Title
Numerical range of a matrix associated with the graph of a trigonometric polynomial (Structural study of operators via spectra or numerical ranges)

Author(s)
Nakazato, Hiroshi

Citation
数理解析研究所講究録 (2012), 1778: 78-87

Issue Date
2012-02

URL
http://hdl.handle.net/2433/171793

Type
Departmental Bulletin Paper

Textversion
publisher

Kyoto University
Numerical range of a matrix associated with the graph of a trigonometric polynomial

Hiroshi Nakazato (Hirosaki University)

Abstract

We present a determinantal representation of a hyperbolic ternary form associated with a trigonometric polynomial. The result is obtained by a joint work with Professor Mao-Ting Chien.

Keywords: curves, numerical range, curves
Mathematics Subject Classification: 15A60, 14Q05

1. Lax-Fiedler conjecture

Suppose that $A$ is an $n \times n$ complex matrix. The numerical range $W(A)$ of $A$ is defined as

$$W(A) = \{\xi^*A\xi : \xi \in \mathbb{C}^n, \xi^*\xi = 1\}.$$ \hspace{1cm} (1.1)

In 1918 Toeplitz introduced this set $W(A)$. He characterized $\partial W(A)$ by

$$\max\{\Re(e^{-i\theta}z) : z \in W(A)\} = \max\sigma(H(\theta : A)),$$ \hspace{1cm} (1.2)

where

$$H(\theta : A) = \frac{1}{2}(e^{-i\theta}A + e^{i\theta}A^*),$$

$$\sigma(H) = \{\lambda \in \mathbb{R} : \det(\lambda I - H) = 0\},$$ \hspace{1cm} (1.3)

for $H = H^*$. In 1919 Hausdorff proved the simply connectedness of the range $W(A)$. The simply connectedness of the numerical range is also valid for a linear matrix pencil $A\lambda + B$ with $0 \notin W(A)$ ([20]). To compute the eigenvalues of $H(\theta : A)$ we introduce a ternary form

$$F_A(t, x, y) = \det(tI_n + x/2(A + A^*) - yi/2(A - A^*)).$$ \hspace{1cm} (1.4)

By the equation

$$\det(tI_n - H(\theta : A)) = F_A(t, -\cos\theta, -\sin\theta),$$

this ternary form determines the eigenvalues of $H(\theta)$ for every angle $\theta$. 
In 1951, Kippenhahn [15] showed that

$$W(A) = \text{Conv}\{(X + iY : (X, Y) \in \mathbb{R}^2, Xx + Yy + 1 = 0 \}.$$ 

By this result, the boundary of the numerical range $W(A)$ lies on the dual curve of the algebraic curve $F(1, x, y) = 0$ when $W(A)$ is strictly convex.

The form $F_A(t, x, y)$ satisfies (i) $F_A(1, 0, 0) > 0$ and (ii) For every $(x_0, y_0) \in \mathbb{R}^2$, the equation $F_A(t, x_0, y_0) = 0$ in $t$ has $n$ real solutions counting the multiplicities of the solutions. In 1981, Fiedler [11] conjectured: If $F(t, x, y)$ is a real ternary form of degree $n$ and satisfies (i) $F(1, 0, 0) = c > 0$ and (ii) For every $(x_0, y_0) \in \mathbb{R}^2$, the equation $F(t, x_0, y_0) = 0$ in $t$ has $n$ real solutions counting the multiplicities of the solutions, then there exists an $n \times n$ complex matrix $A$ with

$$F(t, x, y) = c \det(tI_n + x/2(A + A^*) - yi/2(A - A^*)). \quad (1.5)$$

If a ternary form $F(t, x, y)$ satisfies the above conditions (i) and (ii), then the form is said to be hyperbolic with respect to $(1, 0, 0)$ ([1]). Before Fiedler’s formulation, Lax [16] conjectured more strong result in 1958: the above conditions (i), (ii) for $F$ implies the existence of a pair of real symmetric matrices $H, K$ satisfying

$$F(t, x, y) = c \det(tI_n + xH + yK). \quad (1.6)$$

In 2007, Helton and Vinnikov [13] showed that the Lax conjecture is true (cf. [17]). Hence the Fiedler conjecture is true.

We shall consider the determinantal representations of a homogeneous polynomial. Whether a complex homogeneous polynomial $F(x_1, x_2, \ldots, x_m)$ ($m \geq 2$) with Degree $n$ in $m$ indeterminates $x_1, \ldots, x_m$ can be represented as

$$F(x_1, x_2, \ldots, x_n) = \det(x_1A_1 + x_2A_2 + \cdots + x_nA_n), \quad (1.7)$$

for some $n \times n$ complex matrices $A_1, A_2, \ldots, A_n$ or not?

In the case $m = 2$, the form $F$ is expressed as

$$\prod_{j=1}^{n}(\alpha_jx_1 + \beta_jx_2).$$

Hence the diagonal matrices $A_1 = \text{diag}(\alpha_1, \ldots, \alpha_n)$, $A_2 = \text{diag}(\beta_1, \ldots, \beta_n)$ satisfy (1.7). The following results are known.
Theorem [A. C. Dixon, 1901, [9]] For every (non-zero) complex ternary form $F(t, x, y)$ of degree $n$, there are $n \times n$ complex symmetric matrices $A_1, A_2, A_3$ satisfying

$$F(t, x, y) = \det(tA_1 + xA_2 + yA_3).$$

Theorem [L. E. Dickson, 1920, [10]] A generic homogeneous polynomials in $m$ variables of degree $n$ has a representation

$$\det(x_1A_1 + x_2A_2 + \ldots + x_mA_m) = 0$$

by $n \times n$ matrices $A_1, A_2, \ldots, A_m$ if and only if

1. $m = 3$ (curves),
2. $m = 4$ and $n = 2, 3$ (surfaces),
3. $m = 4$ and $n = 2$ (threefolds).

Theorem [V. Vinnikov, 1993. [21]] An irreducible real algebraic curve $F(t, x, y) = 0$ has a representation

$$\det(tH_1 + xH_2 + yH_3) = 0,$$

(1.8)

by Hermitian matrices $H_1, H_2, H_3$.

We remark that if $H_1$ in (1.8) is positive definite, then the real ternary form $\det(tH_1 + xH_2 + yH_3)$ has the property (i) and (ii) mentioned in the above. In such a case, we have the equation

$$\det(tH_1 + xH_2 + yH_3) = \det(H_1)\det(tI + xH_1^{-1/2}H_2H_1^{-1/2} + yH_1^{-1/2}H_3H_1^{-1/2}).$$

An analogous object of $W(A)$ for a linear operator in an indefinite space satisfies some convexity property (cf. [2], [3], [19]).

We shall consider the joint numerical range of Hermitian matrices. Suppose that $\{H_1, H_2, \ldots, H_m\}$ is an ordered $m$-ple of $n \times n$ Hermitian matrices. The joint numerical range $W(H_1, H_2, \ldots, H_m)$ is defined as

$$W(H_1, H_2, \ldots, H_m) = \{(\xi^*H_1\xi, \xi^*H_2\xi, \ldots, \xi^*H_m\xi) : \xi \in \mathbb{C}^n, \xi^*\xi = 1\}.$$  

(1.9)

If $m = 3$, $n \geq 3$, the set $W(H_1, H_2, H_3) \subset \mathbb{R}^3$ is convex. In the case $H_3 = H_1^2 + H_2^2 + i(H_1H_2 - H_2H_1)$, the joint numerical range $W(H_1, H_2, H_3)$ is known as the Davis-Wielandt shell of a matrix $A = H_1 + iH_2$. By using the convexity
of the joint numerical range $W(H_1, H_2, (H_1 + iH_2)^*(H_1 + iH_2))$ for $n \geq 3$, we can prove the convexity of the generalized numerical range

$$W_q(A) = \{ \eta^* A \xi : \xi, \eta \in \mathbb{C}^n, \xi^* \xi = 1, \eta^* \eta = 1, \eta^* \xi = q \}$$

for an $n \times n$ matrix $A$ and a real number $0 \leq q \leq 1$ (cf. [18], [5], [6]). In the case $q = 1$, the range $W_q(A)$ coincides with the numerical range $W(A)$. The set $W(H_1, H_2, H_3, H_4)$ is not necessarily convex.

Example Let

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$H_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_4 = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and let $$\Pi = \{(1, x, y, z) : (x, y, z) \in \mathbb{R}^3\}.$$ Then we have

$$W(H_1, H_2, H_3, H_4) = \{(1, x, y, z) : x^2 + y^2 + z^2 = 1\}.$$

Suppose that $\Delta = \text{Conv}(W(H_1, H_2, \ldots, H_m))$ contains $(0, 0, \ldots, 0)$ as an interior point. Then the set

$$\hat{\Delta} = \{(X_1, X_2, \ldots, X_m) \in \mathbb{R}^m, X_1 x_1 + X_2 x_2 + \ldots + X_m x_m + 1 \geq 0, \text{ for}$$

$$(x_1, x_2, \ldots, x_m) \in W(H_1, H_2, \ldots, H_m)\}$$

is a compact convex set. Its boundary point $(X_1, X_2, \ldots, X_m)$ satisfies

$$\det(I_n + X_1 H_1 + X_2 H_2 + \ldots + X_m H_m) = 0, \quad \det(I_n + t[X_1 H_1 + X_2 H_2 + \ldots + X_m H_m]) > 0$$

for $0 \leq t < 1$. The connected component of the set

$$\{(Y_1, Y_2, \ldots, Y_m) \in \mathbb{R}^m : \det(I_n + Y_1 H_1 + Y_2 H_2 + \ldots + Y_m H_m) \neq 0\},$$

containing $(0, 0, \ldots, 0)$ corresponds to the cross section of the positive cone

$$\{K = (a_{ij}) \in M_n(\mathbb{C}) : K = K^*, \xi^* K \xi > 0 \text{ for } \xi \in \mathbb{C}^n, \xi \neq 0\},$$

with the affine plane

$$\{I_n + Y_1 H_1 + Y_2 H_2 + \ldots + Y_m H_m : (Y_1, Y_2, \ldots, Y_m) \in \mathbb{R}^m\}.$$
Are there an m-ple of Hermitian matrices $H_1, H_2, \ldots, H_m$ and a constant $c$ satisfying

$$F(x_0, x_1, x_2, \ldots, x_m) = c\det(x_0I_n + x_1H_1 + x_2H_2 + \ldots + x_mH_m),$$

(1.13)

if $F$ is a form of degree $n$ hyperbolic with respect to $(1, 0, \ldots, 0)$?

**Example 1** Suppose that

$$F(t, x_1, x_2, x_3, x_4) = t^2 - (x_1^2 + x_2^2 + x_3^2 + x_4^2).$$

Then the form $F$ is hyperbolic with respect to $(1, 0, 0, 0, 0)$. There is no ordered set $(H_2, H_3, H_4)$ of $2 \times 2$ Hermitian matrices satifying

$$t^2 - (x_1^2 + x_2^2 + x_3^2 + x_4^2) = \det(tI_2 + x_1H_1 + x_2H_2 + x_3H_3 + x_4H_4).$$

In fact we assume that there exist such Hermitian matrices $H_1, H_2, H_3, H_4$. For every point $(x_1, x_2, x_3, x_4)$, we have

$$x_0^2 - (x_1^2 + x_2^2 + x_3^2 + x_4^2) = (x_0 + \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2})(x_0 - \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}) = 0$$

and hence $\text{tr}(x_1H_1 + x_2H_2 + x_3H_3 + x_4H_4) = 0$. Thus the Hermitian matrix $x_1H_1 + x_2H_2 + x_3H_3 + x_4H_4$ is expressed as

$$L_1(x_1, x_2, x_3, x_4)\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix} + L_2(x_1, x_2, x_3, x_4)\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix} + L_3(x_1, x_2, x_3, x_4)\begin{pmatrix}0 & i \\ -i & 0\end{pmatrix},$$

where $L_j(x_1, x_2, x_3, x_4)$ ($j = 1, 2, 3$) are linear functionals. We should have

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = L_1(x_1, x_2, x_3, x_4)^2 + L_2(x_1, x_2, x_3, x_4)^2 + L_3(x_1, x_2, x_3, x_4)^2.$$

However this equation is impossible since the rank of the quadratic form in the right-hand side is less than or equal to 3 and the rank of the quadratic form in the left-hand side is 4. Thus the expression as (1.9) is impossible.

**Example 2** Suppose that

$$F(t, x_1, x_2, x_3, x_4, x_5) = t^3 - t(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2).$$

Then the form $F$ is hyperbolic with respect to $(1, 0, 0, 0, 0, 0)$. The form $F(t, x_1, x_2, x_3, x_4, 0)$ is realized as

$$\det\left(\begin{array}{ccc} t & x_1 + ix_2 & x_3 + ix_4 \\ x_1 - ix_2 & t & 0 \\ x_3 - ix_4 & 0 & t \end{array}\right).$$

Probably the form $F$ itself can not be realized as $\det(tI_3 + x_1H_1 + x_2H_2 + x_3H_3 + x_4H_4 + x_5H_5)$ by $3 \times 3$ Hermitian matrices $H_1, H_2, H_3, H_4, H_5$. I can not so far prove such a non existence.
2. Henrion’s method using Bezoutians

Consider the two polynomials in $s$, there are coefficients $\alpha_j, \beta_j$ so that

\[
\phi_1(s) = \sum_{j=0}^{m} \alpha_j s^j, \tag{2.1}
\]

\[
\phi_2(s) = \sum_{j=0}^{m} \beta_j s^j. \tag{2.2}
\]

The Bezoutian matrix of (2.1) and (2.2) is the $m \times m$ matrix

\[
\text{Bez} = (g_{i,j}), \ 1 \leq i, j \leq m,
\]

where

\[
g_{i,j} = \sum_{0 \leq k \leq \min(i-1,j-1)} (\alpha_{i+j-1-k} \beta_k - \alpha_k \beta_{i+j-1-k}). \tag{2.3}
\]

The entries $g_{i,j}$ are characterized as

\[
\frac{\phi_1(s)\phi_2(t) - \phi_2(s)\phi_1(t)}{s-t} = \sum_{i,j=1}^{m} g_{i,j} s^{i-1}t^{j-1}.
\]

For example, when $m = 4$, the $4 \times 4$ Bezoutian matrix

\[
\text{Bez} = \{(g_{i,j}), \ 1 \leq i, j \leq 4\} \tag{2.4}
\]

is symmetric with entries

\[
\begin{align*}
g_{11} &= \alpha_1\beta_0 - \alpha_0\beta_1, & g_{12} &= \alpha_2\beta_0 - \alpha_0\beta_2, \\
g_{13} &= \alpha_3\beta_0 - \alpha_0\beta_3, & g_{14} &= \alpha_4\beta_0 - \alpha_0\beta_4, \\
g_{22} &= \alpha_3\beta_0 + \alpha_2\beta_1 - \alpha_1\beta_2 - \alpha_0\beta_3, & g_{23} &= \alpha_4\beta_0 + \alpha_3\beta_1 - \alpha_1\beta_3 - \alpha_0\beta_4, \\
g_{24} &= \alpha_4\beta_1 - \alpha_1\beta_4, & g_{33} &= \alpha_4\beta_1 + \alpha_3\beta_2 - \alpha_2\beta_3 - \alpha_1\beta_4 \\
g_{34} &= \alpha_4\beta_2 - \alpha_2\beta_4, & g_{44} &= \alpha_4\beta_3 - \alpha_3\beta_4
\end{align*}
\]

The two polynomials $\phi_1(s), \phi_2(s)$ have a non-constant common divisor $\psi(s)$ if and only if $\det(\text{Bez}) = 0$.

Henrion [12] provided a more elementary method in the case $F(t, x, y) = 0$ is a rational curve. Henrion started from a parametrized form

\[
x = \phi(s), \quad y = \psi(s), \tag{2.1}
\]

of the rational curve $F(1, x, y) = 0$ by real rational functions in $s$.

We express the rational functions $\phi(s), \psi(s)$

\[
\phi(s) = \frac{f(s)}{h(s)}, \quad \psi(s) = \frac{g(s)}{h(s)}, \tag{2.2}
\]
by real polynomials \( f(s), g(s), h(s) \)

We have

\[
L_1(s) = h(s)x - f(s) = 0, \quad (2.3)
\]
\[
L_2(s) = h(s)y - g(s) = 0. \quad (2.4)
\]

By these equations, he constructed real symmetric matrices \( H_1, H_2, H_3 \) satisfying

\[
F(t, x, y) = \det(tH_1 + xH_2 + yH_3)
\]

by using Bezoutians.

We shall treat the rational curve \( F(1, x, y) = 0 \) given as the graph of a trigonometric polynomial

\[
z(\theta) = c_{-n} \exp(-in\theta) + \ldots + c_0 + \ldots + c_n \exp(in\theta) = \sum_{j=-n}^{n} c_j \exp(\sqrt{-1} j\theta), \quad (2.5)
\]

\((n = 1, 2, \ldots)\)

Then we can obtain a real ternary form \( F(t, x, y) \) of degree \( 2n \) satisfying

\[
F(1, \Re(z(\theta)), \Im(z(\theta))) = 0
\]

\((0 \leq \theta \leq 2\pi)\). One method to obtain the non-homogeneous \( f(x, y) = F(1, x, y) \) is given as the following. We set \( z = x + iy \) and \( w = x - iy \) and \( u = \exp(i\theta) \). We have

\[
M_1(u) = -zu^m + c_m u^{2m} + \ldots + c_0 u^m + \ldots + c_{-m} = 0,
\]
\[
M_2(u) = -wu^m + \overline{c_{-m}} u^{2m} + \ldots + \overline{c_0} u^m + \ldots + \overline{c_m} = 0,
\]

By using Sylvester determinant, we can eliminate \( u \) from these equations and obtain the polynomial \( f(x, y) \). However this method does not provide us a method to construct Hermitian matrices \( H_1, H_2, H_3 \) satisfying \((1.6)\).

We have another problem. When the form \( F(t, x, y) \) associated with the trigonometric polynomial \((2.5)\) is hyperbolic with respect to \((1, 0,0)\)? By the condition \( F(1,0,0) > 0 \), the graph of the trigonometric polynomial does not pass through the origin \(0\) in the Gaussian plane. In an early step, the author supposed the condition

\[
|c_n| > \sum_{j=-n}^{n-1} |c_j|
\]

for the form \( F(t, x, y) \) to be hyperbolic with respect to \((1,0,0)\).

In a letter to the author, Prof. T. Nakazi provided a general condition for the form \( F(t, x, y) \) to be hyperbolic
under the condition
\[
\begin{align*}
  c_n > 0, \\
  \frac{d \text{Arg}(z(\theta))}{d \theta} > 0
\end{align*}
\] (2.6),

(0 \leq \theta \leq 2\pi).

Nakazi’s condition: The equation
\[
c_n z^{2n} + \cdots + c_0 z^n + \cdots + c_{-n} = c_n \prod_{j=1}^{2n} (z - \alpha_j),
\] (2.7)
holds for \(|\alpha_j| < 1\) (j = 1, 2, \ldots, 2n). His condition is deduced from Rouché’s theorem.

**Theorem**[8] If a trigonometric polynomial
\[
z(\theta) = \sum_{j=-n}^{n} c_j \exp(\sqrt{-1}j\theta)
\]
satisfies the condition
\[
c_n z^{2n} + \cdots + c_0 z^n + \cdots + c_{-n} = c_n \prod_{j=1}^{2n} (z - \alpha_j),
\]
for \(|\alpha_j| < 1\), then the rational curve obtained as the graph of \(z(\theta) = x(\theta) + iy(\theta)\) is realized as
\[
\det(H_1 + xH_2 + yH_3) = 0
\]
for some 2\(n\) \(\times\) 2\(n\) real symmetric matrices \(H_2, H_3\) and a positive definite real symmetric matrix \(H_1\).

To prove the positivity of the Hermitian matrix \(H_1\), Hermite’s classical theorem on zeros of a polynomial plays an important role. Let
\[
p(z) = \sum_{j=0}^{n} \gamma_j z^j
\]
be a polynomial in \(z\) with the leading coefficient \(\gamma_n \neq 0\). We define two polynomials \(\phi_1(z)\) and \(\phi_2(z)\) by
\[
\phi_1(z) = \sum_{j=0}^{n} \Re(\gamma_j) z^j, \quad \phi_2(z) = \sum_{j=0}^{n} \Im(\gamma_j) z^j.
\]
The Bezout matrix of \(\phi_2(z)\) and \(\phi_1(z)\) is positive definite if and only if the roots of \(p(z)\) are contained in the upper half plane \(\Im(z) > 0\) (cf. [14], [22]). The graph
of a special trigonometric polynomial is treated in [7]. A special rational curve associated with a nilpotent Toeplitz matrix is treated in [4].

**Example** We give an example to illustrate Hermite’s theorem. Let \( p(z) = (z - 2i)(z - i) = z^2 - 3iz - 2 \), \( \phi_2(z) = 0 \cdot z^2 - 3z + 0 \), \( \phi_1(z) = z^2 + 0 \cdot z - 2 \). Then the corresponding Bezoutian matrix is given by

\[
\begin{pmatrix}
6 & 0 \\
0 & 3
\end{pmatrix}
\]

**References**


