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SYMMETRIES ON REDUCED BERS BOUNDARIES

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This note is based on the author's talk in RIMS on 23 June 2011.
There are several ways to compactify Teichmüller space. In this
note, we shall focus on one of them, the Bers compactification of Te-
ichmüller space, which was first introduced by Bers in [1]. This com-
 pactification played, and still plays, an important role in the study of
Kleinian groups. Indeed, first examples of (finitely generated) geomet-
rically infinite Kleinian groups were found on the boundary of the Bers
compactification.

Let us explain what is the Bers compactification first. Let \( S \) be a
closed orientable surface of genus \( g \geq 2 \). (More in general, we can
consider any hyperbolic surface of finite type.) Let \( AH(S) \) denote the
space of faithful discrete representations of \( \pi_1(S) \) into \( \text{PSL}_2\mathbb{C} \) mod-
ulo conjugacy. We endow \( AH(S) \) with the topology induced from the
representation space. The interior of \( AH(S) \) is know to coincide with
the space of quasi-Fuchsian representations of \( \pi_1(S) \), which we denote
by \( QF(S) \), by work of Ahlfors, Bers and Sullivan. Also, the simulta-
nous uniformisation due to Ahlfors and Bers implies that \( QF(S) \)
is parametrised as \( qf : T(S) \times T(\bar{S}) \rightarrow QF(S) \), where \( T(S) \) denotes
the Teichmüller space of \( S \) and \( T(\bar{S}) \) the Teichmüller space of \( S \) with
orientation reversing markings. Now, fix any point \( m_0 \) in \( T(S) \), and
consider the slice \( qf(\{m_0\} \times T(\bar{S})) \). Bers showed that this slice is rela-
tively compact in \( AH(S) \) for any \( m_0 \). Its closure \( qf(\{m_0\} \times T(\bar{S})) \)
is called the Bers compactification of the Teichmüller space \( T(S) \), identi-
fying \( T(S) \) with \( T(\bar{S}) \) by the complex conjugation, and its boundary,
which we denote by \( \partial^{B}_{m_0}T(S) \), is called the Bers boundary of the Te-
ichmüller space.

A drawback of this compactification is the fact that it depends on the
base point \( m_0 \) in the first factor of the parametrisation. Indeed, Ker-
ckhoff and Thurston showed in [7] that there exist two points \( m_0, m_1 \)
for which there is no continuous extension of the natural identification
between \( qf(\{m_0\} \times T(\bar{S})) \) and \( qf(\{m_1\} \times T(\bar{S})) \) to the Bers bound-
aries. Moreover, they also showed that the action of the mapping class
group on \( T(S) \) does not extend continuously to the Bers boundary.
Looking at their proof of these facts, we can see that the cause of these phenomena lies in the fact that Bers boundaries contain non-trivial quasi-conformal deformation spaces. Therefore, we can expect if we mod out Bers boundaries by collapsing every quasi-conformal deformation space into a point, these phenomena would disappear. We call thus obtained spaces the reduced Bers boundaries. Let us state the definition more formally.

**Definition 1.** For the Teichmüller space $\mathcal{T}(S)$ of $S$, let $q_m : \mathcal{T}(S) \to AH(S)$ be the Bers embedding with basepoint at $m_0$. Let $\partial_{m_0} B \mathcal{T}(S)$ be the boundary of $\mathcal{T}(S)$ an equivalence relation $\sim$ such that two points $x, y \in \partial_{m_0} B \mathcal{T}(S)$ are $\sim$-equivalent if and only if they are quasi-conformally conjugate to each other. We consider the quotient space $\partial_{m_0} B \mathcal{T}(S)/ \sim$, which we call the reduced Bers boundary with basepoint at $m_0$ and denote by $\partial_{m_0} B \mathcal{T}(S)$. We also consider the reduced Bers compactification with basepoint at $m_0$, which is $\mathcal{T}(S) \cup \partial_{m_0} B \mathcal{T}(S)$ endowed with the quotient topology induced from the Bers compactification $\mathcal{T}(S) \cup \partial_{m_0} B \mathcal{T}(S)$. As it is clear from the context which Teichmüller space we are talking about, we omit $\mathcal{T}(S)$ and use the symbols $\partial_{m_0} B$ and $\partial_{m_0} B$ for simplicity.

According to McMullen [6], Thurston already considered this space back in 1980's, and conjectured that this space is independent of the basepoint. We have shown that this is indeed the case.

**Theorem 2.** Let $m_1, m_2$ be two points in $\mathcal{T}(S)$. Then there is a homeomorphism from $\partial_{m_1} B$ to $\partial_{m_2} B$ which is an extension of the natural identification between $qf(m_1 \times \mathcal{T}(\bar{S}))$ and $qf(m_2 \times \mathcal{T}(\bar{S}))$.

As a corollary, we see that the mapping class group or the extended mapping class group acts on $\partial_{m_0} B$ for any $\partial_{m_0} B \in \mathcal{T}(S)$ as an extension of the natural action on $\mathcal{T}(S)$.

Now, let us fix a basepoint $m_0$ once and for all, and omitting the subscript of $\partial_{m_0} B$, denote it by $\partial_{m_0} B$. By the ending lamination theorem proved by Brock-Canary-Minsky [2], and the invariance of ending laminations under quasi-conformal deformations, we see that for any fixed $m_0$, there is an injection $e$ from $\partial_{m_0} B$ to the unmeasured lamination $\mathcal{UML}(S)$.

To understand its image, we introduce the following subspace.

**Definition 3.** We set $\mathcal{UML}_0(S)$ to be the subset of $\mathcal{UML}(S)$ consisting of unmeasured laminations $\lambda$ such that for each component $\lambda_0$ of $\lambda$ that is not a simple closed curve, every frontier component of the minimal supporting surface of $\lambda_0$ is contained in $\lambda$. 
Then, by using the result of [8], we can easily see that the image of $e$ coincides with $\mathcal{UML}_0(S)$. Still, we see that $e$ is far from a homeomorphism. Indeed, we can show the following.

**Proposition 4.** Neither $e$ nor $e^{-1}$ is continuous.

As a topological space, $\partial^{RB}$ is not Hausdorff, and more strongly is not $T_1$ either. This kind of space may look very hard to deal with. On the other hand, this non-separability is useful in showing that there are not many symmetries in $\partial^{RB}$.

Inspired by Papadopoulos’ work [9], we have shown the following.

**Theorem 1.** Suppose that $\xi(S) > 4$ (i.e. $\dim T(S) > 2$). Let $f : \partial^{RB} \to \partial^{RB}$ be a homeomorphism. Then there exists a diffeomorphism $h : S \to S$ which induces $f$ on $\partial^{RB}$. Furthermore, unless $S$ is a closed surface of genus 2, two diffeomorphisms $h, h' : S \to S$ inducing the same homeomorphism on $\partial^{RB}$ are isotopic.

To prove this theorem, we shall introduce the notion of the adherence height.

**Definition 5.** A point $b$ in $\partial^{RB}$ is said to be unilaterally adherent to $a$ in $\partial^{RB}$ if every neighbourhood of $b$ contains $a$. (We are not excluding the possibility that $a$ is also unilaterally adherent to $b$ although we say “unilaterally”. We put this adverb to distinguish our definition from that of “adherence” by Papadopoulos [9], which is symmetric with regard to $a$ and $b$.) Let $T = (a_0, \ldots, a_n)$ be an ordered subset of $\partial^{RB}$. The set $T$ is said to be an adherence tower if $a_j$ is unilaterally adherent to $a_1, \ldots, a_{j-1}$, and we call $n$ the length of $T$. We define the adherence height of $a \in \partial^{RB}$ to be the supremum of the lengths of the adherence towers starting from $a$. We denote the adherence height of $a$ by a.h.$(a)$.

Obviously, any auto-homeomorphism of $\partial^{RB}$ preserves the adherence height.

Furthermore, by using results in Kleinian groups, we can show the following lemma and proposition.

**Lemma 6.** For a point $a \in \partial^{RB}$, we have a.h.$(a) = \dim \pi^{-1}(a)/2$.

Let $C(S)$ denote the curve complex of $S$. Then by considering barycentres, each simplex of $C(S)$ is regarded as a point in $\mathcal{UML}_0(S)$. We denote the restriction of $e^{-1}$ on the set of simplices of $C(S)$ by $\iota$.

**Proposition 7.** Let $f : \partial^{RB} \to \partial^{RB}$ be a homeomorphism. Then, there is a simplicial automorphism $f' : C(S) \to C(S)$ such that $\iota(f'(c)) = f(\iota(c))$ for every simplex $c$ of $C(S)$.
Now, applying the results of Ivanov, Luo and Korkmaz, we see that there is a diffeomorphism $g$ inducing $f'$ as above.

Finally, we can show that $g$ induces the same homeomorphism as $f$ in the entire $\partial^{RB}$ by using the following lemma.

**Lemma 8.** Let $b$ be a point in $\partial^{RB}$ with $a.h.(b) = k$. Then there is a sequence $\{a_i\}$ in $\partial^{reg}$ which converges to $b$, such that for any point $d$ other than $b$ that is contained in the limit of $\{a_i\}$, we have $a.h.(d) < a.h.(b)$.

**REFERENCES**