Problem session

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I. Finite representations of knot groups.

The method of mapping knot groups onto finite groups is a very effective method for distinguishing the groups (see [10, 11, 3]). So, it is natural to ask if this method is always successful at distinguishing the groups (see [11, Page 30]).

**Problem 1** (1) Can we distinguish knot groups by counting the numbers of transitive representations of the knot groups to the symmetric group $S_n$ of degree $n$? To be precise, for a knot group $G$ and a positive integer $n$, let $R(G; n)$ be the set of transitive representations of $G$ to $S_n$ modulo post composition of inner automorphisms of $S_n$. Then its cardinality $|R(G; n)|$ is of course an invariant of the knot group. Is the family of invariants, $\{ |R(G; n)| \}_n$, a complete invariant of the knot group? Namely, for two non-isomorphic knot groups $G_1$ and $G_2$, can we always find a positive integer $n$ such that $|R(G_1; n)| \neq |R(G_2; n)|$?

(2) When a meridian, $\mu$, of $G$ is specified, we can refine $R(G; n)$ as follows. Let $(n_1, n_2, \cdots, n_k)$ be a sequence of positive integers such that $n_1 + n_2 + \cdots + n_k = n$ and $n_1 \leq n_2 \leq \cdots \leq n_k$. Let $R(G, \mu; n_1, n_2, \cdots, n_k)$ be the subset of $R(G; n)$ consisting of those representations which map $\mu$ to a product of mutually disjoint cyclic permutations of length $n_1, n_2, \ldots, n_k$. Then is the family of the invariants, $\{ |R(G, \mu; n_1, n_2, \cdots, n_k)| \}_k$, a complete invariant of $(G, \mu)$?

(3) We can also consider the homology of branched/unbranched coverings associated with transitive representations of $G$ to finite symmetric groups. Is the combination of the invariants $\{ |R(G; n)| \}_n$ (resp. $\{ |R(G, \mu; n_1, n_2, \cdots, n_k)| \}_k$) and the homology of associate finite branched/unbranched coverings a complete invariant of $G$ (resp. $(G, \mu)$)?

**Remark 2** In [3], we had to distinguish various pairs of mutants, and this was carried out by using the above methods with the help of Kodama’s software [2].

Problem 1 motivates the following problem.

**Problem 3** Is it true that two non-isomorphic knot groups have non-isomorphic profinite completions?

II. Simple loops on bridge spheres.

We present variations of the problems on Heegaard splittings of 3-manifolds raised by Y. Minsky [1, Question 5.4]. For a knot $K$ in the 3-sphere $S^3$, let $(S^3, K) = (B^3_i, t_i) \cup (B^3_j, t_j)$ be an $n$-bridge decomposition of $K$ and set $S := \partial B^3_i \setminus t_i (= \partial B^3_j \setminus t_j)$.

**Problem 4** (1) Which essential simple loop in $S$ is null-homotopic in $S^3 \setminus K$?

(2) Which essential simple loops in $S$ are mutually homotopic in $S^3 \setminus K$?

Let $\mathcal{M}(S)$ and $\mathcal{M}(B^3_i, t_i)$ $(i = 1, 2)$, respectively, be the mapping class groups $\pi_0\text{Diff}(S)$ and $\pi_0\text{Diff}(B^3_i, t_i)$. For each $i = 1, 2$, let $\mathcal{M}_0(B^3_i, t_i)$ be the subgroup of

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\[ \mathcal{M}(B_i^3, t_i) \] which consists of elements which induce the identity element in the outer-automorphism group \( \text{Out}(\pi_1(B_i^3 \setminus t_i)) \). Let \( \Gamma \) be the subgroup of \( \mathcal{M}(S) \) generated by \( \mathcal{M}_0(B_i^3, t_i) \cup \mathcal{M}_0(B_i^3, t_2) \). Let \( \Delta_i \ (i = 1, 2) \) be the set of essential simple loops in \( S \) which bounds a disk in \( B_i^3 \setminus t_i \), and let \( \Delta \) be the union of \( \Delta_1 \) and \( \Delta_2 \). Note that \( \Delta \) is a subcomplex of the curve complex \( C^{(0)}(S) \) of \( S \).

**Observation 5** Any simple loop in \( \Gamma \Delta \) is null-homotopic.

**Problem 6** Is the converse true if the bridge decomposition is “complicated enough”?

Let \( \mathcal{PML}(S) \) be the projective measured lamination space of \( S \). Though the action of \( \mathcal{M}(S) \) on \( \mathcal{PML}(S) \) is ergodic, the action of \( \mathcal{M}_0(B_i^3, t_i) \) on \( \mathcal{PML}(S) \) would have a non-empty domain of discontinuity for each \( i = 1, 2 \) (see Masur [9]).

**Problem 7** If the bridge decomposition of \( K \) is “complicated enough”, then does the action of \( \Gamma(\subset \mathcal{M}(S)) \) on \( \mathcal{PML}(S) \) have a nonempty domain of discontinuity?

**Problem 8** Is \( \Gamma \) isomorphic to the free product of \( \mathcal{M}_0(B_1^3, t_1) \) and \( \mathcal{M}_0(B_2^3, t_2) \)?

**Problem 9** Let \( \Omega(\Gamma) \) be the domain of discontinuity of the action of \( \Gamma \) on \( \mathcal{PML}(S) \). If a loop \( c \) on \( S \) belongs to the intersection of \( \Omega(\Gamma) \) and \( C^{(0)}(S) \), then is \( c \) not null-homotopic in \( S^3 \setminus K \)?

**Problem 10** Can we find an open set \( U \) in \( \mathcal{PML}(S) \) such that any loop which belongs to the intersection of \( U \) and \( C^{(0)}(S) \) is not null-homotopic in \( S^3 \setminus K \)?

Let \( \Delta^* \) be the closure in \( \mathcal{PML}(S) \) of the set of loops in \( C^{(0)}(S) \) which is null-homotopic in \( S^3 \setminus K \).

**Problem 11** Does \( \Delta^* \) have measure 0?

**Remark 12** For 2-bridge spheres of 2-bridge links, Problems 4 - 11 are solved affirmatively (see [4, 5, 6, 7, 8]). In particular, for a 2-bridge link \( K(r) \), the action of \( \Gamma \) on \( \mathcal{PML}(S) \) has the domain of discontinuity, and the union of two intervals \( I_1 \cup I_2 \) in Figure 1 forms a fundamental domain.
References


