Anti-Loewner matrices; Numerical radius and unitarity

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We review results on two topics by Hidaka and Sano; Sano and A. Uchiyama. For details, we refer [4, 5].

1 Anti-Loewner matrices

Let \( f \) be a positive \( C^1 \) function on \((0, \infty)\). Let \( H^n \) be the subspace of \( \mathbb{C}^n \) consisting of all \( x = (x_1, \ldots, x_n)^T \in \mathbb{C}^n \) for which \( \sum_{i=1}^{n} x_i = 0 \). An \( n \times n \) Hermitian matrix \( A \) is said to be conditionally positive definite (c.p.d. for short) if

\[
\langle x, Ax \rangle \geq 0 \quad \text{for all} \quad x \in H^n,
\]
and conditionally negative definite (c.n.d. for short) if \(-A\) is c.p.d.

For positive numbers \( t_1, \ldots, t_n \), the matrices

\[
K_f(t_1, \ldots, t_n) = \begin{bmatrix}
\frac{f(t_i) + f(t_j)}{t_i + t_j}
\end{bmatrix}
\]

have been of some interest. We call it an anti-Loewner matrix. Kwong showed that if \( f \) is a non-negative operator monotone function on \((0, \infty)\) then all \( K_f \) are p.s.d. On the other hand, it is shown in [3] that if \( f \) is operator convex on \([0, \infty)\) with \( f(0) \leq 0 \), or \( f(t) = tg(t) \) for an operator convex function \( g \) with \( f''(0) \geq 0 \) then all \( K_f \) are c.n.d.

Recently, Audenaert in [2] gives a characterisation of functions \( f \) for which all \( K_f \) are p.s.d; by [2, Theorem 2.1], for a positive \( C^1 \) function \( f \) on \((0, \infty)\), all \( K_f \) are p.s.d. if and only if \( f(\sqrt{t})\sqrt{t} \) is matrix monotone of any order \( n \), i.e., operator monotone. Hence, such a function \( f \) is of the form

\[
f(t) = \frac{\alpha}{t} + \beta t + \int_{0}^{\infty} \frac{t}{\lambda + t^2} d\nu(\lambda),
\]

(1.1)
where \( \alpha, \beta \geq 0 \) and \( \nu \) is a positive measure on \((0, \infty)\).

Here are our complementary results in [4]:

**Theorem 1.1.** Let \( f \) be a positive, differentiable function on \((0, \infty)\) with \( f(0) = f'(0) = 0 \) and \( t_1, t_2, \ldots, t_n > 0 \) given. Suppose that \( K_f(t_1, \ldots, t_n, t_{n+1}) \) is c.n.d. for any \( t_{n+1} > 0 \). Then \( K_{f(t)/t^2}(t_1, \ldots, t_n) \) is p.s. Conversely, if \( K_{f(t)/t^2}(t_1, \ldots, t_n) \) is p.s., then \( K_f(t_1, \ldots, t_n) \) is c.n.d.

By Audenaert's characterisation (1.1),

**Theorem 1.2.** Let \( f \) be a positive \( C^1 \) function on \((0, \infty)\) with \( f(0) = f'(0) = 0 \). Then all \( K_f \) are c.n.d. if and only if all \( K_{f(t)/t^2} \) are p.s. or \( f \) is of the form

\[
f(t) = \beta t^3 + \int_0^\infty \frac{t^3}{\lambda + t^2} \, d\nu(\lambda),
\]

(1.2)

where \( \beta \geq 0 \) and \( \nu \) is a positive measure on \((0, \infty)\).

In the case where \( f \) is of the form (1.2), we can consider the inverse \( f^{-1} \) of \( f \).

**Corollary 1.3.** Let \( f \) be a positive \( C^1 \) function of the form (1.2). Then \( K_{f^{-1}} \) is infinitely divisible.

**Proposition 1.4.** (1) For a function \( f \) on \((0, \infty)\), \( K_f(t_1, t_2) \) are c.n.d. for all \( t_1, t_2 > 0 \) if and only if \( f(t)/t \) is increasing.

(2) For a non-negative function \( f \) on \((0, \infty)\), \( K_f(t_1, t_2) \) are p.s. for all \( t_1, t_2 > 0 \) if and only if \( f(t)/t \) is decreasing and \( tf(t) \) is increasing.

**Corollary 1.5.** For \( f(t) = t^p \) (\( p \in \mathbb{R} \)) on \((0, \infty)\), the following hold:

(1) \( K_f(t_1, t_2) \) are c.n.d. for all \( t_1, t_2 > 0 \) if and only if \( 1 \leq p \).

(2) \( K_f(t_1, t_2) \) are p.s. for all \( t_1, t_2 > 0 \) if and only if \( -1 \leq p \leq 1 \).

(3) \( K_f(t_1, t_2, t_3) \) are c.n.d. for all \( t_1, t_2, t_3 > 0 \) if and only if \( 1 \leq p \leq 3 \).

2 **Numerical radius and unitarity**

Let \( \mathcal{H} \) be a Hilbert space and \( B(\mathcal{H}) \) denote the set of all bounded linear operators on \( \mathcal{H} \). Here we study the following condition: for an invertible operator \( A \in B(\mathcal{H}) \),

\[
|\langle A\xi, \xi \rangle| \leq 1, \quad |\langle A^{-1}\xi, \xi \rangle| \leq 1
\]
for all unit vectors $\xi \in \mathcal{H}$. In this case, we show that $A$ is unitary. It is clear that $A$ is unitary if $A$ is invertible, $\|A\| \leq 1$, and $\|A^{-1}\| \leq 1$. Hence, our theorem means that the operator norm can be replaced by the numerical radius; for $A \in B(\mathcal{H})$ the numerical range $W(A)$ and the numerical radius $w(A)$ are defined as

$$W(A) = \{\langle A\xi, \xi \rangle : \|\xi\| = 1\},$$

$$w(A) = \sup\{\|\langle A\xi, \xi \rangle\| : \|\xi\| = 1\}.$$  

We remark that the main result already appeared as Corollary 1 to Theorem 1 in [7] and as Theorem B in [6] with a more general result, whose proof seems to be involved.

**Theorem 2.1.** Let $A \in B(\mathcal{H})$ be invertible. If $w(A) \leq 1$ and $w(A^{-1}) \leq 1$, then $A$ is unitary.

**Proof.** Let $A = U|A|$ be the polar decomposition. Since $(A^{-1})^* = (|A|^{-1}U^{-1})^* = U|A|^{-1}$, $w(U|A|^{-1}) = w(A^{-1}) \leq 1$. Let $B := U \frac{|A| + |A|^{-1}}{2}$. Then $w(B) \leq 1$, and $|B| = \frac{|A| + |A|^{-1}}{2} \geq I$. Applying the following lemma, we have $|B| = I$ or $|A| = I$; therefore, $A$ is unitary.

**Lemma 2.2.** Let $B \in B(\mathcal{H})$ be invertible. If $w(B) \leq 1$ and $|B| \geq I$, then $B$ is unitary.

### 参考文献


