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Kyoto University
Matrix inequalities including Furuta inequality via Riemannian mean of $n$-matrices

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Abstract

In this report, we shall obtain a generalization of Furuta inequality via weighted Riemannian mean, a kind of geometric mean, of $n$-matrices. This result is related to Yamazaki's recent results which is a kind of generalizations of Ando-Hiai inequality and Furuta inequality for chaotic order.

1 Introduction

The weighted geometric mean of two positive definite matrices $A$ and $B$ defined by

$$A \#_{\alpha} B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\alpha}A^{\frac{1}{2}}$$

for $\alpha \in [0, 1]$. In particular, we call $A \#_{\frac{1}{2}} B$ (denoted by $A \# B$ simply) the geometric mean of $A$ and $B$. The weighted geometric mean sometimes appears in famous matrix inequalities, for example, Furuta inequality [10] (see also [6, 11, 13, 17, 20]) and Ando-Hiai inequality [1]. We remark that these inequalities hold even in the case of bounded linear operators on a complex Hilbert space. In what follows, we denote $A \geq 0$ if $A$ is a positive semidefinite matrix (or operator), and we denote $A > 0$ if $A$ is a positive definite matrix (or operator).

Theorem 1.A (Satellite form of Furuta inequality [10, 17]).

$$A \geq B \geq 0 \text{ with } A > 0 \text{ implies } A^{-r} \#_{\frac{1+r}{p+r}} B^{p} \leq B \leq A \text{ for } p \geq 1 \text{ and } r \geq 0.$$

Theorem 1.B (Ando-Hiai inequality [1]). For $A, B > 0$,

$$A \#_{\alpha} B \leq I \text{ for } \alpha \in (0, 1) \text{ implies } A^{r} \#_{\alpha} B^{r} \leq I \text{ for } r \geq 1.$$

For $A, B > 0$, it is well known that chaotic order $\log A \geq \log B$ is weaker than usual order $A \geq B$ since $\log t$ is a matrix (or operator) monotone function. The following result is known as the Furuta inequality for chaotic order.

Theorem 1.C (Furuta inequality for chaotic order [7, 12]). Let $A, B > 0$. Then the following assertions are mutually equivalent;
(i) \( \log A \geq \log B \),

(ii) \( A^{-p} \frac{1}{p} B^p \leq I \) for all \( p \geq 0 \),

(iii) \( A^{-r} \frac{1}{p+r} B^p \leq I \) for all \( p \geq 0 \) and \( r \geq 0 \).

It has been a longstanding problem to extend the (weighted) geometric mean for three or more positive definite matrices. Many authors attempt to find a natural extension, for example, Ando-Li-Mathias’ mean and its refinement [2, 5, 15, 16] and Riemannian mean (or the least squares mean) [4, 18, 19]. We remark that Ando-Li-Mathias [2] originally proposed the following ten properties (P1)–(P10) which should be required for a reasonable geometric mean \( \mathfrak{G} \) of positive definite matrices. We note that, in [2], they require continuity from above as (P5).

Let \( P_m(\mathbb{C}) \) be the set of \( m \times m \) positive definite matrices on \( \mathbb{C} \). Let \( A_i, A'_i, B_i \in P_m(\mathbb{C}) \) for \( i = 1, \ldots, n \) and let \( \omega = (w_1, \ldots, w_n) \) be a probability vector. Then

(P1) Consistency with scalars. If \( A_1, \ldots, A_n \) commute with each other, then
\[
\mathfrak{G}(\omega; A_1, \ldots, A_n) = A_1^{w_1} \cdots A_n^{w_n}.
\]

(P2) Joint homogeneity. For positive numbers \( a_i > 0 \) (\( i = 1, \ldots n \)),
\[
\mathfrak{G}(\omega; a_1 A_1, \ldots, a_n A_n) = a_1^{w_1} \cdots a_n^{w_n} \mathfrak{G}(\omega; A_1, \ldots, A_n).
\]

(P3) Permutation invariance. For any permutation \( \pi \) on \( \{1, \ldots n\} \),
\[
\mathfrak{G}(\omega; A_1, \ldots, A_n) = \mathfrak{G}(\pi(\omega); A_{\pi(1)}, \ldots, A_{\pi(n)}),
\]
where \( \pi(\omega) = (w_{\pi(1)}, \ldots, w_{\pi(n)}) \).

(P4) Monotonicity. If \( B_i \leq A_i \) for each \( i = 1, \ldots n \), then
\[
\mathfrak{G}(\omega; B_1, \ldots, B_n) \leq \mathfrak{G}(\omega; A_1, \ldots, A_n).
\]

(P5) Continuity. For each \( i = 1, \ldots n \), let \( \{A_i^{(k)}\}_{k=1}^{\infty} \) be positive definite matrix sequences such that \( A_i^{(k)} \rightarrow A_i \) as \( k \rightarrow \infty \). Then
\[
\mathfrak{G}(\omega; A_1^{(k)}, \ldots, A_n^{(k)}) \rightarrow \mathfrak{G}(\omega; A_1, \ldots, A_n) \text{ as } k \rightarrow \infty.
\]

(P6) Congruence invariance. For any invertible matrix \( S \),
\[
\mathfrak{G}(\omega; S^* A_1 S, \ldots, S^* A_n S) = S^* \mathfrak{G}(\omega; A_1, \ldots, A_n) S.
\]
Joint concavity.

\[ \mathcal{G}(\omega; \lambda A_1 + (1-\lambda)A'_1, \ldots, \lambda A_n + (1-\lambda)A'_n) \geq \lambda \mathcal{G}(\omega; A_1, \ldots, A_n) + (1-\lambda)\mathcal{G}(\omega; A'_1, \ldots, A'_n) \text{ for } 0 \leq \lambda \leq 1. \]

Self-duality. \( \mathcal{G}(\omega; A_1^{-1}, \ldots, A_n^{-1})^{-1} = \mathcal{G}(\omega; A_1, \ldots, A_n) \)

Determinant identity. \( \det \mathcal{G}(\omega; A_1, \ldots, A_n) = \prod_{i=1}^{n} (\det A_i)^{w_i} \)

The arithmetic-geometric-harmonic mean inequality.

\[ \left( \sum_{i=1}^{n} w_i A_i^{-1} \right)^{-1} \leq \mathcal{G}(\omega; A_1, \ldots, A_n) \leq \sum_{i=1}^{n} w_i A_i. \]

For \( A, B \in P_m(\mathbb{C}) \), Riemannian metric between \( A \) and \( B \) is defined as \( \delta_2(A, B) = \| \log A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \|_2 \), where \( \| X \|_2 = (\text{tr} X^*X)^{\frac{1}{2}} \) (details are in [3]). By using Riemannian metric, Riemannian mean is defined as follows:

**Definition 1** ([3, 4, 18, 19]). Let \( A_1, \ldots, A_n \in P_m(\mathbb{C}) \) and \( \omega = (w_1, \ldots, w_n) \) be a probability vector. Then weighted Riemannian mean \( \mathfrak{G}_\delta(\omega; A_1, \ldots, A_n) \in P_m(\mathbb{C}) \) is defined by

\[ \mathfrak{G}_\delta(\omega; A_1, \ldots, A_n) = \arg \min_{X \in P_m(\mathbb{C})} \sum_{i=1}^{n} w_i \delta_2^2(A_i, X), \]

where \( \arg \min f(X) \) means the point \( X_0 \) which attains minimum value of the function \( f(X) \). In particular, we call \( \mathfrak{G}_\delta(\omega; A_1, \ldots, A_n) \) (denoted by \( \mathfrak{G}_\delta(A_1, \ldots, A_n) \) simply) Riemannian mean if \( \omega = \left( \frac{1}{n}, \ldots, \frac{1}{n} \right) \).

We remark that \( \mathfrak{G}_\delta(\omega; A, B) = A#_{\alpha} B \) for \( \alpha \in [0, 1] \) and \( \omega = (1-\alpha, \alpha) \) since the property \( \delta_2(A, A#_{\alpha} B) = \alpha \delta_2(A, B) \) holds.

It is shown in [3, 4, 18, 19] that weighted Riemannian mean satisfies (P1)--(P10) (see also [21]). We remark that Riemannian mean has a stronger property (P5') than (P5).

(P5') Non-expansive.

\[ \delta_2(\mathfrak{G}_\delta(\omega; A_1, \ldots, A_n), \mathfrak{G}_\delta(\omega; B_1, \ldots, B_n)) \leq \sum_{i=1}^{n} w_i \delta_2(A_i, B_i). \]

Very recently, Yamazaki [21] has obtained an excellent generalization of Theorems 1.B and 1.C via weighted Riemannian mean \( \mathfrak{G}_\delta \) of \( n \)-matrices. We recall that \( \omega = (w_1, \ldots, w_n) \) is a probability vector if the components satisfy \( \sum_i w_i = 1 \) and \( w_i > 0 \) for \( i = 1, \ldots, n \).
Theorem 1.D ([21]). Let $A_1, \ldots, A_n \in P_m(\mathbb{C})$ and $\omega = (w_1, \ldots, w_n)$ be a probability vector. Then
\[ \mathfrak{G}_{\delta}(\omega; A_1, \ldots, A_n) \leq I \implies \mathfrak{G}_{\delta}(\omega; A_1^p, \ldots, A_n^p) \leq I \text{ for } p \geq 1. \]

Theorem 1.E ([21]). Let $A_1, \ldots, A_n \in P_m(\mathbb{C})$. Then the following assertions are mutually equivalent;

(i) $\log A_1 + \cdots + \log A_n \leq 0$,

(ii) $\mathfrak{G}_{\delta}(A_1^p, \ldots, A_n^p) \leq I$ for all $p > 0$,

(iii) $\mathfrak{G}_{\delta}(\omega; A_1^{p_1}, \ldots, A_n^{p_n}) \leq I$ for all $p_1, \ldots, p_n > 0$, where $p_{\neq i} = \prod_{j \neq i} p_j$ and
\[ \omega = \left( \frac{p_{\neq 1}}{\sum_{i} p_{\neq i}}, \ldots, \frac{p_{\neq n}}{\sum_{i} p_{\neq i}} \right). \]

Theorems 1.D and 1.E imply Theorems 1.B and 1.C, respectively, since $\mathfrak{G}_{\delta}(\omega; A, B) = A \ast_{\alpha} B$ for $\omega = (1 - \alpha, \alpha)$. Moreover, it has been shown in [21] that Theorem 1.D does not hold for other geometric means satisfying (P1)-(P10).

In this report, corresponding to Theorem 1.E, we shall obtain a generalization of Furuta inequality (Theorem 1.A) via weighted Riemannian mean of n-matrices. Moreover we shall show an extension of Theorem 1.D.

2 Results

Firstly, we show an extension of Theorem 1.D. Theorem 1.D follows from Theorem 2.1 by putting $p_1 = \cdots = p_n = p$.

Theorem 2.1. Let $A_1, \ldots, A_n \in P_m(\mathbb{C})$ and $\omega = (w_1, \ldots, w_n)$ be a probability vector. If $\mathfrak{G}_{\delta}(\omega; A_1, \ldots, A_n) \leq I$, then
\[ \mathfrak{G}_{\delta}(\hat{\omega}; A_1^{p_1}, \ldots, A_n^{p_n}) \leq \mathfrak{G}_{\delta}(\omega; A_1, \ldots, A_n) \leq I \text{ for } p_1, \ldots, p_n \geq 1, \]
where $\hat{\omega} = (\frac{w_1}{p_1}, \ldots, \frac{w_n}{p_n})$ and $\omega' = \frac{\hat{\omega}}{\|\omega\|_1}$.

We remark that $\| \cdot \|_1$ means 1-norm, that is, $\| x \|_1 = \sum_i |x_i|$ for $x = (x_1, \ldots, x_n)$. In order to prove Theorem 2.1, we use the following results.
Theorem 2.A ([18, 19]). Let $A_1, \ldots, A_n \in P_m(\mathbb{C})$ and $\omega = (w_1, \ldots, w_n)$ be a probability vector. Then $X = \mathfrak{G}_\delta(\omega; A_1, \ldots, A_n)$ is the unique positive solution of the following matrix equation:

$$w_1 \log X^{\frac{-1}{2}} A_1 X^{\frac{1}{2}} + \cdots + w_n \log X^{\frac{-1}{2}} A_n X^{\frac{1}{2}} = 0.$$ 

Theorem 2.B ([21]). Let $A_1, \ldots, A_n \in P_m(\mathbb{C})$ and $\omega = (w_1, \ldots, w_n)$ be a probability vector. Then

$$w_1 \log A_1 + \cdots + w_n \log A_n \leq 0 \implies \mathfrak{G}_\delta(\omega; A_1, \ldots, A_n) \leq I.$$ 

Proof of Theorem 2.1. Let $X = \mathfrak{G}_\delta(\omega; A_1, \ldots, A_n) \leq I$. Then for each $p_1, \ldots, p_n \in [1, 2]$, by Theorem 2.A and Hansen's inequality [14],

$$0 = \frac{1}{||\hat{\omega}'||_1} \sum w_i \log X^{\frac{p_i}{2}} A_i^{-1} X^{\frac{1}{2}} = \frac{1}{||\hat{\omega}'||_1} \sum \frac{w_i}{p_i} \log \left(X^{\frac{p_i}{2}} A_i^{-p_i} X^{\frac{1}{2}}\right)^{p_i}$$

that is,

$$\sum \frac{w_i}{p_i} \log X^{\frac{-1}{2}} A_i^{p_i} X^{\frac{1}{2}} \leq 0. \text{ By applying Theorem 2.B,}$$

$$\mathfrak{G}_\delta(\omega''; A_1^{p_1}, \ldots, A_n^{p_n}) \leq Y \leq X \leq I$$

where $\hat{\omega}' = (\frac{w_1}{p_1}, \ldots, \frac{w_n}{p_n})$ and $\omega' = \frac{\hat{\omega}'}{||\omega'||_1}$. Therefore we have that

$$X \leq I \implies \mathfrak{G}_\delta(\omega; A_1^{p_1}, \ldots, A_n^{p_n}) \leq X \leq I \text{ for } p_1, \ldots, p_n \in [1, 2]. \quad (2.1)$$

Put $Y = \mathfrak{G}_\delta(\omega; A_1^{p_1}, \ldots, A_n^{p_n}) \leq I$. Then by (2.1), we get

$$\mathfrak{G}_\delta(\omega''; A_1^{p_1} p_1', \ldots, A_n^{p_n} p_n') \leq Y \leq X \leq I$$

for $p_1', \ldots, p_n' \in [1, 2]$, where $\omega'' = (\frac{w_1}{p_1 p_1'}, \ldots, \frac{w_n}{p_n p_n'})$ and $\omega' = \frac{\omega''}{||\omega''||_1}$. Therefore, by putting $q_i = p_i p_i'$ for $i = 1, \ldots, n$, we have that

$$X \leq I \implies \mathfrak{G}_\delta(\omega''; A_1^{q_1}, \ldots, A_n^{q_n}) \leq X \leq I \text{ for } q_1, \ldots, q_n \in [1, 4]. \quad (2.2)$$

where $\hat{\omega}'' = (\frac{w_1}{q_1}, \ldots, \frac{w_n}{q_n})$ and $\omega'' = \frac{\hat{\omega}''}{||\omega''||_1}$.

By repeating the same way from (2.1) to (2.2), we have the conclusion. \(\square\)

Theorem 2.1 also implies generalized Ando-Hiai inequality [9] since $\mathfrak{G}_\delta(\omega; A, B) = A \#_\alpha B$ for $\omega = (1 - \alpha, \alpha)$ and $\omega' = \left(\frac{\alpha}{1 - \alpha + \alpha}, \frac{1 - \alpha}{1 - \alpha + \alpha}\right) = \left(\frac{1 - \alpha}{(1 - \alpha)s + \alpha r}, \frac{\alpha}{(1 - \alpha)s + \alpha r}\right).$
Theorem 2.2. Let $A_1, \ldots, A_n \in P_m(\mathbb{C})$ and $\omega = (w_1, \ldots, w_n)$ be a probability vector. For each $i = 1, \ldots, n$ and $q \in \mathbb{R}$, if
$$\mathfrak{G}_\delta(\omega; A_1^{p_1}, A_2^{p_2}, \ldots, A_n^{p_n}) \leq A_i^q \quad \text{for } p_1, \ldots, p_n \in \mathbb{R} \text{ with } p_i > q,$$
then
$$\mathfrak{G}_\delta(\omega'; A_1^{p_1}, A_2^{p_2}, \ldots, A_n^{p_n}) \leq A_i^q \quad \text{for } p_i' \geq p_i,$$
where $\omega' = (w_1, \ldots, w_i - \frac{q}{p_i}, w_{i+1}, \ldots, w_n)$ and $\omega' = \frac{\omega'}{\|\omega'\|_1}$.

Proof. We may assume $i = 1$ by permutation invariance of $\mathfrak{G}_\delta$.

For $p_1, \ldots, p_n \in \mathbb{R}$ with $p_1 \geq q$, $\mathfrak{G}_\delta(\omega; A_1^{p_1}, A_2^{p_2}, \ldots, A_n^{p_n}) \leq A_i^q$ if and only if
$$\mathfrak{G}_\delta(\omega; A_1^{p_1}, A_2^{p_2}, \ldots, A_n^{p_n}) \leq I.$$

By applying Theorem 2.1,
$$\mathfrak{G}_\delta(\omega'; A_1^{p_1}, A_2^{p_2}, \ldots, A_n^{p_n}) \leq I,$$
holds for $\frac{p_1 - q}{p_1 - q} \geq 1$, where $\omega' = (\frac{p_1 - q}{p_1 - q} w_1, w_2, \ldots, w_n)$. Therefore
$$\mathfrak{G}_\delta(\omega'; A_1^{p_1}, A_2^{p_2}, \ldots, A_n^{p_n}) \leq \mathfrak{G}_\delta(\omega; A_1^{p_1}, A_2^{p_2}, \ldots, A_n^{p_n}) \leq A_i^q$$
holds for $p_i' \geq p_i$. \hfill $\square$

Next, we show our main result. The following Theorem 2.3 is a generalization of Theorem 1.A, and also a parallel result to (i) $\implies$ (iii) in Theorem 1.E.
Theorem 2.3. Let $A_1, \ldots, A_n \in P_m(\mathbb{C})$ and $w_1, \ldots, w_n > 0$. If

$$A_1^{w_1} \geq A_n^{w_n} > 0$$

and

$$\frac{w_1}{p_1 - q_1} \log A_1^{-\frac{w_1}{p_1 - q_1}} A_1^{p_1} A_1^{-\frac{w_1}{p_1 - q_1}} + \cdots + \frac{w_{n-1}}{p_{n-1} - q_{n-1}} \log A_{n-1}^{-\frac{w_{n-1}}{p_{n-1} - q_{n-1}}} A_{n-1}^{p_{n-1}} A_{n-1}^{-\frac{w_{n-1}}{p_{n-1} - q_{n-1}}} + \frac{w_n}{p_n - q_n} \log A_n^{p_n - q_n} \leq 0$$

hold for $q_i \in \mathbb{R}$, $p_i > q_i$ and $i = 1, \ldots, n$, then

$$(\omega', A_1^{p_1'}, A_2^{p_2'}, \ldots, A_n^{p_n'}) \leq (\omega; A_1^{p_1}, A_2^{p_2}, \ldots, A_n^{p_n}) \leq A_n^{p_n}$$

for all $p_i' \geq p_i$ and $i = 1, \ldots, n$, where

$$w_i = \frac{w_i}{p_i - q_i}, \quad p_i' = \frac{p_i}{p_i - q_i}, \quad p_i'' = \frac{p_i}{p_i - q_i}.$$

Proof. Applying Theorem 2.2 to (2.4), we have

$$\mathfrak{G}_\delta(\omega; A_1^{-\frac{w_1}{p_1 - q_1}} A_1^{p_1} A_1^{-\frac{w_1}{p_1 - q_1}} A_2^{p_2} \cdots A_{n-1}^{p_{n-1}} A_{n-1}^{-\frac{w_{n-1}}{p_{n-1} - q_{n-1}}} A_n^{p_n - q_n}) \leq I,$$

so that by (2.3),

$$X_0 \equiv \mathfrak{G}_\delta(\omega; A_1^{p_1}, A_2^{p_2} \cdots A_{n-1}^{p_{n-1}} A_n^{p_n}) \leq A_n^{p_n} \leq A_1^{p_1}.$$

By applying Theorem 2.2 to (2.5) and by (2.3),

$$X_1 \equiv \mathfrak{G}_\delta(\omega_1; A_1^{p_1'}, A_2^{p_2'}, \ldots, A_n^{p_n}) \leq X_0 \leq A_n^{p_n} \leq A_2^{p_2}$$

for $p_1' \geq p_1$, where

$$\omega_1 = \left( \frac{w_1}{p_1' - q_1}, \frac{w_2}{p_2' - q_2}, \ldots, \frac{w_n}{p_n' - q_n} \right) \quad \text{and} \quad \omega_1 = \frac{\omega}{\|\omega\|_1}.$$  

By applying Theorem 2.2 to (2.6) and by (2.3),

$$X_2 \equiv \mathfrak{G}_\delta(\omega_2; A_1^{p_1'}, A_2^{p_2'}, A_3^{p_3} \cdots) \leq X_1 \leq X_0 \leq A_n^{p_n} \leq A_3^{p_3}$$

for $p_1' \geq p_1$ and $p_2' \geq p_2$, where

$$\omega_2 = \left( \frac{w_1}{p_1' - q_1}, \frac{w_2}{p_2' - q_2}, \frac{w_3}{p_3' - q_3}, \ldots, \frac{w_n}{p_n' - q_n} \right) \quad \text{and} \quad \omega_2 = \frac{\omega_2}{\|\omega_2\|_1}.$$  

By repeating this argument, we can get

$$X_n \equiv \mathfrak{G}_\delta(\omega_n; A_1^{p_1'}, \ldots, A_n^{p_n'}) \leq X_{n-1} \leq X_0 \leq A_n^{p_n}$$

for $p_i' \geq p_i$ for $i = 1, \ldots, n$, where

$$\omega_n = \left( \frac{w_1}{p_1' - q_1}, \ldots, \frac{w_n}{p_n' - q_n} \right).$$  

Remark. (i) in Theorem 1.E, that is, log $A_1 + \cdots + \log A_n \leq 0$ holds if and only if

$$\frac{1}{p_1} \log A_1^{p_1} + \cdots + \frac{1}{p_n} \log A_n^{p_n} \leq 0 \quad \text{for every } p_i > 0 \text{ and } i = 1, \ldots, n.$$
Therefore we recognize that Theorem 2.3 implies (i) $\implies$ (iii) in Theorem 1.E by putting $q_1 = \ldots = q_n = 0$ and $w_1 = \ldots = w_n = 1$ since

$$\frac{1}{p_i} = \frac{1}{p_i} + \cdots + \frac{1}{p_n} = \frac{p \neq i}{\sum_j p \neq j} \quad \text{for } i = 1, \ldots, n$$

ensures $\omega = \frac{\hat{\omega}}{\|\hat{\omega}\|_1} = \left(\frac{1}{p_1 + q}, \ldots, \frac{1}{p_{n-1} + q}, \frac{n-1}{p_n - q}\right)$.

It is well known that we have a variant from Theorem 1.A by replacing $A, B$ with $A^q, B^q$ and $p, r$ with $\frac{p}{q}, \frac{r}{q}$ in Theorem 1.

**Theorem 2.D** ([8]). Let $A > 0$, $B \geq 0$ and $q > 0$. Then

$$A^q \geq B^q \implies A^{-r} \#_{\frac{q}{q+p}} B^p \leq B^q \leq A^q$$

for $p \geq q$ and $r \geq 0$.

Here we show that Theorem 2.3 is a generalization of Furuta inequality via weighted Riemannian mean of $n$-matrices. Precisely, we show that Theorem 2.3 ensures the following Theorem 2.4 and Theorem 2.4 is a generalization of Theorem 2.D.

**Theorem 2.4.** Let $A_1, \ldots, A_n \in P_n(\mathbb{C})$ and $q > 0$. Then $A_i^q \geq A_n^q > 0$ for $i = 1, \ldots, n-1$ implies

$$\mathcal{G}_s(\omega; A_1^{-p_1}, \ldots, A_{n-1}^{-p_{n-1}}, A_n^{p_n}) \leq A_n^q \leq A_i^q$$

(2.7)

for all $p_i \geq 0$, $i = 1, \ldots, n-1$ and $p_n > q$, where $\widehat{\omega} = \left(\frac{1}{p_1 + q}, \ldots, \frac{1}{p_{n-1} + q}, \frac{n-1}{p_n - q}\right)$ and $\omega = \frac{\widehat{\omega}}{\|\widehat{\omega}\|_1}$.

**Proof.** Assume that $A_i^q \geq A_n^q > 0$ for $q > 0$ and $i = 1, \ldots, n-1$. Then $A_i^q \geq A_n^q > 0$ implies $\log A_i \geq \log A_n$. By (i) $\implies$ (iii) in Theorem 1.C, $\log A_i \geq \log A_n$ implies $A_i^{-p_i} \#_{\frac{q}{q+p_i}} A_n^{p_n} \leq I$ for all $p_i \geq 0$. This is equivalent to $A_n^{-q} \#_{\frac{q}{q+p_i}} A_i^{p_i} \geq I$, that is,

$$(A_n^q A_i^{p_i} A_n^{-q}) \frac{1}{p_i + q} \geq A_i^q.$$

By taking logarithm, we have

$$\frac{1}{p_i + q} \log A_n^{-q} (A_i^{-1})^{p_i} A_i^{q} + \frac{1}{p_n - q} \log A_n^{p_n-q} \leq 0$$

(2.8)

for all $p_i \geq 0$, $i = 1, \ldots, n-1$ and $p_n > q$. Summing up (2.8) for $i = 1, \ldots, n-1$, we have

$$\frac{1}{p_1 + q} \log A_n^{-q} (A_1^{-1})^{p_1} A_n^{q} + \cdots$$

$$+ \frac{1}{p_{n-1} + q} \log A_n^{-q} (A_{n-1}^{-1})^{p_{n-1}} A_n^{q} + \frac{n-1}{p_n - q} \log A_n^{p_n-q} \leq 0$$

(2.9)
By applying Theorem 2.3 to \((A_i^{-1})^{-q} \geq A_n^q > 0\) and (2.9), we can obtain
\[
\mathfrak{G}_\delta(\omega; A_1^{-p_1}, \ldots, A_{n-1}^{-p_{n-1}}, A_n^p) \leq A_n^q \leq A_i^q
\]
for all \(p_i \geq 0 > -q, i = 1, \ldots, n - 1\) and \(p_n > q\).

Proof of Theorem 2.D. Put \(n = 2, p_1 = r\) and \(p_2 = p\) in Theorem 2.4. Then \(\hat{\omega} = \left(\frac{1}{r+q}, \frac{1}{p-q}\right)\) and \(\omega = \left(\frac{p-q}{p+r}, \frac{q+r}{p+r}\right)\). Therefore we obtain the desired result.

\[
\square
\]

3 3-matrices case

In this section, for the sake of readers' convenience, we state 3-matrices case of Theorems 2.3 and 2.4.

Corollary 3.1. Let \(A, B, C \in P_m(\mathbb{C})\) and \(w_1, w_2, w_3 > 0\). If
\[
A^{q_1} \geq C^{q_3} > 0, \quad B^{q_2} \geq C^{q_3} > 0,
\]
and
\[
\frac{w_1}{p_1-q_1} \log C^{-\frac{q_3}{2}} A^{p_1} C^{-\frac{q_3}{2}} + \frac{w_2}{p_2-q_2} \log C^{-\frac{q_3}{2}} B^{p_2} C^{-\frac{q_3}{2}} + \frac{w_3}{p_3-q_3} \log C^{-\frac{q_3}{2}} C^{p_3} C^{-\frac{q_3}{2}} \leq 0
\]
hold for \(q_i \in \mathbb{R}, p_i > q_i\) and \(i = 1, 2, 3\), then
\[
\mathfrak{G}_\delta(\omega'; A^{p_1}, B^{p_2}, C^{p_3}) \leq \mathfrak{G}_\delta(\omega; A^{p_1}, B^{p_2}, C^{p_3}) \leq C^{q_3}
\]
for all \(p_i' \geq p_i\) and \(i = 1, 2, 3\), where \(\hat{\omega} = \left(\frac{w_1}{p_1-q_1}, \frac{w_2}{p_2-q_2}, \frac{w_3}{p_3-q_3}\right), \hat{\omega}' = \left(\frac{w_1}{p_1-q_1}, \frac{w_2}{p_2-q_2}, \frac{w_3}{p_3-q_3}\right), \omega = \frac{\hat{\omega}}{\|\omega\|_1}\) and \(\omega' = \frac{\hat{\omega}'}{\|\omega'\|_1}\).

Corollary 3.2. Let \(A, B, C \in P_m(\mathbb{C})\) and \(q > 0\). Then \(A^q \geq C^q > 0\) and \(B^q \geq C^q > 0\) implies
\[
\mathfrak{G}_\delta(\omega; A^{-r}, B^{-s}, C^p) \leq C^q \leq A^q \quad \text{(or} \ B^q)\]
for \(r \geq 0, s \geq 0\) and \(p > q\), where \(\hat{\omega} = \left(\frac{1}{r+q}, \frac{1}{s+q}, \frac{2}{p-q}\right)\) and \(\omega = \frac{\hat{\omega}}{\|\hat{\omega}\|_1}\).
References


[10] T. Furuta, $A \geq B \geq 0$ assures $(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0$, $p \geq 0$, $q \geq 1$ with $(1+2r)q \geq p + 2r$, Proc. Amer. Math. Soc., 101 (1987), 85–88.


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