<table>
<thead>
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<th>Title</th>
<th>Anti-Loewner matrices: Numerical radius and unitarity (Structural study of operators via spectra or numerical ranges)</th>
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<tbody>
<tr>
<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2012), 1778: 117-120</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2012-02</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/171789">http://hdl.handle.net/2433/171789</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
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Anti-Loewner matrices;
Numerical radius and unitarity

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We review results on two topics by Hidaka and Sano; Sano and A. Uchiyama. For details, we refer [4, 5].

1 Anti-Loewner matrices

Let $f$ be a positive $C^1$ function on $(0, \infty)$. Let $H^n$ be the subspace of $\mathbb{C}^n$ consisting of all $x = (x_1, \ldots, x_n)^T \in \mathbb{C}^n$ for which $\sum_{i=1}^{n} x_i = 0$. An $n \times n$ Hermitian matrix $A$ is said to be conditionally positive definite (c.p.d. for short) if

$$\langle x, Ax \rangle \geq 0 \quad \text{for all} \quad x \in H^n,$$

and conditionally negative definite (c.n.d. for short) if $-A$ is c.p.d.

For positive numbers $t_1, \ldots, t_n$, the matrices

$$K_f(t_1, \ldots, t_n) = \begin{bmatrix} f(t_i) + f(t_j) \end{bmatrix}_{t_i + t_j}$$

have been of some interest. We call it an anti-Loewner matrix. Kwong showed that if $f$ is a non-negative operator monotone function on $(0, \infty)$ then all $K_f$ are p.s.d. On the other hand, it is shown in [3] that if $f$ is operator convex on $[0, \infty)$ with $f(0) \leq 0$, or $f(t) = tg(t)$ for an operator convex function $g$ with $f''(0) \geq 0$ then all $K_f$ are c.n.d.

Recently, Audenaert in [2] gives a characterisation of functions $f$ for which all $K_f$ are p.s.d; by [2, Theorem 2.1], for a positive $C^1$ function $f$ on $(0, \infty)$, all $K_f$ are p.s.d. if and only if $f(\sqrt{t})\sqrt{t}$ is matrix monotone of any order $n$, i.e., operator monotone. Hence, such a function $f$ is of the form

$$f(t) = \frac{\alpha}{t} + \beta t + \int_{0}^{\infty} \frac{t}{\lambda + t^2} d\nu(\lambda),$$

(1.1)
where $\alpha, \beta \geq 0$ and $\nu$ is a positive measure on $(0, \infty)$.

Here are our complementary results in [4]:

**Theorem 1.1.** Let $f$ be a positive, differentiable function on $(0, \infty)$ with $f(0) = f'(0) = 0$ and $t_1, t_2, \ldots, t_n > 0$ given. Suppose that $K_f(t_1, \ldots, t_n, t_{n+1})$ is c.n.d. for any $t_{n+1} > 0$. Then $K_{f(t)/t^2}(t_1, \ldots, t_n)$ is p.s.d. Conversely, if $K_{f(t)/t^2}(t_1, \ldots, t_n)$ is p.s., then $K_f(t_1, \ldots, t_n)$ is c.n.d.

By Audenaert's characterisation (1.1),

**Theorem 1.2.** Let $f$ be a positive $C^1$ function on $(0, \infty)$ with $f(0) = f'(0) = 0$. Then all $K_f$ are c.n.d. if and only if all $K_{f(t)/t^2}$ are p.s.d. or $f$ is of the form

$$f(t) = \beta t^3 + \int_0^\infty \frac{t^3}{\lambda + t^2} d\nu(\lambda), \quad (1.2)$$

where $\beta \geq 0$ and $\nu$ is a positive measure on $(0, \infty)$.

In the case where $f$ is of the form (1.2), we can consider the inverse $f^{-1}$ of $f$.

**Corollary 1.3.** Let $f$ be a positive $C^1$ function of the form (1.2). Then $K_{f^{-1}}$ is infinitely divisible.

**Proposition 1.4.** (1) For a function $f$ on $(0, \infty)$, $K_f(t_1, t_2)$ are c.n.d. for all $t_1, t_2 > 0$ if and only if $f(t)/t$ is increasing.

(2) For a non-negative function $f$ on $(0, \infty)$, $K_f(t_1, t_2)$ are p.s.d. for all $t_1, t_2 > 0$ if and only if $f(t)/t$ is decreasing and $tf(t)$ is increasing.

**Corollary 1.5.** For $f(t) = t^p$ ($p \in \mathbb{R}$) on $(0, \infty)$, the following hold:

(1) $K_f(t_1, t_2)$ are c.n.d. for all $t_1, t_2 > 0$ if and only if $1 \leq p$.

(2) $K_f(t_1, t_2)$ are p.s.d. for all $t_1, t_2 > 0$ if and only if $-1 \leq p \leq 1$.

(3) $K_f(t_1, t_2, t_3)$ are c.n.d. for all $t_1, t_2, t_3 > 0$ if and only if $1 \leq p \leq 3$.

2 Numerical radius and unitarity

Let $\mathcal{H}$ be a Hilbert space and $B(\mathcal{H})$ denote the set of all bounded linear operators on $\mathcal{H}$. Here we study the following condition: for an invertible operator $A \in B(\mathcal{H})$,

$$|\langle A\xi, \xi \rangle| \leq 1, \quad |\langle A^{-1}\xi, \xi \rangle| \leq 1$$
for all unit vectors $\xi \in \mathcal{H}$. In this case, we show that $A$ is unitary. It is clear that $A$ is unitary if $A$ is invertible, $\|A\| \leq 1$, and $\|A^{-1}\| \leq 1$. Hence, our theorem means that the operator norm can be replaced by the numerical radius; for $A \in B(\mathcal{H})$ the numerical range $W(A)$ and the numerical radius $w(A)$ are defined as

$$W(A) = \{ \langle A\xi, \xi \rangle : \|\xi\| = 1 \},$$

$$w(A) = \sup \{ |\langle A\xi, \xi \rangle| : \|\xi\| = 1 \}.$$ 

We remark that the main result already appeared as Corollary 1 to Theorem 1 in [7] and as Theorem B in [6] with a more general result, whose proof seems to be involved.

**Theorem 2.1.** Let $A \in B(\mathcal{H})$ be invertible. If $w(A) \leq 1$ and $w(A^{-1}) \leq 1$, then $A$ is unitary.

**Proof.** Let $A = U|A|$ be the polar decomposition. Since $(A^{-1})^* = (|A|^{-1}U^{-1})^* = U|A|^{-1}$, $w(U|A|^{-1}) = w(A^{-1}) \leq 1$. Let $B := U\frac{|A| + |A|^{-1}}{2}$. Then $w(B) \leq 1$, and $|B| = \frac{|A| + |A|^{-1}}{2} \geq I$. Applying the following lemma, we have $|B| = I$ or $|A| = I$; therefore, $A$ is unitary.

**Lemma 2.2.** Let $B \in B(\mathcal{H})$ be invertible. If $w(B) \leq 1$ and $|B| \geq I$, then $B$ is unitary.

参考文献


