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Anti-Loewner matrices; Numerical radius and unitarity

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We review results on two topics by Hidaka and Sano; Sano and A. Uchiyama. For details, we refer [4, 5].

1 Anti-Loewner matrices

Let f be a positive C^1 function on $(0,\infty)$. Let H^n be the subspace of $\mathbb{C}^n$ consisting of all $x=(x_1,\ldots,x_n)^T\in\mathbb{C}^n$ for which $\sum_{i=1}^{n} x_i=0$. An $n\times n$ Hermitian matrix A is said to be *conditionally positive definite* (c.p.d. for short) if

$$\langle x,Ax\rangle\geqq 0 \quad \text{for all} \quad x\in H^n,$$

and *conditionally negative definite* (c.n.d. for short) if $-A$ is c.p.d.

For positive numbers $t_1,\ldots,t_n$, the matrices

$$K_f(t_1,\ldots,t_n)=\left[\frac{f(t_i)+f(t_j)}{t_i+t_j}\right]$$

have been of some interest. We call it an *anti-Loewner matrix.* Kwong showed that if f is a non-negative operator monotone function on $(0,\infty)$ then all K_f are p.s.d. On the other hand, it is shown in [3] that if f is operator convex on $[0,\infty)$ with $f(0)\leqq 0$, or $f(t)=tg(t)$ for an operator convex function g with $f''(0)\geqq 0$ then all K_f are c.n.d.

Recently, Audenaert in [2] gives a characterisation of functions f for which all K_f are p.s.d; by [2, Theorem 2.1], for a positive C^1 function f on $(0,\infty)$, all K_f are p.s.d. if and only if $f(\sqrt{t})\sqrt{t}$ is matrix monotone of any order n, i.e., operator monotone. Hence, such a function f is of the form

$$f(t)=\frac{\alpha}{t}+\beta t+\int_0^\infty \frac{t}{\lambda+t^2}\,d\nu(\lambda), \tag{1.1}$$

where $\alpha, \beta \geqq 0$ and ν is a positive measure on $(0, \infty)$.

Here are our complementary results in [4]:

Theorem 1.1. Let f be a positive, differentiable function on $(0, \infty)$ with $f(0) = f'(0) = 0$ and $t_1, t_2, \ldots, t_n > 0$ given. Suppose that $K_f(t_1, \ldots, t_n, t_{n+1})$ is c.n.d. for any $t_{n+1} > 0$. Then $K_{f(t)/t^2}(t_1, \ldots, t_n)$ is p.s.d. Conversely, if $K_{f(t)/t^2}(t_1, \ldots, t_n)$ is p.s.d., then $K_f(t_1, \ldots, t_n)$ is c.n.d.

By Audenaert's characterisation (1.1),

Theorem 1.2. Let f be a positive C^1 function on $(0, \infty)$ with $f(0) = f'(0) = 0$. Then all K_f are c.n.d. if and only if all $K_{f(t)/t^2}$ are p.s.d. or f is of the form

$$f(t) = \beta t^3 + \int_0^\infty \frac{t^3}{\lambda + t^2} \, d\nu(\lambda), \tag{1.2}$$

where $\beta \geqq 0$ and ν is a positive measure on $(0, \infty)$.

In the case where f is of the form (1.2), we can consider the inverse f^{-1} of f.

Corollary 1.3. Let f be a positive C^1 function of the form (1.2). Then $K_{f^{-1}}$ is infinitely divisible.

Proposition 1.4. (1) For a function f on $(0, \infty)$, $K_f(t_1, t_2)$ are c.n.d. for all $t_1, t_2 > 0$ if and only if $f(t)/t$ is increasing.
(2) For a non-negative function f on $(0, \infty)$, $K_f(t_1, t_2)$ are p.s.d. for all $t_1, t_2 > 0$ if and only if $f(t)/t$ is decreasing and $tf(t)$ is increasing.

Corollary 1.5. For $f(t) = t^p$ $(p \in \mathbb{R})$ on $(0, \infty)$, the following hold:
(1) $K_f(t_1, t_2)$ are c.n.d. for all $t_1, t_2 > 0$ if and only if $1 \leqq p$.
(2) $K_f(t_1, t_2)$ are p.s.d. for all $t_1, t_2 > 0$ if and only if $-1 \leqq p \leqq 1$.
(3) $K_f(t_1, t_2, t_3)$ are c.n.d. for all $t_1, t_2, t_3 > 0$ if and only if $1 \leqq p \leqq 3$.

2 Numerical radius and unitarity

Let $\mathcal{H}$ be a Hilbert space and $B(\mathcal{H})$ denote the set of all bounded linear operators on $\mathcal{H}$. Here we study the following condition: for an invertible operator $A \in B(\mathcal{H})$,

$$|\langle A\xi, \xi\rangle| \leqq 1, \quad |\langle A^{-1}\xi, \xi\rangle| \leqq 1$$

for all unit vectors $\xi \in \mathcal{H}$. In this case, we show that A is unitary. It is clear that A is unitary if A is invertible, $\|A\| \leqq 1$, and $\|A^{-1}\| \leqq 1$. Hence, our theorem means that the operator norm can be replaced by the numerical radius; for $A \in B(\mathcal{H})$ the numerical range $W(A)$ and the numerical radius $w(A)$ are defined as

$$\begin{aligned} W(A) &= \{\langle A\xi, \xi\rangle : \|\xi\| = 1\}, \\ w(A) &= \sup\{|\langle A\xi, \xi\rangle| : \|\xi\| = 1\}. \end{aligned}$$

We remark that the main result already appeared as Corollary 1 to Theorem 1 in [7] and as Theorem B in [6] with a more general result, whose proof seems to be involved.

Theorem 2.1. Let $A \in B(\mathcal{H})$ be invertible. If $w(A) \leqq 1$ and $w(A^{-1}) \leqq 1$, then A is unitary.

Proof. Let $A = U|A|$ be the polar decomposition. Since $(A^{-1})^* = (|A|^{-1}U^{-1})^* = U|A|^{-1}$, $w(U|A|^{-1}) = w(A^{-1}) \leqq 1$. Let $B := U\dfrac{|A| + |A|^{-1}}{2}$. Then $w(B) \leqq 1$, and $|B| = \dfrac{|A| + |A|^{-1}}{2} \geqq I$. Applying the following lemma, we have $|B| = I$ or $|A| = I$; therefore, A is unitary.

■

Lemma 2.2. Let $B \in B(\mathcal{H})$ be invertible. If $w(B) \leqq 1$ and $|B| \geqq I$, then B is unitary.

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