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Numerical range of a matrix associated with the graph of a trigonometric polynomial

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Abstract

We present a determinantal representation of a hyperbolic ternary form associated with a trigonometric polynomial. The result is obtained by a joint work with Professor Mao-Ting Chien.

Keywords: curves, numerical range, curves
Mathematics Subject Classification: 15A60, 14Q05

1. Lax-Fiedler conjecture

Suppose that $A$ is an $n \times n$ complex matrix. The numerical range $W(A)$ of $A$ is defined as

$$ W(A) = \{ \xi^* A \xi : \xi \in \mathbb{C}^n, \xi^* \xi = 1 \}. $$

(1.1)

In 1918 Toeplitz introduced this set $W(A)$. He characterized $\partial W(A)$ by

$$ \max \{ \Re(e^{-i\theta}z) : z \in W(A) \} = \max \sigma(H(\theta : A)), $$

(1.2)

where

$$ H(\theta : A) = \frac{1}{2} (e^{-i\theta} A + e^{i\theta} A^*), $$

$$ \sigma(H) = \{ \lambda \in \mathbb{R} : \det(\lambda I - H) = 0 \}, $$

(1.3)

for $H = H^*$. In 1919 Hausdorff proved the simply connectedness of the range $W(A)$. The simply connectedness of the numerical range is also valid for a linear matrix pencil $A\lambda + B$ with $0 \notin W(A)$ ([20]). To compute the eigenvalues of $H(\theta : A)$ we introduce a ternary form

$$ F_A(t, x, y) = \det(tI_n + x/2(A + A^*) - yi/2(A - A^*)). $$

(1.4)

By the equation

$$ \det(tI_n - H(\theta : A)) = F_A(t, -\cos \theta, -\sin \theta), $$

this ternary form determines the eigenvalues of $H(\theta)$ for every angle $\theta$. 
In 1951, Kippenhahn [15] showed that

\[ W(A) = \text{Conv}\{X + iY : (X, Y) \in \mathbb{R}^2, Xx + Yy + 1 = 0 \} \text{ is a tangent of} \]

\[ F(1, x, y) = 0 \].

By this result, the boundary of the numerical range \( W(A) \) lies on the dual curve of the algebraic curve \( F(1, x, y) = 0 \) when \( W(A) \) is strictly convex.

The form \( F_A(t, x, y) \) satisfies (i) \( F_A(1,0,0) > 0 \) and (ii) For every \((x_0, y_0) \in \mathbb{R}^2\), the equation \( F_A(t, x_0, y_0) = 0 \) in \( t \) has \( n \) real solutions counting the multiplicities of the solutions. In 1981, Fiedler [11] conjectured: If \( F(t, x, y) \) is a real ternary form of degree \( n \) and satisfies (i) \( F(1,0,0) = c > 0 \) and (ii) For every \((x_0, y_0) \in \mathbb{R}^2\), the equation \( F(t, x_0, y_0) = 0 \) in \( t \) has \( n \) real solutions counting the multiplicities of the solutions, then there exists an \( n \times n \) complex matrix \( A \) with

\[ F(t, x, y) = c \det(tI_n + x/2(A + A^*) - yi/2(A - A^*)). \] (1.5)

If a ternary form \( F(t, x, y) \) satisfies the above conditions (i) and (ii), then the form is said to be hyperbolic with respect to \((1,0,0)\) ([1]). Before Fiedler's formulation, Lax [16] conjectured more strong result in 1958: the above conditions (i), (ii) for \( F \) implies the existence of a pair of real symmetric matrices \( H, K \) satisfying

\[ F(t, x, y) = c \det(tI_n + xH + yK). \] (1.6)

In 2007, Helton and Vinnikov [13] showed that the Lax conjecture is true (cf. [17]). Hence the Fiedler conjecture is true.

We shall consider the determinantal representations of a homogeneous polynomial. Whether a complex homogeneous polynomial \( F(x_1, x_2, \ldots, x_m) \) \((m \geq 2)\) with Degree \( n \) in \( m \) indeterminates \( x_1, \ldots, x_m \) can be represented as

\[ F(x_1, x_2, \ldots, x_n) = \det(x_1A_1 + x_2A_2 + \cdots + x_nA_n), \] (1.7)

for some \( n \times n \) complex matrices \( A_1, A_2, \ldots, A_n \) or not?

In the case \( m = 2 \), the form \( F \) is expressed as

\[ \prod_{j=1}^{n}(\alpha_jx_1 + \beta_jx_2). \]

Hence the diagonal matrices \( A_1 = \text{diag}(\alpha_1, \ldots, \alpha_n), A_2 = \text{diag}(\beta_1, \ldots, \beta_n) \) satisfy (1.7). The following results are known.
Theorem [ A. C. Dixon, 1901, [9]] For every (non-zero) complex ternary form $F(t, x, y)$ of degree $n$, there are $n \times n$ complex symmetric matrices $A_1, A_2, A_3$ satisfying

$$F(t, x, y) = \det(tA_1 + xA_2 + yA_3).$$

Theorem [L. E. Dickson, 1920, [10]] A generic homogeneous polynomials in $m$ variables of degree $n$ has a representation

$$\det(x_1 A_1 + x_2 A_2 + \ldots + x_m A_m) = 0$$

by $n \times n$ matrices $A_1, A_2, \ldots, A_m$ if and only if

1. $m = 3$ (curves),
2. $m = 4$ and $n = 2, 3$ (surfaces),
3. $m = 4$ and $n = 2$ (threefolds).

Theorem [V. Vinnikov, 1993. [21]] An irreducible real algebraic curve $F(t, x, y) = 0$ has a representation

$$\det(tH_1 + xH_2 + yH_3) = 0,$$

by Hermitian matrices $H_1, H_2, H_3$.

We remark that if $H_1$ in (1.8) is positive definite, then the real ternary form $\det(tH_1 + xH_2 + yH_3)$ has the property (i) and (ii) mentioned in the above. In such a case, we have the equation

$$\det(tH_1 + xH_2 + yH_3) = \det(H_1)\det(tI + xH_1^{-1/2}H_2H_1^{-1/2} + yH_1^{-1/2}H_3H_1^{-1/2}).$$

An analogous object of $W(A)$ for a linear operator in an indefinite space satisfies some convexity property (cf. [2], [3], [19]).

We shall consider the joint numerical range of Hermitian matrices. Suppose that $\{H_1, H_2, \ldots, H_m\}$ is an ordered $m$-ple of $n \times n$ Hermitian matrices. The joint numerical range $W(H_1, H_2, \ldots, H_m)$ is defined as

$$W(H_1, H_2, \ldots, H_m) = \{ (\xi^*H_1\xi, \xi^*H_2\xi, \ldots, \xi^*H_m\xi) : \xi \in \mathbb{C}^n, \xi^*\xi = 1 \}.$$  

If $m = 3$, $n \geq 3$, the set $W(H_1, H_2, H_3) \subset \mathbb{R}^3$ is convex. In the case $H_3 = H_1^2 + H_2^2 + i(H_1H_2 - H_2H_1)$, the joint numerical range $W(H_1, H_2, H_3)$ is known as the Davis-Wielandt shell of a matrix $A = H_1 + iH_2$. By using the convexity
of the joint numerical range \( W(H_1, H_2, (H_1 + iH_2)^*(H_1 + iH_2)) \) for \( n \geq 3 \), we can prove the convexity of the generalized numerical range

\[
W_q(A) = \{ \eta^*A\xi : \xi, \eta \in \mathbb{C}^n, \xi^*\xi = 1, \eta^*\eta = 1, \eta^*\xi = q \}
\]

for an \( n \times n \) matrix \( A \) and a real number \( 0 \leq q \leq 1 \) (cf. [18], [5], [6]). In the case \( q = 1 \), the range \( W_q(A) \) coincides with the numerical range \( W(A) \). The set \( W(H_1, H_2, H_3, H_4) \) is not necessarily convex.

Example Let

\[
H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

\[
H_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_4 = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

and let

\[
\Pi = \{(1, x, y, z) : (x, y, z) \in \mathbb{R}^3\}.
\]

Then we have

\[
W(H_1, H_2, H_3, H_4) = \{(1, x, y, z) : x^2 + y^2 + z^2 = 1 \}.
\]

Suppose that \( \Delta = \text{Conv}(W(H_1, H_2, \ldots, H_m)) \) contains \((0, 0, \ldots, 0)\) as an interior point. Then the set

\[
\hat{\Delta} = \{(X_1, X_2, \ldots, X_m) \in \mathbb{R}^m, X_1x_1 + X_2x_2 + \ldots + X_mx_m + 1 \geq 0, \text{for}
\]

\[
(x_1, x_2, \ldots, x_m) \in W(H_1, H_2, \ldots, H_m) \}
\]

is a compact convex set. Its boundary point \((X_1, X_2, \ldots, X_m)\) satisfies

\[
\det(I_n + X_1H_1 + X_2H_2 + \ldots + X_mH_m) = 0, \quad \det(I_n + t[X_1H_1 + X_2H_2 + \ldots + X_mH_m]) > 0
\]

for \( 0 \leq t < 1 \). The coonnccted ccomponent of the set

\[
\{(Y_1, Y_2, \ldots, Y_m) \in \mathbb{R}^m : \det(I_n + Y_1H_1 + Y_2H_2 + \ldots + Y_mH_m) \neq 0\}, \quad (1.10)
\]

containing \((0, 0, \ldots, 0)\) corresponds to the cross section of the positive cone

\[
\{K = (a_{ij}) \in M_n(\mathbb{C}) : K = K^*, \xi^*K\xi > 0 \text{ for } \xi \in \mathbb{C}^n, \xi \neq 0\}, \quad (1.11)
\]

with the affine plane

\[
\{I_n + Y_1H_1 + Y_2H_2 + \ldots + Y_mH_m : (Y_1, Y_2, \ldots, Y_m) \in \mathbb{R}^m\}. \quad (1.12)
\]
Are there an m-ple of Hermitian matrices $H_1, H_2, \ldots, H_m$ and a constant $c$ satisfying
\[ F(x_0, x_1, x_2, \ldots, x_m) = c \det(x_0 I_n + x_1 H_1 + x_2 H_2 + \ldots + x_m H_m), \] (1.13)
if $F$ is a form of degree $n$ hyperbolic with respect to $(1, 0, \ldots, 0)$?

Example 1  Suppose that
\[ F(t, x_1, x_2, x_3, x_4) = t^2 - (x_1^2 + x_2^2 + x_3^2 + x_4^2). \]
Then the form $F$ is hyperbolic with respect to $(1, 0, 0, 0, 0)$. There is no ordered set $(H_2, H_3, H_4)$ of $2 \times 2$ Hermian matrices satifying
\[ t^2 - (x_1^2 + x_2^2 + x_3^2 + x_4^2) = \det(tI_2 + x_1 H_1 + x_2 H_2 + x_3 H_3 + x_4 H_4). \]
In fact we assume that there exist such Hermian matrices $H_1, H_2, H_3, H_4$. For every point $(x_1, x_2, x_3, x_4)$, we have
\[
x_0^2 - (x_1^2 + x_2^2 + x_3^2 + x_4^2) = (x_0 + \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2})(x_0 - \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}) = 0
\]
and hence $\text{tr}(x_1 H_1 + x_2 H_2 + x_3 H_3 + x_4 H_4) = 0$. Thus the Hermian matrix $x_1 H_1 + x_2 H_2 + x_3 H_3 + x_4 H_4$ is expressed as
\[
L_1(x_1, x_2, x_3, x_4) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + L_2(x_1, x_2, x_3, x_4) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + L_3(x_1, x_2, x_3, x_4) \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},
\]
where $L_j(x_1, x_2, x_3, x_4)$ $(j = 1, 2, 3)$ are linear functionals. We should have
\[
x_1^2 + x_2^2 + x_3^2 + x_4^2 = L_1(x_1, x_2, x_3, x_4)^2 + L_2(x_1, x_2, x_3, x_4)^2 + L_3(x_1, x_2, x_3, x_4)^2.
\]
However this equation is impossible since the rank of the quadratic form in the right-hand side is less than or equal to 3 and the rank of the quadratic form in the left-hand side is 4. Thus the expression as (1.9) is impossible.

Example 2  Suppose that
\[ F(t, x_1, x_2, x_3, x_4, x_5) = t^3 - t(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2). \]
Then the form $F$ is hyperbolic with respect to $(1, 0, 0, 0, 0, 0)$. The form $F(t, x_1, x_2, x_3, x_4, 0)$ is realized as
\[
\det\left( \begin{pmatrix} t & x_1 + ix_2 & x_3 + ix_4 \\ x_1 - ix_2 & t & 0 \\ x_3 - ix_4 & 0 & t \end{pmatrix} \right)
\]
Probably the form $F$ itself can not be realized as $\det(tI_3 + x_1 H_1 + x_2 H_2 + x_3 H_3 + x_4 H_4 + x_5 H_5)$ by $3 \times 3$ Hermian matrices $H_1, H_2, H_3, H_4, H_5$. I can not so far prove such a non existence.
2. Henrion's method using Bezoutians

Consider the two polynomials in $s$, there are coefficients $\alpha_j, \beta_j$ so that

$$\phi_1(s) = \sum_{j=0}^{m} \alpha_j s^j, \quad (2.1)$$

$$\phi_2(s) = \sum_{j=0}^{m} \beta_j s^j. \quad (2.2)$$

The Bezoutian matrix of (2.1) and (2.2) is the $m \times m$ matrix

$$\text{Bez} = (g_{i,j}), \quad 1 \leq i, j \leq m,$$

where

$$g_{i,j} = \sum_{0 \leq k \leq \min(i-1,j-1)} (\alpha_{i+j-1-k} \beta_k - \alpha_k \beta_{i+j-1-k}). \quad (2.3)$$

The entries $g_{i,j}$ are characterized as

$$\frac{\phi_1(s) \phi_2(t) - \phi_2(s) \phi_1(t)}{s-t} = \sum_{i,j=1}^{m} g_{i,j} s^{i-1} t^{j-1}.$$

For example, when $m = 4$, the $4 \times 4$ Bezoutian matrix

$$\text{Bez} = \{(g_{ij}), \quad 1 \leq i, j \leq 4\} \quad (2.4)$$

is symmetric with entries

$$\begin{align*}
g_{11} &= \alpha_1 \beta_0 - \alpha_0 \beta_1, \quad g_{12} = \alpha_2 \beta_0 - \alpha_0 \beta_2, \\
g_{13} &= \alpha_3 \beta_0 - \alpha_0 \beta_3, \quad g_{14} = \alpha_4 \beta_0 - \alpha_0 \beta_4, \\
g_{22} &= \alpha_2 \beta_1 + \alpha_3 \beta_0 - \alpha_0 \beta_3 - \alpha_1 \beta_2 - \alpha_0 \beta_2, \quad g_{23} = \alpha_4 \beta_0 + \alpha_3 \beta_1 - \alpha_1 \beta_3 - \alpha_0 \beta_3, \\
g_{24} &= \alpha_4 \beta_1 - \alpha_3 \beta_2 - \alpha_2 \beta_3 - \alpha_1 \beta_4, \quad g_{33} = \alpha_4 \beta_1 + \alpha_3 \beta_2 - \alpha_2 \beta_3 - \alpha_1 \beta_4, \\
g_{34} &= \alpha_4 \beta_2 - \alpha_3 \beta_4, \quad g_{44} = \alpha_4 \beta_3 - \alpha_3 \beta_4.
\end{align*}$$

The two polynomials $\phi_1(s), \phi_2(s)$ have a non-constant common divisor $\psi(s)$ if and only if $\det(\text{Bez}) = 0$.

Henrion [12] provided a more elementary method in the case $F(t, x, y) = 0$ is a rational curve. Henrion started from a parametrized form

$$x = \phi(s), \quad y = \psi(s), \quad (2.1)$$

of the rational curve $F(1, x, y) = 0$ by real rational functions in $s$.

We express the rational functions $\phi(s), \psi(s)$

$$\phi(s) = \frac{f(s)}{h(s)} \quad \psi(s) = \frac{g(s)}{h(s)}, \quad (2.2)$$
by real polynomials $f(s), g(s), h(s)$

We have

\[ L_1(s) = h(s)x - f(s) = 0, \]
\[ L_2(s) = h(s)y - g(s) = 0. \]

(2.3)
(2.4)

By these equations, he constructed real symmetric matrices $H_1, H_2, H_3$ satisfying

\[ F(t, x, y) = \det(tH_1 + xH_2 + yH_3) \]

by using Bezoutians.

We shall treat the rational curve $F(1, x, y) = 0$ given as the graph of a trigonometric polynomial

\[ z(\theta) = c_{-n} \exp(-in\theta) + \ldots + c_0 + \ldots + c_n \exp(in\theta) = \sum_{j=-n}^{n} c_j \exp(\sqrt{-1}j\theta), \]

(2.5)

($n = 1, 2, \ldots$)

Then we can obtain a real ternary form $F(t, x, y)$ of degree $2n$ satisfying

\[ F(1, \Re(z(\theta)), \Im(z(\theta))) = 0 \]

($0 \leq \theta \leq 2\pi$). One method to obtain the non-homogeneous $f(x, y) = F(1, x, y)$ is given as the following. We set $z = x + iy$ and $w = x - iy$ and $u = \exp(i\theta)$. We have

\[ M_1(u) = -zu^m + c_m u^{2m} + \ldots + c_0 u^m + \ldots + c_{-m} = 0, \]
\[ M_2(u) = -wu^m + \overline{c_{-m}} u^{2m} + \ldots + \overline{c_0} u^m + \ldots + \overline{c_m} = 0, \]

By using Sylvester determinant, we can eliminate $u$ from these equations and obtain the polynomial $f(x, y)$. However this method does not provide us a method to construct Hermitian matrices $H_1, H_2, H_3$ satisfying (1.6).

We have another problem. When the form $F(t, x, y)$ associated with the trigonometric polynomial (2.5) is hyperbolic with respect to $(1, 0, 0)$? By the condition $F(1, 0, 0) > 0$, the graph of the trigonometric polynomial does not pass through the origin 0 in the Gaussian plane. In an early step, the author supposed the condition

\[ |c_n| > \sum_{j=-n}^{n-1} |c_j| \]

for the form $F(t, x, y)$ to be hyperbolic with respect to $(1, 0, 0)$.

In a letter to the author, Prof. T. Nakazi provided a general condition for the form $F(t, x, y)$ to be hyperbolic.
under the condition

\begin{align}
\frac{d\text{Arg}(z(\theta))}{d\theta} > 0 \\
c_n > 0,
\end{align}

(0 \leq \theta \leq 2\pi).

Nakazi's condition: The equation

\begin{equation}
c_nz^{2n} + \cdots + c_0z^n + \cdots + c_{-n} = c_n \prod_{j=1}^{2n}(z - \alpha_j), \tag{2.7}
\end{equation}

holds for \( |\alpha_j| < 1 \) \((j = 1, 2, \ldots, 2n)\). His condition is deduced from Rouché's theorem.

**Theorem**[8] If a trigonometric polynomial

\[ z(\theta) = \sum_{j=-n}^{n} c_j \exp(i j \theta) \]

satisfies the condition

\begin{equation}
c_nz^{2n} + \cdots + c_0z^n + \cdots + c_{-n} = c_n \prod_{j=1}^{2n}(z - \alpha_j), \tag{2.7}
\end{equation}

for \( |\alpha_j| < 1 \), then the rational curve obtained as the graph of \( z(\theta) = x(\theta) + iy(\theta) \) is realized as

\[ \det(H_1 + xH_2 + yH_3) = 0 \]

for some \( 2n \times 2n \) real symmetric matrices \( H_2, H_3 \) and a positive definite real symmetric matrix \( H_1 \).

To prove the positivity of the Hermitian matrix \( H_1 \), Hermite's classical theorem on zeros of a polynomial plays an important role. Let

\[ p(z) = \sum_{j=0}^{n} \gamma_j z^j \]

be a polynomial in \( z \) with the leading coefficient \( \gamma_n \neq 0 \). We define two polynomials \( \phi_1(z) \) and \( \phi_2(z) \) by

\[ \phi_1(z) = \sum_{j=0}^{n} \Re(\gamma_j) z^j, \quad \phi_2(z) = \sum_{j=0}^{n} \Im(\gamma_j) z^j. \]

The Bezout matrix of \( \phi_2(z) \) and \( \phi_1(z) \) is positive definite if and only if the roots of \( p(z) \) are contained in the upper half plane \( \Im(z) > 0 \) (cf. [14], [22]). The graph
of a special trigonometric polynomial is treated in [7]. A special rational curve associated with a nilpotent Toeplitz matrix is treated in [4].

**Example** We give an example to illustrate Hermite’s theorem. Let \( p(z) = (z - 2i)(z - i) = z^2 - 3iz - 2 \), \( \phi_2(z) = 0 \cdot z^2 - 3z + 0 \), \( \phi_1(z) = z^2 + 0 \cdot z - 2 \). Then the corresponding Bezoutian matrix is given by

\[
\begin{pmatrix}
0 & 3 \\
0 & 3
\end{pmatrix}
\]

**References**


