac{d^2}{dy^2} + \frac{d^2}{dz^2} + \frac{d^2}{dx^2}) \\
\rho(Y - \frac{d^2}{dy^2}) &= \frac{dv}{dy} + \beta (\frac{d^2}{dx^2} + \frac{d^2}{dz^2} + \frac{d^2}{dy^2}) \\
\rho(Z - \frac{d^2}{dz^2}) &= \frac{dw}{dz} + \beta (\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}) \\
\end{align*}
\]

Stokes comments on Poisson’s (7-9) \( P_1 \) as follows:

On this supposition we shall get the value of \( \frac{d\Psi t}{dt} \) from that of \( R_1' - K \) in the equations of page 140 by putting

\[
\frac{d\chi t}{dt} = \frac{1}{3\chi t} \frac{d\chi t}{dt}.
\]

We have therefore

\[
\alpha \frac{d\chi t}{dt} = \frac{\alpha}{3} (K - 5k) \frac{d\chi t}{dt}.
\]

Putting now for \( \beta + \beta' \) its value \( 2 \alpha k \), and for \( \frac{1}{\chi t} \frac{d\chi t}{dt} \) its value given by equation (6)\(^6\), the expression for \( \omega \), page 152,\(^7\) becomes

\[
\omega = p - \alpha \frac{d\psi t}{dt} - \frac{\beta + \beta'}{\chi t} \frac{d\chi t}{dt} = p - \left( \frac{\alpha}{3} (K - 5k) + 2 \alpha k \right) \frac{d\chi t}{dt} = p + \frac{\alpha}{3} (K + k) \left( \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} \right).
\]

Observing that \( \alpha (K + k) = \beta \), this value of \( \omega \) reduces Poisson’s equation (7-9) \( P_1 \)\(^{(=5)} \) to the equation (12)\(_S\) of this paper. ([22, p.119]).

Namely, by using \( \alpha (K + k) = \beta \) in (3), we get the following:

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{dw}{dx} = \frac{dx}{dx} + \frac{\partial}{\partial x} \left( \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} \right), \\
\frac{dx}{dx} = \frac{dx}{dx} + \frac{\partial}{\partial x} \left( \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} \right), \\
\frac{dx}{dx} = \frac{dx}{dx} + \frac{\partial}{\partial x} \left( \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} \right), \\
\end{array} \right.
\end{align*}
\]

then (5)\(^{(=7-9)P_1\)} turns out:

\[
\begin{align*}
\rho \left( \frac{\partial u}{\partial t} - X \right) + \frac{\partial u}{\partial x} + \alpha (K + k) \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + \frac{\alpha}{3} (K + k) \left( \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} \right) &= 0, \\
\rho \left( \frac{\partial v}{\partial t} - Y \right) + \frac{\partial v}{\partial y} + \alpha (K + k) \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) + \frac{\alpha}{3} (K + k) \left( \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} \right) &= 0, \\
\rho \left( \frac{\partial w}{\partial t} - Z \right) + \frac{\partial w}{\partial z} + \alpha (K + k) \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) + \frac{\alpha}{3} (K + k) \left( \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} \right) &= 0, \\
\Rightarrow (12)_S \left\{ \begin{array}{l}
\rho \left( \frac{\partial u}{\partial t} - X \right) + \frac{\partial u}{\partial x} - \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) - \frac{1}{2} \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} \right) = 0, \\
\rho \left( \frac{\partial v}{\partial t} - Y \right) + \frac{\partial v}{\partial y} - \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) - \frac{1}{2} \frac{\partial}{\partial y} \left( \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} \right) = 0, \\
\rho \left( \frac{\partial w}{\partial t} - Z \right) + \frac{\partial w}{\partial z} - \mu \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) - \frac{1}{2} \frac{\partial}{\partial z} \left( \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} \right) = 0.
\end{array} \right.
\end{align*}
\]

Here, we remark that the succession from (7-9) \( P_1 \) to (12)_S means that the Stokes’ equations come from Poisson’s linear equations, however, Poisson’s proper equations contain both compressible and incompressible fluid, taking no notice of the Navier’s equations including both linear and non-linear terms until Rayleigh [20] in 1883. (cf. Table 1.)

\(^5\) (7-9) \( P_1 \) means the equation number with chapter of Poisson [17]  
\(^6\) Poisson[17, p.141]  
\(^7\) p.141
3.1 A progenitor of gas theory after Poisson and Stokes.

Even after Poisson, Saint-Venent and Stokes, we can cite the progenitors of microscopically descriptive, hydromechanical equations, which are specialists in gas theories, in which they describe the hydrodynamic equations, and they contribute to fix the tensor and equations of NS, so we have to trace them. cf. Table 1, 2.

Maxwell [12] had presented between late 1865 and early 1866, the original equations calculating his original coefficient, with which his tensor coincides with Poisson and Stokes, and his gas theory prior to Kirchhoff [8] in 1876 and Boltzmann [1] in 1895. Maxwell says as follows:

\[ \xi^2 \rho = p - \frac{M}{6k \rho \Theta_2} \left( \frac{d}{dx} - \frac{dv}{dy} - \frac{dw}{dz} \right) \]

which similar expressions for \( \eta^2 \rho \) and \( \zeta^2 \rho \). By transformation of coordinates we can easily obtain the expressions for \( \xi \eta \), \( \eta \zeta \) and \( \xi \zeta \). They are of the form

\[ \zeta \xi \rho = -\frac{M}{6k \rho \Theta_2} \left( \frac{d}{dx} + \frac{dw}{dz} \right) \]

Having thus obtained the values of the pressures in different directions we may substitute them in the equation of motion

\[ \begin{cases} \rho \frac{\partial u}{\partial x} + \frac{\partial}{\partial x}(\rho \xi^2) + \frac{\partial}{\partial y}(\rho \eta^2) + \frac{\partial}{\partial z}(\rho \zeta^2) = X \rho, \\ \rho \frac{\partial v}{\partial y} + \frac{\partial}{\partial y}(\rho \xi \eta) + \frac{\partial}{\partial z}(\rho \eta \zeta) = Y \rho, \\ \rho \frac{\partial w}{\partial z} + \frac{\partial}{\partial z}(\rho \xi \zeta) + \frac{\partial}{\partial x}(\rho \zeta \xi) = Z \rho. \end{cases} \]

This becomes as follows:

\[ \begin{cases} \rho \frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} + \frac{1}{3} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right] = \rho X, \\ \rho \frac{\partial v}{\partial y} + \frac{\partial}{\partial y} \left[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} + \frac{1}{3} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right] = \rho Y, \\ \rho \frac{\partial w}{\partial z} + \frac{\partial}{\partial z} \left[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} + \frac{1}{3} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right] = \rho Z. \end{cases} \]

Maxwell states as follows:

This is the equation of motion in the direction of x. The other equations may be written down by symmetry. The form of the equations is identical

- with that deduced by Poisson from the theory of elasticity by supposing the strain to be constantly relaxed at the given rate
- and the ratio of the coefficients of \( \nabla^2 \) to \( \frac{\partial}{\partial \rho \partial \rho} \) agrees with that given by Professor Stokes, which means (10) equals (12) above.

The quantity \( \frac{\mu}{k \rho \Theta_2} \) is the coefficient of viscosity or of internal friction and is denoted by \( \mu \) in the writings of Professor Stokes and in my paper on the Viscosity of Air and other Gases. [13, pp.261-262].

3.2. Law of Volumes.

In late 1865 or early 1866, Maxwell proposed this paper. It was likely that Boltzmann had got his idea from this paper.

Footnotes:
8(4) The Equation (9) in [17, p.139], which we cite as (5) (7-9) above.
9(4) Stokes [22]
10(4) 1844-1906.
$u$, $v$, $w$ are the components of the mean velocity of all the molecules which are at a given instant in a given element of volume, hence there is no motion of translation. 

$\xi$, $\eta$, $\zeta$ are the components of the relative velocity of one of these molecules with respect to the mean velocity, the 'velocity of agitation of molecules'.

In the case of a single gas in motion let $Q$ be the total energy of a single molecule then

\[
Q = \frac{1}{2}M\left\{(u + \xi)^2 + (v + \eta)^2 + (w + \zeta)^2 + \beta(\xi^2 + \eta^2 + \zeta^2)\right\}
\]

and

\[
\frac{\delta Q}{\delta t} = M(uX + vY + wZ).
\]

The general equation becomes

\[
\frac{1}{2} \rho \frac{\partial}{\partial t} \left\{u^2 + v^2 + w^2 + (1 + \beta)(\xi^2 + \eta^2 + \zeta^2)\right\}
+ \frac{d}{dx}(u\rho \xi^2 + v\rho \xi \eta + w\rho \xi \zeta) + \frac{d}{dy}(u\rho \xi \eta + v\rho \eta^2 + w\rho \eta \zeta) + \frac{d}{dz}(u\rho \xi \zeta + v\rho \eta \zeta + w\rho \zeta^2)
+ \frac{1}{2}(1 + \beta)(\xi^2 + \eta^2 + \zeta^2)(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}) = \rho(uX + vY + wZ).
\]

Substituting the values of $\rho X$, $\rho Y$ and $\rho Z$ with $\frac{d\xi}{dx}$, $\frac{d\eta}{dy}$ and $\frac{d\zeta}{dz}$ and dividing by $\rho$ of both hand-side, then

\[
\frac{1}{2} \frac{\partial}{\partial t} (1 + \beta)(\xi^2 + \eta^2 + \zeta^2)
+ \xi^2 \frac{du}{dx} + \eta^2 \frac{dv}{dy} + \zeta^2 \frac{dw}{dz} + \eta \zeta \left(\frac{dv}{dz} + \frac{dw}{dy}\right) + \zeta \xi \left(\frac{dw}{dx} + \frac{du}{dz}\right) + \xi \eta \left(\frac{du}{dy} + \frac{dv}{dx}\right)
+ \frac{1}{2}(1 + \beta)(\xi^2 + \eta^2 + \zeta^2)(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}) = 0.
\]

If we set $\mathcal{R} \equiv \frac{\mathcal{R}}{1(1+\beta)}$, then we get the second, linear term of the left-hand-side by Maxwell is written by tensor

\[
\begin{bmatrix}
\rho \xi^2 & \rho \xi \eta & \rho \xi \zeta \\
\rho \xi \eta & \rho \eta^2 & \rho \eta \zeta \\
\rho \xi \zeta & \rho \eta \zeta & \rho \zeta^2
\end{bmatrix}
= -\mathcal{R}
\begin{bmatrix}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\
\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\
\frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}
\end{bmatrix}
\]

which is Maxwell called it 'general tensor'.

3.3. Determination of the inequality of pressure in a medium.

Maxwell constructs the tensor with his viscosity coefficient as follows:

\[
\begin{bmatrix}
\rho \xi^2 & \rho \xi \eta & \rho \xi \zeta \\
\rho \xi \eta & \rho \eta^2 & \rho \eta \zeta \\
\rho \xi \zeta & \rho \eta \zeta & \rho \zeta^2
\end{bmatrix}
= \begin{bmatrix}
p - \frac{M}{8\alpha_1 \beta_1} (2 \frac{du}{dx} + \frac{dv}{dy} - \frac{dw}{dz}) - \frac{M}{8\alpha_2 \beta_2} (\frac{dv}{dy} + \frac{dw}{dz}) - \frac{M}{8\alpha_3 \beta_3} (\frac{du}{dx} + \frac{dw}{dz}) \\
- \frac{M}{8\alpha_1 \beta_1} (\frac{dv}{dy} + \frac{dw}{dz}) - \frac{M}{8\alpha_2 \beta_2} (2 \frac{du}{dx} - \frac{dv}{dy} - \frac{dw}{dz}) - \frac{M}{8\alpha_3 \beta_3} (\frac{dv}{dy} - \frac{dw}{dz}) \\
- \frac{M}{8\alpha_1 \beta_1} (\frac{dw}{dz} + \frac{dw}{dz}) - \frac{M}{8\alpha_2 \beta_2} (\frac{dv}{dy} + \frac{dw}{dz}) - \frac{M}{8\alpha_3 \beta_3} (\frac{du}{dx} + \frac{dv}{dy})
\end{bmatrix}
\]

(11)

Here, it tells of the equivalent in the structure between (11) and (8). If we set $\mathcal{R} \equiv \frac{Mp}{6k\rho \Theta}$, then these equations are completely equal to (221)$_B$ (= (24)) by Boltzmann. These facts state that Boltzmann had got his idea of special form of hydromechanics from Maxwell.
4. 'Lectures on Gas theory' by Boltzmann

In general, according to Ukai [23], we can state the Boltzmann equations as follows: 11

$$\partial_t f + v \cdot \nabla_x f = Q(f, g), \quad t > 0, \quad x, v \in \mathbb{R}^n(n \geq 3), \quad x = (x, y, z), \quad v = (\xi, \eta, \zeta),$$

$$Q(f, g)(t, x, v) = \int_{S^2} \int_{\mathbb{R}^3} B(v - v_*, \sigma)(g(u'_*)f(v') - g(v'_*)f(v))d\sigma dv_*,$n

$$g(v'_*) = g(t, x, v'_*), \quad v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma, \quad v'' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma, \quad \sigma \in S^{n-1}$$

where,

- \( f = f(t, x, v) \) is interpretable as many meanings such as
  - density distribution of a molecule
  - number density of a molecule
  - probability density of a molecule
- \( f(v) \) means \( f(t, x, v) \) as abbreviating \( t \) and \( x \) in the same time and place with \( f(v') \)
- \( Q(f, g) \) of the right-hand-side of (12) is the Boltzmann bilinear collision operator.
- \( v \cdot \nabla_x f \) is the transport operator,
- \( B(\xi, \eta, \zeta) \) of the right-hand-side in (13) is the non-negative function of collision cross-section.
- \( Q(f, g)(t, x, v) \) is expressed in brief as \( Q(f) \).
- \( (v, v_*) \) and \( (v', v'_*) \) are the velocities of a molecule before and after collision.
- According to Ukai [24], the transport operators are expressed with two sort of terms like Boltzmann's descriptions: (114)_B and (115)_B including the collision term \( \nabla_v \cdot (Ff) \) by exterior force \( F \) as follow: 12

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (Ff) = Q(f)$$

$$Q(f) = \int_{S^2} \int_{\mathbb{R}^3} B(v - v_*, \sigma)(f(u'_*)f(v') - f(v'_*)f(v))d\sigma dv_*$$

where, \( v \cdot \nabla_x f + \nabla_v \cdot (Ff) \) are transport operators operating under the exterior force: \( F(t, x, v) = (F_1, F_2, F_3) \). The right-hand side of (15) is expressed in brief as \( Q(f) \) meaning \( Q(f)(t, x, v) \).

4.1. Reduction of the partial differential equations for \( f \) and \( F \).

We show the Figure 6 in the last page of our paper, which defines the model of the collision between the molecule \( m_1 \) calling the point of it and the molecule \( m \) wich we call the point \( m \). The instant when the molecule \( m \) passes vertically thought the disc of \( m_1 \) molecule, is defined as collision. We show Boltzmann's definitions as follow:

We fix our attention on the parallelepiped representing all space points whose coordinates lie between the limits 13

$$[x, x + dx], \quad [y, y + dy], \quad [z, z + dz], \quad do = dx dy dz$$

We now construct a second rectangular parallelepiped, which include all points whose coordinates lie between the limits

$$[\xi, \xi + d\xi], \quad [\eta, \eta + d\eta], \quad [\zeta, \zeta + d\zeta]$$

We set its volume equal to

$$d\xi d\eta d\zeta = d\omega$$

(16)

and we call it the parallelepiped \( d\omega \). The molecules that are in \( do \) at the time \( t \) and whose velocity points lie in \( d\omega \) at the same time will again be called the specified molecules, or the \( "dn \) molecules." Their number is clearly proportional to the product \( do \cdot d\omega \). Then all volume elements immediately adjacent to \( do \) find themselves subject to similar conditions,

---


12(9) In the Boltzmann' original equations, they are used with two terms like (114)_B, (115)_B. We can refer the General lecture in the autumn meeting of MSJ by S.Ukai [24]: The study of Boltzmann equations: past and future, MSJ, 23, September, 2010.

13(9) \( (\cdot)_{B} \) in the top of the equation or expression means the number cited in Boltzmann[2] in below of our paper.
<table>
<thead>
<tr>
<th>no</th>
<th>symbol</th>
<th>defined content of conformation in modeling of collision. cf. Figure 6 in the last page.</th>
<th>cf. m</th>
<th>m&lt;sub&gt;n&lt;/sub&gt;</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$X_{i}$, $Y_{j}$, $Z_{k}$</td>
<td>(18) The component of accelerating force of a molecule in a coordinate direction.</td>
<td>m</td>
<td>m&lt;sub&gt;n&lt;/sub&gt;</td>
</tr>
<tr>
<td>2</td>
<td>$m_{i}X_{i}$, $m_{j}Y_{j}$, $m_{k}Z_{k}$</td>
<td>(18) The component of the external force acting on any m-molecule.</td>
<td>m</td>
<td>m&lt;sub&gt;n&lt;/sub&gt;</td>
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<tr>
<td>3</td>
<td>$\xi_{i}, \eta_{i}, \zeta_{i}$</td>
<td>(98) The component of velocity of any m-molecule in a coordinate direction.</td>
<td>m</td>
<td>m&lt;sub&gt;n&lt;/sub&gt;</td>
</tr>
<tr>
<td>4</td>
<td>$f$</td>
<td>(99) $F = f(x, y, z, \xi, \eta, \zeta, \tau)$, different only with velocity of $f$.</td>
<td>m</td>
<td>m&lt;sub&gt;n&lt;/sub&gt;</td>
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<tr>
<td>5</td>
<td>$f_{1}$</td>
<td>(99) $F = f(x, y, z, \xi, \eta, \zeta, \tau)$, different only with velocity of $F$.</td>
<td>m</td>
<td>m&lt;sub&gt;n&lt;/sub&gt;</td>
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<tr>
<td>6</td>
<td>$F$</td>
<td>(100) $F = F(x, y, z, \xi, \eta, \zeta, \tau)$, different only with velocity of $F$.</td>
<td>m</td>
<td>m&lt;sub&gt;n&lt;/sub&gt;</td>
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<tr>
<td>7</td>
<td>$F_{1}$</td>
<td>(103) $F = F(x, y, z, \xi, \eta, \zeta, \tau)$, different only with velocity of $F_{1}$.</td>
<td>m</td>
<td>m&lt;sub&gt;n&lt;/sub&gt;</td>
</tr>
<tr>
<td>8</td>
<td>$\xi_{1}, \eta_{1}, \zeta_{1}$</td>
<td>(102) The component of velocity of any m&lt;sub&gt;1&lt;/sub&gt;-molecule in a coordinate direction.</td>
<td>m</td>
<td>m&lt;sub&gt;n&lt;/sub&gt;</td>
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<tr>
<td>9</td>
<td>$b$</td>
<td>(104) The moving distance of an m-molecule to an m&lt;sub&gt;1&lt;/sub&gt;-molecule during $dt$.</td>
<td>Fig. 6</td>
<td>m&lt;sub&gt;n&lt;/sub&gt;</td>
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<tr>
<td>10</td>
<td>$\sigma$</td>
<td>(104) The limit of the length of a line.</td>
<td>Fig. 6</td>
<td>m&lt;sub&gt;n&lt;/sub&gt;</td>
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<tr>
<td>11</td>
<td>$\epsilon$</td>
<td>(104) An angle formed between a line $b$ and a line $m_{1}H$, where, $\epsilon$ is the angle between the two planes through the direction of relative motion, one parallel to $P_{1}P_{2}$ along $b$, and the other to the abscissa axis.</td>
<td>Fig. 6</td>
<td>m&lt;sub&gt;n&lt;/sub&gt;</td>
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<tr>
<td>12</td>
<td>$\xi_{1}, \eta_{1}, \zeta_{1}$</td>
<td>(108) The component of velocity of a molecule after the collision.</td>
<td>m</td>
<td>m&lt;sub&gt;n&lt;/sub&gt;</td>
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<td>13</td>
<td>$\chi_{1}$</td>
<td>(109) $\chi_{1}$ The length of a line after the collision.</td>
<td>Fig. 6</td>
<td>m&lt;sub&gt;n&lt;/sub&gt;</td>
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<td>14</td>
<td>$e'$</td>
<td>(109) An angle formed between a line $b$ and a line $m_{1}H$ after the collision.</td>
<td>Fig. 6</td>
<td>m&lt;sub&gt;n&lt;/sub&gt;</td>
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<td>15</td>
<td>$\phi_{1}$</td>
<td>(102) We set $do = d\phi_{1}$ in which the m-molecules lie, and we always call this parallelepiped the parallelepiped $do$.</td>
<td>m</td>
<td>m&lt;sub&gt;n&lt;/sub&gt;</td>
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<td>16</td>
<td>$d\phi_{1}$</td>
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<td>17</td>
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<td>44</td>
<td>$d\phi_{1}$</td>
<td>(97) $d\phi_{1}$ In which the m-molecules lie, and we always call this parallelepiped the parallelepiped $d\phi_{1}$.</td>
<td>m</td>
<td>m&lt;sub&gt;n&lt;/sub&gt;</td>
</tr>
</tbody>
</table>
so that in a parallelepiped twice as large there will be twice as many molecules. We can therefore set this number equal to

\[(99)_B \quad dn = f(x, y, z, \xi, \eta, \zeta, t) do\omega = f do\omega\]

Similarly the number of \(m_1\)-molecules that satisfy the conditions \((97)_B\) and \((98)_B\) at time \(t\) will be:

\[(100)_B \quad dN = F(x, y, z, \xi, \eta, \zeta, t) do\omega = F do\omega\]

The two functions \(f\) and \(F\) completely characterize the state of motion, the mixing ratio, and the velocity distribution at all places in the gas mixture. We shall allow a very short time \(dt\) to elapse, and during this time we keep the size and position of \(do\) and \(d\omega\) completely unchanged. The number of \(m\)-molecules that satisfy the conditions \((97)_B\) and \((98)_B\) at time \(t + dt\) is, according to Equation \((99)_B\),

\[dn' = f(x, y, z, \xi, \eta, \zeta, t + dt) do\omega = f do\omega\]

and the total increase experienced by \(dn\) during time \(dt\) is

\[(101)_B \quad dn' - dn = \frac{\partial f}{\partial t} do\omega dt.\]

\(\xi, \eta, \zeta\) are the rectangular coordinates of the velocity point. Although this is only an imaginary point, still it moves like the molecule itself in space. Since \(X, Y, Z\) are the components of the accelerating force, \(^{14}\) we have:

\[\frac{d\xi}{dt} = X, \quad \frac{d\eta}{dt} = Y, \quad \frac{d\zeta}{dt} = Z\]

(18)

4.2. **Four different causes bringing up increase of \(dn\).**

Boltzmann explains an increase of \(dn\) as a result of the following *four different causes* of \(V_1, V_2, V_3\) and \(V_4:\)

- \(V_1:\) increment by *transport* through \(do\)
- \(V_2:\) increment by *transport* of external force
- \(V_3:\) increment as a result of *collisions* of \(m\)-molecules with \(m_1\)-molecules
- \(V_4:\) increment by *collision* of molecules with each other

We extract an outline by the Boltzmann \(^1\) as follows:

The number \(dn\) experiences an increase as a result of *four different causes.*

1. \((V_1: \text{increase going out through } do):\) All \(m\)-molecules whose velocity points lie in \(d\omega\) move in the \(x\)-direction with velocity \(\xi\), in the \(y\)-direction with velocity \(\eta\), and in the \(z\)-direction with velocity \(\zeta\).

Hence through the left side of the parallelepiped \(do\) facing the negative abscissa direction there will enter during time \(dt\) as many molecules satisfying the condition \((98)_B\) as may be found, at the beginning of \(dt\), in a parallelepiped of base \(dydz\) and height \(\xi dt\).\(^{15}\) viz. \(\xi f(x, y, z, \xi, \eta, \zeta, t) dydz do\omega dt\) molecules. Likewise, for the number of \(m\)-molecules that satisfying \((98)_B\) and go out through the opposite face of \(do\) during time \(dt\), the value:

\[\xi \cdot f(x + dx, y, z, \xi, \eta, \zeta, t) dydz do\omega dt\]

By similar arguments for the four other sides of the parallelepiped, one finds that during time \(dt\),

\[\left(\xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + \zeta \frac{\partial f}{\partial z}\right) do\omega dt\]

more molecules satisfying \((98)_B\) enter \(do\) than leave it. This is therefore the increase \(V_1\) which \(dn\) experiences as a result of motion of the molecules during time \(dt\).

\[V_1 = \left(\xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + \zeta \frac{\partial f}{\partial z}\right) do\omega dt\]

\(\xi, \eta, \zeta\) are the rectangular coordinates of the velocity point. Although this is only an imaginary point, still it moves like the molecule itself in space. Since \(X, Y, Z\) are the components of the accelerating force, \(^{14}\) we have:

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\[V_1 = \left(\xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + \zeta \frac{\partial f}{\partial z}\right) do\omega dt\]

\(^{14}\) Da \(X, Y, Z\) die Componenten der beschleunigenden Kraft sind, so ist: \(^{15}\) Boltzmann \(^1\) p.103.

\(^{15}\) \(\xi\) : the \(x\)-direction with velocity multiplied by \(dt\) becomes the length of an edge of which consists a parallelepiped with a base \(dydz\).
(2) $(V_2: \text{increase by external force;})$ As a result of the action of external forces, the velocity components of all the molecules change with time, and hence the velocity points of the molecules in $d\omega$ will move. Some velocity points will leave $d\omega$, others will come in, and since we always include in the number $dn$ only those molecules whose velocity points lie in $d\omega$, $dn$ likewise be changed for this reason.

$$V_2 = -\left( X \frac{\partial f}{\partial \xi} + Y \frac{\partial f}{\partial \eta} + Z \frac{\partial f}{\partial \zeta} \right) \, d\omega \, dt$$

(20)

Boltzmann defines the effects of collisions as follows:

(3) $(V_3: \text{increase as a result of collisions of } m\text{-molecules with } m_1\text{-molecules;})$ Those of our $dn$ molecules that undergo a collision during the time $dt$ will clearly have in general different velocity components after the collision.

- (Decrease:) Their velocity points will therefore be expected, as it were, from the parallelepiped by the collision, and thrown into a completely different parallelepiped. The number $dn$ will thereby be decreased.
- (Increase:) On the other hand, the velocity points of $m\text{-molecules in other parallelepipeds will be thrown into } d\omega \text{ by collisions, and } dn \text{ will thereby increase.}$
- (Total increase by collision between } m\text{-molecules and } m_1\text{-molecules:) It is now a question of finding this total increase } V_3 \text{ experienced by } dn \text{ during time } dt \text{ as a result of the collisions taking place between any } m\text{-molecules and any } m_1\text{-molecules.}

For this purpose we shall fix our attention on a very small fraction of the total number $\nu_1$ of collisions undergone by our $dn$ molecules during time $dt$ with $m_1\text{-molecules. We construct a third parallelepiped which includes all points whose coordinates lie between the limits}

$$(102) \quad [\xi_1, \xi_1 + d\xi_1], \quad [\eta_1, \eta_1 + d\eta_1], \quad [\zeta_1, \zeta_1 + d\zeta_1]$$

Its volume is

$$d\omega_1 = d\xi_1 \, d\eta_1 \, d\zeta_1$$

(21)

It constitutes the parallelepiped $d\omega_1$. By analogy with Equation (100)$_B$, the number of $m_1\text{-molecules in } d\omega \text{ whose velocity points lie in } d\omega_1 \text{ at time } t \text{ is:}

$$(103)_B \quad dN_1 = F_1 \, d\omega_1,$$

where $F_1$ is an abbreviation for $F(x, y, z, \xi_1, \eta_1, \zeta_1)$.

Boltzmann defines a passage of an $m\text{-point by an } m_1\text{-point as follows:}

(a) (How to pass:) We define a passage of an $m\text{-point by an } m_1\text{-point as that instant of time when distance between the points has its smallest value; thus } m \text{ would pass through the plane through } m_1 \text{ perpendicular to the direction } g, \text{ if no interaction took place between the two molecules.}$

(b) ($\nu_2:$ the number of passages of an $m\text{-point by an } m_1\text{-point:) Hence, } \nu_2 \text{ is equal to the number of passages of an } m\text{-point by an } m_1\text{-point that occurs during time } dt, \text{ such that the smallest distance between the two molecules is less than } \sigma.$

(c) (A plane } E:\) In order to find this number, we draw through each $m_1\text{-point a plane } E \text{ moving with } m_1, \text{ perpendicular to the direction of } g, \text{ and a line } G, \text{ which parallel to this direction.}$

(d) (When a passage ends:) As soon as an $m\text{-point crosses } E, \text{ a passage take place between it and the } m_1\text{-point.}$

(e) (A line } m_1X:\) We draw through each $m_1\text{-point a line } m_1X \text{ parallel to the positive abscissa direction and similarly directed.}$

(f) (Half-plane:) The half-plane bounded by $G, \text{ which contains the latter line, cuts } E \text{ in the line } m_1H, \text{ which of course again contains each } m_1\text{-point.}$
(g) \( (b \text{ and } \epsilon : ) \) Furthermore, we draw from each \( m_1 \)-point in each of the plane \( E \) a line of length \( b \), which forms an angle \( \epsilon \) with the line \( m_1 H \).

(h) (Rectangles of surface area \( R \) formed by \( b \) and \( \epsilon : ) \) All points of the plane \( E \) for which \( b \) and \( \epsilon \) lie between the limits

\[
(104)_B \quad [b, \ b + db], \quad [\epsilon, \ \epsilon + d\epsilon]
\]

form a rectangle of surface area \( R = bdbd\epsilon \).

In his Figure 6, \(^{16}\) the intersections of all these lines with a sphere circumscribed about \( m_1 \) are shown. The large circle (shown as an ellipse) lies in the plane \( E \); the circular arc \( GXH \) lies in the half-plane defined above. In each of planes \( E \), an equal and identically situated rectangle will be found. We consider for the moment only those passages of an \( m \)-point by an \( m_1 \)-point in which the first point penetrates one of the rectangles \( R \).

Below, Boltzmann calculates \( V_3 \) in order of \( \Pi \to \nu_3 \to \nu_2 \to \nu_1 \to i_1 \to V_3 \). At first,

\[
\Pi = Rgdt = bdbd\epsilon gdt, \quad \sum \Pi = dN_1 \Pi = \frac{F_1 d\omega_1 g bdbd\epsilon dt}{dN_1 (103)_B \Pi}
\]

Since these volumes are infinitesimal, and lie infinitely close to the point with coordinates \( x, y, z \), then by analogy with Equation (99)\(_B \) the number of \( m \)-points (i.e., \( m \)-molecules whose velocity points lie in \( d\omega \)) that are initially in the volumes \( \sum \Pi \) is equal to:

\[
(105)_B \quad \nu_3 = f \omega \sum \Pi = f F_1 d\omega d\omega_1 g bdbd\epsilon dt
\]

This is at the same time the number of \( m \)-points that pass an \( m \)-point during time \( dt \) at a distance between \( b \) and \( b + db \), in such a way that the angle \( \epsilon \) lie between \( \epsilon \) and \( \epsilon + d\epsilon \).

By \( \nu_2 \) we mean the number of \( m \)-points that pass an \( m_1 \)-point at any distance less than \( \sigma \) during \( dt \). We find \( \nu_2 \) by integrating the differential expression \( \nu_2 \) over \( \epsilon \) from 0 to \( 2\pi \), and over \( b \) from 0 to \( \sigma \).

\[
(106)_B \quad \nu_2 = \int_{0}^{\sigma} db \int_{0}^{2\pi} \nu_2 d\epsilon = d\omega d\omega_1 d\epsilon \int_{0}^{\sigma} db \int_{0}^{2\pi} \epsilon \cdot \epsilon \cdot f \cdot F_1.
\]

The number denoted by \( \nu_1 \) of all collisions of our \( dn \) molecules during \( dt \) with \( m_1 \)-molecules is therefore found by integrating over the three variable \( \xi_1, \eta_1, \zeta_1 \) whose differentials occur in \( d\omega_1 \), from \(-\infty \) to \(+\infty \); we indicate this a single integral sign:

\[
(107)_B \quad \nu_1 = \int_{-\infty}^{\infty} \nu_2 d\omega_1 = do \cdot d\omega \cdot dt \int_{-\infty}^{\infty} d\omega_1 \int_{0}^{\sigma} db \int_{0}^{2\pi} f F_1 g b d\epsilon
\]

We shall consider again those collisions between \( m \)-molecules and \( m_1 \)-molecules, whose number was denoted by \( \nu_3 \) and is given by Equation (105)\(_B \).

These are the collisions that occur in unit time in the volume element \( do \) in such a way the following conditions are satisfied:

- The velocity components of the \( m \)-molecules and the \( m_1 \)-molecules lie between the limits (98)\(_B \) and (102)\(_B \), respectively, before the interaction begins.
- We denote by \( b \) the closest distance of approach that would be attained if the molecules did not interact but retained the velocities they had before the collision.

The total increment \( i_1 \) experienced by \( dn \) as a result of collisions of \( m \)-molecules with \( m_1 \)-molecules is founded by integrating over \( \epsilon \) from 0 to \( 2\pi \), over \( b \) from 0 to \( \sigma \),

---

\(^{16}(\gamma)\) We show this Figure 6 in the last page of our paper citing [2, p.107], which is equal to [1, p.117], however, we must correct the symbol \( R \) by \( H \) of [1, p.117].
and over \( \xi_1, \eta_1, \zeta_1 \) from \(-\infty\) to \(+\infty\). We shall write the result of this integration in the form:

\[
(111)_B \quad i_1 = -\frac{\partial f}{\partial t} + \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + \zeta \frac{\partial f}{\partial z} + X_1 \frac{\partial f}{\partial x} + Y_1 \frac{\partial f}{\partial y} + Z_1 \frac{\partial f}{\partial z}
\]

Of course we cannot perform explicitly the integration with respect to \( b \) and \( \epsilon \) since the variable \( \xi', \eta', \zeta' \) and \( \xi_1', \eta_1', \zeta_1' \) occurring in \( f' \) and \( F_1' \) are functions of \( \xi, \eta, \zeta, \xi_1, \eta_1, \zeta_1, b \) and \( \epsilon \), which cannot be computed until the force law is given.\(^{17}\)

The difference \( i_1 - \nu_1 \) expresses the net increase of \( df \) during time \( dt \) as a result of collisions of \( m \)-molecules with \( m_1 \)-molecules. It is therefore the total increase \( V_3 \) experienced by \( df \) as a result of these collisions, and one has

\[
(112)_B \quad V_3 = i_1 - \nu_1 = -\frac{\partial f}{\partial t} + \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + \zeta \frac{\partial f}{\partial z} + X_1 \frac{\partial f}{\partial x} + Y_1 \frac{\partial f}{\partial y} + Z_1 \frac{\partial f}{\partial z}
\]

(4) ( \( V_4 \) : increment by collision of molecules with each other) The increment \( V_4 \) experienced by \( df \) as a result of collisions of \( m \)-molecules with each other is found from Equation (112)\(_B\) by a simple permutation. One now uses \( \xi_1, \eta_1, \zeta_1 \) and \( \xi_1', \eta_1', \zeta_1' \) for the velocity components of the other \( m \)-molecule before and after the collision, respectively, and one writes \( f_1 \) and \( f_1' \) for

\[
f_1 = f(x, y, z, \xi_1, \eta_1, \zeta_1, t) \quad \text{and} \quad f_1' = f(x, y, z, \xi_1', \eta_1', \zeta_1', t)
\]

Then:

\[
(113)_B \quad V_4 = -\frac{\partial f}{\partial t} + \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + \zeta \frac{\partial f}{\partial z} + X_1 \frac{\partial f}{\partial x} + Y_1 \frac{\partial f}{\partial y} + Z_1 \frac{\partial f}{\partial z}
\]

4.3. Formulation of Boltzmann's transport equations.

According to Boltzmann\(^{2}\), his equations (so-called transport equations) are the following:\(^{18}\)

Since now \( V_1 + V_2 + V_3 + V_4 \) is equal to the increment \( df' - df \) of \( df \) during time \( dt \), and this according to Equation (101)\(_B\) must be equal to \( \frac{\partial f}{\partial t} \), one obtains on substituting all the appropriate values and dividing by \( \omega dt \) the following partial differential equation for the function \( f \):

\[
(114)_B \quad \frac{\partial f}{\partial t} + \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + \zeta \frac{\partial f}{\partial z} + X_1 \frac{\partial f}{\partial x} + Y_1 \frac{\partial f}{\partial y} + Z_1 \frac{\partial f}{\partial z} = \frac{\int_{0}^{\infty} \int_{0}^{2\pi} \left( f' F_1' - f F_1 \right) b \, \omega \, dt \, db \, d\epsilon}{V_3 + V_4}
\]

Similarly we obtain the equation for \( F \):

\[
(115)_B \quad \frac{\partial F}{\partial t} + \xi \frac{\partial F}{\partial x} + \eta \frac{\partial F}{\partial y} + \zeta \frac{\partial F}{\partial z} + X_1 \frac{\partial F}{\partial x} + Y_1 \frac{\partial F}{\partial y} + Z_1 \frac{\partial F}{\partial z} = \frac{\int_{0}^{\infty} \int_{0}^{2\pi} \left( f' F_1' - f F_1 \right) b \, \omega \, dt \, db \, d\epsilon}{V_3 + V_4}
\]

where,

\[
\begin{align*}
\{ f &= f(x, y, z, \xi, \eta, \zeta, t), \quad f_1 = f(x, y, z, \xi_1, \eta_1, \zeta_1, t), \quad f_1' = f(x, y, z, \xi_1', \eta_1', \zeta_1', t), \quad \} \\
\{ F &= F(x, y, z, \xi, \eta, \zeta, t), \quad F_1 = F(x, y, z, \xi_1, \eta_1, \zeta_1, t), \quad F_1' = F(x, y, z, \xi_1', \eta_1', \zeta_1', t) \}
\end{align*}
\]

Namely, we can verify (114)\(_B\) for \( f \):

\(^{17}\) Hier kann die Integration nach \( b \) und \( \epsilon \) natürlich nicht mehr sofort aus geführt werden, da die in \( f' \) und \( F_1' \) vorkommen den Variablen \( \xi', \eta', \zeta' \) und \( \xi_1', \eta_1', \zeta_1' \) Funktion von \( \xi, \eta, \zeta, \xi_1, \eta_1, \zeta_1, b \) und \( \epsilon \), welche nur berechnet werden können, wenn Virkungsgesetz der während eines Zusammenstosses wirkenden Kräfte gegeben ist. [2, p.112].

\(^{18}\) Boltzmann\(^{1844-1906}\) had put the date in the foreword to part I as September in 1895, part II as August in 1898. We mean the equation number in the left-hand side with (\_\_)\(_B\) the citations from the Boltzmann\(^{2}\) or [1]. We state only the symbol \( f \) instead of \( f_{\infty} \) cf. (107)\(_B\).
### TABLE 4. Combination of function before and after collision

<table>
<thead>
<tr>
<th>item</th>
<th>$V_3$ before</th>
<th>$V_3$ after</th>
<th>$f$ of $V_4$ before</th>
<th>$f$ of $V_4$ after</th>
<th>$F$ of $V_4$ before</th>
<th>$F$ of $V_4$ after</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 function of $m$</td>
<td>$f$</td>
<td>$f'$</td>
<td>$f$</td>
<td>$f'$</td>
<td>$F$</td>
<td>$F'$</td>
</tr>
<tr>
<td>2 function of $m$</td>
<td>$F_1$</td>
<td>$F_2$</td>
<td>$f_1$</td>
<td>$f'$</td>
<td>$F_1$</td>
<td>$F_2$</td>
</tr>
<tr>
<td>3 increment</td>
<td>$f'F_1 - fF_1$</td>
<td>$f'F_1 - fF_1$</td>
<td>$F'F_1 - F'F_1$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[
\frac{V_1 + V_2 + V_3 + V_4}{d\omega dt} = \frac{\partial f}{\partial t} - \left( \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + \zeta \frac{\partial f}{\partial z} \right) + \int_0^\infty\int_0^{2\pi} (f'F_1 - fF_1) g \cdot d\omega_1 dBd\epsilon + \int\int_0^\infty l_0^{2\pi} (f'F_1 - fF_1) g \cdot dw_1 dBd\epsilon.
\]

Similarly we obtain (115) for $F$.

\[
\frac{V_1 + V_2 + V_3 + V_4}{d\omega dt} = \frac{\partial F_1}{\partial t} - \left( \xi \frac{\partial F_1}{\partial x} + \eta \frac{\partial F_1}{\partial y} + \zeta \frac{\partial F_1}{\partial z} \right) + \int_0^\infty\int_0^{2\pi} (f'F_1 - fF_1) g \cdot d\omega_1 dBd\epsilon + \int\int_0^\infty l_0^{2\pi} (f'F_1 - fF_1) g \cdot dw_1 dBd\epsilon.
\]

Here, we confirm the identity with the today’s description of the Boltzmann equations (12) and (13):

\[
\partial_t f + v \cdot \nabla_x f + w \cdot \nabla_v f = Q(f,g) \overline{V_1} \overline{V_2} \overline{V_3}, \overline{V_4},
\]

\[
\partial_t F + v \cdot \nabla_x F + w \cdot \nabla_v F = Q(F,G) \overline{V_1} \tilde{V_2} \overline{V_3}, V_4.
\]

### 4.4. General form of the hydrodynamic equations.

As the general expressions for fluid mechanics, he states that when we substitute for $\frac{\partial f}{\partial t}$ its value from Equation (114)$_B$, it turns into (120)$_B$, (126)$_B$, (140)$_B$, a sum of five terms, each of which has its own physical meaning, as follows:

\[
\begin{align*}
(116)_B \sum_{d\omega,do} \varphi & = \varphi f d\omega d\epsilon, \\
(120)_B \frac{\partial \rho}{\partial t} \sum_{d\omega,do} \varphi & = \left( f \frac{\partial \rho}{\partial t} + \varphi \rho \frac{\partial f}{\partial t} \right) d\omega d\epsilon = \sum_{n=1}^5 A_n(\varphi) d\omega d\epsilon, \\
(171)_B \frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} & = 0
\end{align*}
\]

### 4.5. Special form of the incompressible, hydrodynamic equations.

\[
\begin{align*}
(173)_B \begin{cases}
\rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = \rho X - \frac{\partial (\rho u)}{\partial x} - \frac{\partial (\rho v)}{\partial y} - \frac{\partial (\rho w)}{\partial z}, \\
\rho \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = \rho Y - \frac{\partial (\rho v)}{\partial x} - \frac{\partial (\rho w)}{\partial y} - \frac{\partial (\rho u)}{\partial z}, \\
\rho \left( \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = \rho Z - \frac{\partial (\rho w)}{\partial x} - \frac{\partial (\rho u)}{\partial y} - \frac{\partial (\rho v)}{\partial z}
\end{cases}
\end{align*}
\]
Boltzmann says "these equations as well as Equation (171)\textsuperscript{B}, are only special cases of the general equation (126)\textsuperscript{B} and were derived from it by Maxwell and (following him) by Kirchhoff." Boltzmann concludes that if one collects all these terms, then Equation (126) reduces in this special case to:

$$(177)_{B} \frac{\partial (\rho \overline{\varphi})}{\partial t} + \frac{\partial (\rho \overline{\xi \varphi})}{\partial x} + \frac{\partial (\rho \overline{\eta \varphi})}{\partial y} + \frac{\partial (\rho \overline{\zeta \varphi})}{\partial z} - \rho \left[ X \frac{\partial \overline{\varphi}}{\partial \xi} + Y \frac{\partial \overline{\varphi}}{\partial \eta} + Z \frac{\partial \overline{\varphi}}{\partial \zeta} \right] = m \left[ B_{4}(\varphi) + B_{5}(\varphi) \right]$$

Boltzmann states about (177)\textsuperscript{B}:

From this equation Maxwell calculated the viscosity, diffusion, and heat conduction and Kirchhoff therefore calls it the basic equation of the theory. If one sets $\varphi = 1$, he obtains at once the continuity equation (171); for it follows from Equations (134) and (137) that $B_{4}(1) = B_{5}(1) = 0$. Subtraction of the continuity equation, multiplied by $\varphi$, from (177) gives (using the substitution [158]): [1, p.152],

where, (158) : $\xi = \zeta + u, \eta = \eta_{0} + v, \zeta = \zeta_{0} + w$.

$$(178)_{B} \rho \left( \frac{\partial \overline{\varphi}}{\partial t} + \frac{\partial \overline{\xi \varphi}}{\partial x} + \frac{\partial \overline{\eta \varphi}}{\partial y} + \frac{\partial \overline{\zeta \varphi}}{\partial z} \right) + \rho \left( \frac{\partial (\rho \overline{\xi \varphi})}{\partial x} + \frac{\partial (\rho \overline{\eta \varphi})}{\partial y} + \frac{\partial (\rho \overline{\zeta \varphi})}{\partial z} \right) - \rho \left[ X \frac{\partial \overline{\varphi}}{\partial \xi} + Y \frac{\partial \overline{\varphi}}{\partial \eta} + Z \frac{\partial \overline{\varphi}}{\partial \zeta} \right] = m \left[ B_{4}(\varphi) + B_{5}(\varphi) \right]$$

If one denotes the six quantities (179)\textsuperscript{B} : $\rho \overline{\xi_{0}^{2}}, \rho \overline{\eta_{0}^{2}}, \rho \overline{\zeta_{0}^{2}}, \rho \overline{\xi_{0} \eta_{0}}, \rho \overline{\xi_{0} \zeta_{0}}, \rho \overline{\eta_{0} \zeta_{0}}$ by $X_{x}, Y_{y}, Z_{z}, Y_{z} = Z_{u}, Z_{z} = X_{x}, Y_{y} = Y_{z}$, namely, when we use the symmetric tensor, then we get the following:

$$\left[ \begin{array}{ccc} \rho \overline{\xi_{0}^{2}} & \rho \overline{\eta_{0}^{2}} & \rho \overline{\zeta_{0}^{2}} \\ \rho \overline{\xi_{0} \eta_{0}} & \rho \overline{\xi_{0} \zeta_{0}} & \rho \overline{\eta_{0} \zeta_{0}} \end{array} \right] = \left[ \begin{array}{ccc} X_{x} & X_{y} & X_{z} \\ Y_{x} & Y_{y} & Y_{z} \\ Z_{x} & Z_{y} & Z_{z} \end{array} \right] = \rho \left[ \begin{array}{ccc} P_{1} & T_{2} & T_{3} \\ T_{3} & P_{2} & T_{1} \\ T_{1} & T_{2} & P_{3} \end{array} \right]$$

(23)

$$(180)_{B} \rho \left( \frac{\partial \overline{\varphi}}{\partial t} + \frac{\partial \overline{\xi \varphi}}{\partial x} + \frac{\partial \overline{\eta \varphi}}{\partial y} + \frac{\partial \overline{\zeta \varphi}}{\partial z} \right) + \rho \left( \frac{\partial (\rho \overline{\xi \varphi})}{\partial x} + \frac{\partial (\rho \overline{\eta \varphi})}{\partial y} + \frac{\partial (\rho \overline{\zeta \varphi})}{\partial z} \right) - \rho \left[ X \frac{\partial \overline{\varphi}}{\partial \xi} + Y \frac{\partial \overline{\varphi}}{\partial \eta} + Z \frac{\partial \overline{\varphi}}{\partial \zeta} \right] = \rho X, \rho Y, \rho Z$$

These are not NS equations for lack of the pressure term. Moreover (181)\textsuperscript{B} : $p = \rho \overline{\xi_{0}^{2}} = \rho \overline{\eta_{0}^{2}} = \rho \overline{\zeta_{0}^{2}}, \overline{\xi_{0} \eta_{0}} = \overline{\xi_{0} \zeta_{0}} = \overline{\eta_{0} \zeta_{0}} = 0$. Here, he assumes that from the supposition of isotropy and homogeneity, $p = \frac{1}{3}(X_{x} + Y_{y} + Z_{z})$, which is the same as the principle by Saint-Venant or Stokes.

He deduces a special case of the hydrodynamic equations as follows:

For the present, we assume as a fact of experience that in gases the normal pressure is always nearly equal in all directions, and that tangential elastic forces are very small, so that Equations (181) are approximately true. Substitution of the values given by this equation into Equation (173) yields:

$$(183)_{B} \rho \left( \frac{\partial \overline{\varphi}}{\partial t} + \frac{\partial \overline{\xi \varphi}}{\partial x} + \frac{\partial \overline{\eta \varphi}}{\partial y} + \frac{\partial \overline{\zeta \varphi}}{\partial z} \right) + \rho \left( \frac{\partial (\rho \overline{\xi \varphi})}{\partial x} + \frac{\partial (\rho \overline{\eta \varphi})}{\partial y} + \frac{\partial (\rho \overline{\zeta \varphi})}{\partial z} \right) - \rho \left[ X \frac{\partial \overline{\varphi}}{\partial \xi} + Y \frac{\partial \overline{\varphi}}{\partial \eta} + Z \frac{\partial \overline{\varphi}}{\partial \zeta} \right] = 0$$

which are the so-called Euler equations in incompressible condition of (171)\textsuperscript{B}.

$$(185)_{B} \rho \left( \frac{\partial \overline{\varphi}}{\partial t} + \frac{\partial (\rho \overline{\xi \varphi})}{\partial x} + \frac{\partial (\rho \overline{\eta \varphi})}{\partial y} + \frac{\partial (\rho \overline{\zeta \varphi})}{\partial z} \right) - \rho \left[ X \frac{\partial \overline{\varphi}}{\partial \xi} + Y \frac{\partial \overline{\varphi}}{\partial \eta} + Z \frac{\partial \overline{\varphi}}{\partial \zeta} \right] = 0$$

We set the values of (23) as follows, which is the same tensor as Stokes:

$$(220)_{B} \left[ \begin{array}{ccc} \rho \overline{\xi_{0}^{2}} & \rho \overline{\eta_{0}^{2}} & \rho \overline{\zeta_{0}^{2}} \\ \rho \overline{\xi_{0} \eta_{0}} & \rho \overline{\xi_{0} \zeta_{0}} & \rho \overline{\eta_{0} \zeta_{0}} \\ \rho \overline{\xi_{0} \eta_{0}} & \rho \overline{\xi_{0} \zeta_{0}} & \rho \overline{\eta_{0} \zeta_{0}} \end{array} \right] = \rho \left[ \begin{array}{ccc} p - 2R \left( \frac{\partial \overline{\varphi}}{\partial \xi} + \frac{\partial \overline{\varphi}}{\partial \eta} + \frac{\partial \overline{\varphi}}{\partial \zeta} \right) \right] - R \left( \frac{\partial \overline{\varphi}}{\partial \xi} + \frac{\partial \overline{\varphi}}{\partial \eta} + \frac{\partial \overline{\varphi}}{\partial \zeta} \right) - R \left( \frac{\partial \overline{\varphi}}{\partial \xi} + \frac{\partial \overline{\varphi}}{\partial \eta} + \frac{\partial \overline{\varphi}}{\partial \zeta} \right)$$

where $R = \frac{1}{t} \frac{\partial \overline{\varphi}}{\partial t}$.
From (220)\(_B\), we calculate the components of (185)\(_B\) as follows:

\[
\begin{bmatrix}
\frac{\partial (\rho u_x)}{\partial x} & \frac{\partial (\rho u_y)}{\partial x} & \frac{\partial (\rho u_z)}{\partial x} \\
\frac{\partial (\rho u_x)}{\partial y} & \frac{\partial (\rho u_y)}{\partial y} & \frac{\partial (\rho u_z)}{\partial y} \\
\frac{\partial (\rho u_x)}{\partial z} & \frac{\partial (\rho u_y)}{\partial z} & \frac{\partial (\rho u_z)}{\partial z}
\end{bmatrix}
= \begin{bmatrix}
p \frac{\partial (\rho u_x)}{\partial x} - \frac{\partial \mathcal{R}}{\partial x} \left( \frac{\partial (\rho u_x)}{\partial x} + \frac{\partial (\rho u_y)}{\partial y} + \frac{\partial (\rho u_z)}{\partial z} \right) - \mathcal{R} \frac{\partial (\rho u_y)}{\partial y} - \mathcal{R} \frac{\partial (\rho u_z)}{\partial z} \\
- \mathcal{R} \left( \frac{\partial (\rho u_x)}{\partial x} + \frac{\partial (\rho u_y)}{\partial y} + \frac{\partial (\rho u_z)}{\partial z} \right) - \frac{\partial \mathcal{R}}{\partial x} \left( \frac{\partial (\rho u_x)}{\partial x} + \frac{\partial (\rho u_y)}{\partial y} + \frac{\partial (\rho u_z)}{\partial z} \right) - \mathcal{R} \frac{\partial (\rho u_y)}{\partial y} - \mathcal{R} \frac{\partial (\rho u_z)}{\partial z} \\
- \mathcal{R} \left( \frac{\partial (\rho u_x)}{\partial x} + \frac{\partial (\rho u_y)}{\partial y} + \frac{\partial (\rho u_z)}{\partial z} \right) - \frac{\partial \mathcal{R}}{\partial x} \left( \frac{\partial (\rho u_x)}{\partial x} + \frac{\partial (\rho u_y)}{\partial y} + \frac{\partial (\rho u_z)}{\partial z} \right) - \mathcal{R} \frac{\partial (\rho u_y)}{\partial y} - \mathcal{R} \frac{\partial (\rho u_z)}{\partial z}
\end{bmatrix}
\]

Then, substitution of these values into the equations of motion (185)\(_B\) yields:

\[
\begin{align}
(221)\_B & \begin{cases}
\rho \frac{\partial u_x}{\partial t} + \frac{\partial \mathcal{R}}{\partial x} \left( \frac{\partial (\rho u_x)}{\partial x} + \frac{\partial (\rho u_y)}{\partial y} + \frac{\partial (\rho u_z)}{\partial z} \right) - \mathcal{R} \frac{\partial (\rho u_y)}{\partial y} - \mathcal{R} \frac{\partial (\rho u_z)}{\partial z} = \rho X = 0, \\
\rho \frac{\partial u_y}{\partial t} + \frac{\partial \mathcal{R}}{\partial x} \left( \frac{\partial (\rho u_x)}{\partial x} + \frac{\partial (\rho u_y)}{\partial y} + \frac{\partial (\rho u_z)}{\partial z} \right) - \mathcal{R} \frac{\partial (\rho u_y)}{\partial y} - \mathcal{R} \frac{\partial (\rho u_z)}{\partial z} = \rho Y = 0, \\
\rho \frac{\partial u_z}{\partial t} + \frac{\partial \mathcal{R}}{\partial x} \left( \frac{\partial (\rho u_x)}{\partial x} + \frac{\partial (\rho u_y)}{\partial y} + \frac{\partial (\rho u_z)}{\partial z} \right) - \mathcal{R} \frac{\partial (\rho u_y)}{\partial y} - \mathcal{R} \frac{\partial (\rho u_z)}{\partial z} = \rho Z = 0
\end{cases}
\end{align}
\]

We can interpret that as the special cases, Boltzmann have deduced the \textit{NS} equations after substituting the tensor (220)\(_B\) to (173)\(_B\), for lack of pressure terms. Here, we remark that from Maxwell's viscosity coefficient: \( \mathcal{R} \equiv \frac{M}{2\mu} P \), we get the tensor (220)\(_B\), which equals to (11). The equations (9) equals (185)\(_B\) and (10) equals (221)\(_B\) (= (24)) except for the symbol of viscosity coefficient.

5. Conclusions. Contributions to the \textit{NS} equations

Basically, the \textit{NS} equations were deduced from Newton's kinetic equation (the second law of motion): \( F = mr \),\(^{20}\) however Boltzmann's gas equations were not deduced from it, but he extended the ideas of gas theory including the problem of gas collision by its progenitors Maxwell and Kirchhoff. In fact, Boltzmann had confessed his fear the authority in the preface of the Part II of his book (cf. Appendix).

When we consider the contributions by Boltzmann to the \textit{NS} equations, Boltzmann shows the Euler equations and the \textit{NS} equation as the special case of his general hydrodynamic equations. He verified the validity of the Euler equations and the \textit{NS} equations, which were recognized in 1934 at latest by Prandtl [19, p.259], and at the epoch about one hundred years after Navier's paper [15], read by the referees in 1822 and published in \textit{Mémoires de L'Academie des Sciences de l'Institute de France} in 1827.

Maxwell in 1865, Boltzmann in 1895 and Prandtl[18, 19] in 1904 both used the "well-known hydrodynamic equations" and at latest in 1929, used the nomenclature of "Navier-Stokes equations", using the two-constant not of Navier, but of Saint-Venant, Stokes, and expanded by Maxwell, Kirchhoff and Boltzmann. These three persons verified the hydrodynamic equations without the name as Navier-Stokes equations.

In short, we can state that after formulating by Navier (1827) [15], Cauchy (1828) [5], Poisson (1831) [17], Saint-Venant (1843) [21] and Stokes (1849) [22], the topic of hydrodynamic history are rebuilt by Maxwell (1865) [12], Boltzmann (1895) [1] and Prandtl (1927) [19] in the cyclic interval of about 30 years or so.

As the two constants, Saint-Venant had used \( \epsilon \) and \( \delta \), and Stokes \( \mu \) and \( \frac{\epsilon}{\delta} \), while Boltzmann used \( \mathcal{R} \) and \( \frac{\mathcal{R}}{3} \) after tracing Maxwell.

Boltzmann states hydrodynamic equations as well as the Euler equations of (183)\(_B\). According to Boltzmann's description, we can suppose the fact that the then academic society had not fixed yet the name of these equations, up to 1895 or 1898.

\textbf{References}


\(^{20}(\frac{1}{2})\) By d'Alembert's principle in 1756, from the Newton's kinetic equation (the second law of motion): \( F = mr \), d'Alembert proposed \( F - mr = 0 \), where, \( F \) : the force, \( m \) : the gravity, \( r \) : the acceleration. According to his assertion, the problem of kinetic dynamics turns into that of the static dynamics.

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Remark: we use Lu (in French) in the bibliography meaning "read" date by the referees of the journals, for example MAS. In citing the original paragraphs in our paper, the underscores are of ours.