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Matrix inequalities including Furuta inequality via Riemannian mean of \( n \)-matrices

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Abstract

In this report, we shall obtain a generalization of Furuta inequality via weighted Riemannian mean, a kind of geometric mean, of \( n \)-matrices. This result is related to Yamazaki’s recent results which is a kind of generalizations of Ando-Hiai inequality and Furuta inequality for chaotic order.

1 Introduction

The weighted geometric mean of two positive definite matrices \( A \) and \( B \) defined by
\[
A \# \alpha B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\alpha}A^{\frac{1}{2}}
\]
for \( \alpha \in [0, 1] \). In particular, we call \( A \#_{\frac{1}{2}} B \) (denoted by \( A \# B \) simply) the geometric mean of \( A \) and \( B \). The weighted geometric mean sometimes appears in famous matrix inequalities, for example, Furuta inequality [10] (see also [6, 11, 13, 17, 20]) and Ando-Hiai inequality [1]. We remark that these inequalities hold even in the case of bounded linear operators on a complex Hilbert space. In what follows, we denote \( A \geq 0 \) if \( A \) is a positive semidefinite matrix (or operator), and we denote \( A > 0 \) if \( A \) is a positive definite matrix (or operator).

Theorem 1.A (Satellite form of Furuta inequality [10, 17]).
\[
A \geq B \geq 0 \text{ with } A > 0 \implies A^{-r} \#_{\frac{p+r}{p}} B^p \leq B \leq A \text{ for } p \geq 1 \text{ and } r \geq 0.
\]

Theorem 1.B (Ando-Hiai inequality [1]). For \( A, B > 0 \),
\[
A \#_{\alpha} B \leq I \text{ for } \alpha \in (0, 1) \implies A^r \#_{\alpha} B^r \leq I \text{ for } r \geq 1.
\]

For \( A, B > 0 \), it is well known that chaotic order \( \log A \geq \log B \) is weaker than usual order \( A \geq B \) since \( \log t \) is a matrix (or operator) monotone function. The following result is known as the Furuta inequality for chaotic order.

Theorem 1.C (Furuta inequality for chaotic order [7, 12]). Let \( A, B > 0 \). Then the following assertions are mutually equivalent;

\( \quad \)
(i) \( \log A \geq \log B \),

(ii) \( A^{-p} \# B^p \leq I \) for all \( p \geq 0 \),

(iii) \( A^{-r} \# B^p \leq I \) for all \( p \geq 0 \) and \( r \geq 0 \).

It has been a longstanding problem to extend the (weighted) geometric mean for three or more positive definite matrices. Many authors attempt to find a natural extension, for example, Ando-Li-Mathias' mean and its refinement [2, 5, 15, 16] and Riemannian mean (or the least squares mean) [4, 18, 19]. We remark that Ando-Li-Mathias [2] originally proposed the following ten properties (P1)–(P10) which should be required for a reasonable geometric mean \( \mathcal{G} \) of positive definite matrices. We note that, in [2], they require continuity from above as (P5).

Let \( P_m(\mathbb{C}) \) be the set of \( m \times m \) positive definite matrices on \( \mathbb{C} \). Let \( A_i, A'_i, B_i \in P_m(\mathbb{C}) \) for \( i = 1, \ldots, n \) and let \( \omega = (w_1, \ldots, w_n) \) be a probability vector. Then

(P1) Consistency with scalars. If \( A_1, \ldots, A_n \) commute with each other, then

\[
\mathcal{G}(\omega; A_1, \ldots, A_n) = A_1^{w_1} \cdots A_n^{w_n}.
\]

(P2) Joint homogeneity. For positive numbers \( a_i > 0 \) (\( i = 1, \ldots n \)),

\[
\mathcal{G}(\omega; a_1 A_1, \ldots, a_n A_n) = a_1^{w_1} \cdots a_n^{w_n} \mathcal{G}(\omega; A_1, \ldots, A_n).
\]

(P3) Permutation invariance. For any permutation \( \pi \) on \( \{1, \ldots n\} \),

\[
\mathcal{G}(\omega; A_1, \ldots, A_n) = \mathcal{G}(\pi(\omega); A_{\pi(1)}, \ldots, A_{\pi(n)}),
\]

where \( \pi(\omega) = (w_{\pi(1)}, \ldots, w_{\pi(n)}) \).

(P4) Monotonicity. If \( B_i \leq A_i \) for each \( i = 1, \ldots n \), then

\[
\mathcal{G}(\omega; B_1, \ldots, B_n) \leq \mathcal{G}(\omega; A_1, \ldots, A_n).
\]

(P5) Continuity. For each \( i = 1, \ldots n \), let \( \{A_i^{(k)}\}_{k=1}^{\infty} \) be positive definite matrix sequences such that \( A_i^{(k)} \rightarrow A_i \) as \( k \rightarrow \infty \). Then

\[
\mathcal{G}(\omega; A_1^{(k)}, \ldots, A_n^{(k)}) \rightarrow \mathcal{G}(\omega; A_1, \ldots, A_n) \quad \text{as} \quad k \rightarrow \infty.
\]

(P6) Congruence invariance. For any invertible matrix \( S \),

\[
\mathcal{G}(\omega; S^* A_1 S, \ldots, S^* A_n S) = S^* \mathcal{G}(\omega; A_1, \ldots, A_n) S.
\]
(P7) Joint concavity.

\[ \mathfrak{G}(\omega; \lambda A_1 + (1 - \lambda) A_1', \ldots, \lambda A_n + (1 - \lambda) A_n') \geq \lambda \mathfrak{G}(\omega; A_1, \ldots, A_n) + (1 - \lambda) \mathfrak{G}(\omega; A_1', \ldots, A_n') \text{ for } 0 \leq \lambda \leq 1. \]

(P8) Self-duality. \[ \mathfrak{G}(\omega; A_1^{-1}, \ldots, A_n^{-1})^{-1} = \mathfrak{G}(\omega; A_1, \ldots, A_n). \]

(P9) Determinant identity. \[ \det \mathfrak{G}(\omega; A_1, \ldots, A_n) = \prod_{i=1}^{n} (\det A_i)^{w_i}. \]

(P10) The arithmetic-geometric-harmonic mean inequality.

\[ \left( \sum_{i=1}^{n} w_i A_i^{-1} \right)^{-1} \leq \mathfrak{G}(\omega; A_1, \ldots, A_n) \leq \sum_{i=1}^{n} w_i A_i. \]

For \( A, B \in P_m(\mathbb{C}) \), Riemannian metric between \( A \) and \( B \) is defined as \( \delta_2(A, B) = \| \log \frac{1}{2} BA \|_2 \), where \( \| X \|_2 = (\text{tr} X^*X)^{\frac{1}{2}} \) (details are in [3]). By using Riemannian metric, Riemannian mean is defined as follows:

**Definition 1** ([3, 4, 18, 19]). Let \( A_1, \ldots, A_n \in P_m(\mathbb{C}) \) and \( \omega = (w_1, \ldots, w_n) \) be a probability vector. Then weighted Riemannian mean \( \mathfrak{G}_{\delta}(\omega; A_1, \ldots, A_n) \in P_m(\mathbb{C}) \) is defined by

\[ \mathfrak{G}_{\delta}(\omega; A_1, \ldots, A_n) = \arg \min_{X \in P_m(\mathbb{C})} \sum_{i=1}^{n} w_i \delta_2^2(A_i, X), \]

where \( \arg \min f(X) \) means the point \( X_0 \) which attains minimum value of the function \( f(X) \). In particular, we call \( \mathfrak{G}_{\delta}(\omega; A_1, \ldots, A_n) \) (denoted by \( \mathfrak{G}_{\delta}(A_1, \ldots, A_n) \) simply) Riemannian mean if \( \omega = \left( \frac{1}{n}, \ldots, \frac{1}{n} \right) \).

We remark that \( \mathfrak{G}_{\delta}(\omega; A, B) = A \#_{\alpha} B \) for \( \alpha \in [0, 1] \) and \( \omega = (1 - \alpha, \alpha) \) since the property \( \delta_2(A, A \#_{\alpha} B) = \alpha \delta_2(A, B) \) holds.

It is shown in [3, 4, 18, 19] that weighted Riemannian mean satisfies (P1)–(P10) (see also [21]). We remark that Riemannian mean has a stronger property (P5’) than (P5).

(P5’) Non-expansive.

\[ \delta_2(\mathfrak{G}_{\delta}(\omega; A_1, \ldots, A_n), \mathfrak{G}_{\delta}(\omega; B_1, \ldots, B_n)) \leq \sum_{i=1}^{n} w_i \delta_2(A_i, B_i). \]

Very recently, Yamazaki [21] has obtained an excellent generalization of Theorems 1.B and 1.C via weighted Riemannian mean \( \mathfrak{G}_{\delta} \) of \( n \)-matrices. We recall that \( \omega = (w_1, \ldots, w_n) \) is a probability vector if the components satisfy \( \sum_i w_i = 1 \) and \( w_i > 0 \) for \( i = 1, \ldots, n \).
**Theorem 1.D** ([21]). Let $A_1, \ldots, A_n \in P_m(\mathbb{C})$ and $\omega = (w_1, \ldots, w_n)$ be a probability vector. Then

$$\mathcal{G}_\delta(\omega; A_1, \ldots, A_n) \leq I \quad \text{implies} \quad \mathcal{G}_\delta(\omega; A_1^p, \ldots, A_n^p) \leq I \quad \text{for} \ p \geq 1.$$ 

**Theorem 1.E** ([21]). Let $A_1, \ldots, A_n \in P_m(\mathbb{C})$. Then the following assertions are mutually equivalent:

(i) $\log A_1 + \cdots + \log A_n \leq 0,$

(ii) $\mathcal{G}_\delta(A_1, \ldots, A_n) \leq I$ for all $p > 0,$

(iii) $\mathcal{G}_\delta(\omega; A_1^{p_1}, \ldots, A_n^{p_n}) \leq I$ for all $p_1, \ldots, p_n > 0,$ where $p_{\neq i} = \prod_{j \neq i} p_j$ and

$$\omega = \left( \frac{p_{\neq 1}}{\sum_{i} p_{\neq i}}, \ldots, \frac{p_{\neq n}}{\sum_{i} p_{\neq i}} \right).$$

Theorems 1.D and 1.E imply Theorems 1.B and 1.C, respectively, since $\mathcal{G}_\delta(\omega; A, B) = A \#_\alpha B$ for $\omega = (1 - \alpha, \alpha).$ Moreover, it has been shown in [21] that Theorem 1.D does not hold for other geometric means satisfying (P1)-(P10).

In this report, corresponding to Theorem 1.E, we shall obtain a generalization of Furuta inequality (Theorem 1.A) via weighted Riemannian mean of $n$-matrices. Moreover we shall show an extension of Theorem 1.D.

## 2 Results

Firstly, we show an extension of Theorem 1.D. Theorem 1.D follows from Theorem 2.1 by putting $p_1 = \cdots = p_n = p.$

**Theorem 2.1.** Let $A_1, \ldots, A_n \in P_m(\mathbb{C})$ and $\omega = (w_1, \ldots, w_n)$ be a probability vector. If $\mathcal{G}_\delta(\omega; A_1, \ldots, A_n) \leq I,$ then

$$\mathcal{G}_\delta(\omega'; A_1^{p_1}, \ldots, A_n^{p_n}) \leq \mathcal{G}_\delta(\omega; A_1, \ldots, A_n) \leq I \quad \text{for} \ p_1, \ldots, p_n \geq 1,$$

where $\tilde{\omega}' = (\frac{w_1}{p_1}, \ldots, \frac{w_n}{p_n})$ and $\omega' = \frac{\tilde{\omega}'}{||\tilde{\omega}'||_1}.$

We remark that $|| \cdot ||_1$ means 1-norm, that is, $||x||_1 = \sum_i |x_i|$ for $x = (x_1, \ldots, x_n).$ In order to prove Theorem 2.1, we use the following results.
Theorem 2.1 ([18, 19]). Let $A_1, \ldots, A_n \in P_m(\mathbb{C})$ and $\omega = (w_1, \ldots, w_n)$ be a probability vector. Then $X = \mathcal{G}_\delta(\omega; A_1, \ldots, A_n)$ is the unique positive solution of the following matrix equation:

$$w_1 \log X^{\frac{1}{2}} A_1 X^{\frac{1}{2}} + \cdots + w_n \log X^{\frac{1}{2}} A_n X^{\frac{1}{2}} = 0.$$ 

Theorem 2.2 ([21]). Let $A_1, \ldots, A_n \in P_m(\mathbb{C})$ and $\omega = (w_1, \ldots, w_n)$ be a probability vector. Then

$$w_1 \log A_1 + \cdots + w_n \log A_n \leq 0$$

implies $\mathcal{G}_\delta(\omega; A_1, \ldots, A_n) \leq I$.

Proof of Theorem 2.1. Let $X = \mathcal{G}_\delta(\omega; A_1, \ldots, A_n) \leq I$. Then for each $p_1, \ldots, p_n \in [1, 2]$, by Theorem 2.1 and Hansen’s inequality [14],

$$0 = \frac{1}{\|\omega\|_1} \sum w_i \log X^{\frac{1}{2}} A_i^{-1} X^{\frac{1}{2}} = \frac{1}{\|\omega\|_1} \sum \frac{w_i}{p_i} \log (X^{\frac{1}{2}} A_i^{-1} X^{\frac{1}{2}})^{p_i}$$

$$\leq \frac{1}{\|\omega\|_1} \sum \frac{w_i}{p_i} \log X^{\frac{1}{2}} A_i^{-p_i} X^{\frac{1}{2}},$$

that is,

$$\sum \frac{w_i}{p_i} \log X^{\frac{1}{2}} A_i^{-p_i} X^{\frac{1}{2}} \leq 0.$$ 

By applying Theorem 2.2,

$$\mathcal{G}_\delta(\omega'; A_1^{p_1}, \ldots, A_n^{p_n}) \leq X \leq I$$

where $\omega' = (\frac{w_1}{p_1}, \ldots, \frac{w_n}{p_n})$ and $\omega' = \frac{\omega}{\|\omega\|_1}$. Therefore we have that

$$X \leq I \text{ implies } \mathcal{G}_\delta(\omega'; A_1^{p_1}, \ldots, A_n^{p_n}) \leq X \leq I \text{ for } p_1, \ldots, p_n \in [1, 2]. \quad (2.1)$$

Put $Y = \mathcal{G}_\delta(\omega'; A_1^{p_1}, \ldots, A_n^{p_n}) \leq I$. Then by (2.1), we get

$$\mathcal{G}_\delta(\omega''; A_1^{p_1}, \ldots, A_n^{p_n}) \leq Y \leq X \leq I$$

for $p_1, \ldots, p_n \in [1, 2]$, where $\omega'' = (\frac{w_1}{p_1 p_1}, \ldots, \frac{w_n}{p_n p_n})$ and $\omega'' = \frac{\omega'}{\|\omega'\|_1}$. Therefore, by putting $q_i = p_i p_1$ for $i = 1, \ldots, n$, we have that

$$X \leq I \text{ implies } \mathcal{G}_\delta(\omega''; A_1^{q_1}, \ldots, A_n^{q_n}) \leq X \leq I \text{ for } q_1, \ldots, q_n \in [1, 4]. \quad (2.2)$$

where $\omega'' = (\frac{w_1}{q_1}, \ldots, \frac{w_n}{q_n})$ and $\omega'' = \frac{\omega''}{\|\omega''\|_1}$.

By repeating the same way from (2.1) to (2.2), we have the conclusion. \qed

Theorem 2.1 also implies generalized Ando-Hiai inequality [9] since $\mathcal{G}_\delta(\omega; A, B) = A \#_\alpha B$ for $\omega = (1 - \alpha, \alpha)$ and $\omega' = \left( \frac{1 - \alpha}{1 - \alpha + \alpha}, \frac{\alpha}{1 - \alpha + \alpha} \right) = \left( \frac{(1 - \alpha)s}{(1 - \alpha)s + \alpha r}, \frac{\alpha r}{(1 - \alpha)s + \alpha r} \right).$
Theorem 2.C (Generalized Ando-Hiai inequality [9]). Let $A, B > 0$. If $A \#_{\alpha} B \leq I$ for $\alpha \in (0, 1)$, then
\[ A^s \#_{\frac{3}{1-s}} B^t \leq A \#_{\alpha} B \leq I \text{ for } s \geq 1 \text{ and } t \geq 1. \]

The following Theorem 2.2 is a variant from Theorem 2.1.

Theorem 2.2. Let $A_1, \ldots, A_n \in P_m(\mathbb{C})$ and $\omega = (w_1, \ldots, w_n)$ be a probability vector. For each $i = 1, \ldots, n$ and $q \in \mathbb{R}$, if
\[ G_\delta(\omega; A_1^{p_1}, \ldots, A_i^{p_i}, \ldots, A_n^{p_n}) \leq A_i^q \text{ for } p_1, \ldots, p_n \in \mathbb{R} \text{ with } p_i > q, \]
then
\[ G_\delta(\omega; A_1^{p_1}, \ldots, A_i^{p_i-1}, A_i^{p_i}, A_i^{p_i+1}, \ldots, A_n^{p_n}) \leq G_\delta(\omega; A_1^{p_1}, \ldots, A_i^{p_i}, A_i^{p_i+1}, \ldots, A_n^{p_n}) \leq A_i^q \]
for $p_i' \geq p_i$, where $\tilde{\omega}' = (w_1, \ldots, w_{i-1}, \frac{p_i-q}{p_i-q} w_i, w_{i+1}, \ldots, w_n)$ and $\omega' = \frac{\omega'}{\|\omega\|_1}$.

Proof. We may assume $i = 1$ by permutation invariance of $G_\delta$.

For $p_1, \ldots, p_n \in \mathbb{R}$ with $p_1 \geq q$, $G_\delta(\omega; A_1^{p_1}, A_2^{p_2}, \ldots, A_n^{p_n}) \leq A_1^q$ if and only if
\[ G_\delta(\omega; A_1^{p_1-q}, A_2^{p_2}, A_3^{p_3}, \ldots, A_i^{p_i-1}, A_i^{p_i}, A_i^{p_i+1}, \ldots, A_n^{p_n}) \leq I. \]
By applying Theorem 2.1,
\[ G_\delta(\omega'; A_1^{p_1-q}, A_2^{p_2}, A_3^{p_3}, \ldots, A_i^{p_i-1}, A_i^{p_i}, A_i^{p_i+1}, \ldots, A_n^{p_n}) \leq I, \]
holds for $\frac{\tilde{p}_i-q}{\tilde{p}_i-q} \geq 1$, where $\tilde{\omega}' = (\frac{\tilde{p}_1-q}{\tilde{p}_1-q} w_1, w_2, \ldots, w_n)$. Therefore
\[ G_\delta(\omega'; A_1^{p_1}, A_2^{p_2}, \ldots, A_n^{p_n}) \leq G_\delta(\omega; A_1^{p_1}, A_2^{p_2}, \ldots, A_n^{p_n}) \leq A_1^q \]
holds for $p_i' \geq p_i$. \qed

Next, we show our main result. The following Theorem 2.3 is a generalization of Theorem 1.A, and also a parallel result to $(i) \implies (iii)$ in Theorem 1.E.
Theorem 2.3. Let $A_1, \ldots, A_n \in P_m(\mathbb{C})$ and $w_1, \ldots, w_n > 0$. If

$$A_1^{q_1} \geq A_n^{q_n} > 0 \quad (2.3)$$

and

$$\frac{w_1}{p_1 - q_1} \log A_1^{1/2} + \cdots + \frac{w_{n-1}}{p_{n-1} - q_{n-1}} \log A_{n-1}^{1/2} A_n^{q_n} + \frac{w_n}{p_n - q_n} \log A_n^{q_n} \leq 0 \quad (2.4)$$

hold for $q_i \in \mathbb{R}$, $p_i > q_i$ and $i = 1, \ldots, n$, then

$$\otimes_{\delta}(\omega'; A_{1}^{p_{1}'}, \ldots, A_{n}^{p_{n}'}) \leq \otimes_{\delta}(\omega; A_{1}^{p_{1}}, \ldots, A_{n}^{p_{n}}) \leq A_n^{q} \quad (2.5)$$

for all $p_i' \geq p_i$ and $i = 1, \ldots, n$.

Proof. Applying Theorem 2.2 to (2.4), we have

$$\Theta_\delta(\omega; A_{1}^{1/2} A_{2}^{p_{1}'}, \ldots, A_{n}^{p_{n}'}) \leq \Theta_\delta(\omega; A_{1}^{p_{1}'}, A_{2}^{p_{2}'}, \ldots, A_{n}^{p_{n}'}) \leq \Theta_\delta(\omega; A_{1}^{p_{1}'}, A_{2}^{p_{2}'}, \ldots, A_{n}^{p_{n}'}) \leq I,$$

so that by (2.3),

$$X_0 \equiv \Theta_\delta(\omega; A_{1}^{p_{1}'}, \ldots, A_{n}^{p_{n}'}) \leq A_n^{q} \leq A_1^{q_1}. \quad (2.5)$$

By applying Theorem 2.2 to (2.5) and by (2.3),

$$X_1 \equiv \Theta_\delta(\omega_{1}; A_{1}^{p_{1}'}, \ldots, A_{n}^{p_{n}'}) \leq X_0 \leq A_n^{q} \leq A_2^{q_2}. \quad (2.6)$$

for $p_1' \geq p_1$, where $\omega_1 = \left( \frac{w_1}{p_1 - q_1}, \ldots, \frac{w_n}{p_n - q_n} \right)$ and $\omega = \frac{\omega_1}{\|\omega_1\|_1}$. By applying Theorem 2.2 to (2.6) and by (2.3),

$$X_2 \equiv \Theta_\delta(\omega_{2}; A_{1}^{p_{1}'}, A_{2}^{p_{2}'}, \ldots, A_{n}^{p_{n}'}) \leq X_1 \leq X_0 \leq A_n^{q} \leq A_3^{q_3}$$

for $p_1' \geq p_1$ and $p_2' \geq p_2$, where $\omega_2 = \left( \frac{w_1}{p_1 - q_1}, \frac{w_2}{p_2 - q_2}, \ldots, \frac{w_n}{p_n - q_n} \right)$ and $\omega_2 = \frac{\omega_2}{\|\omega_2\|_1}$. By repeating this argument, we can get

$$X_n \equiv \Theta_\delta(\omega_{n}; A_{1}^{p_{1}'}, \ldots, A_{n}^{p_{n}'}) \leq X_{n-1} \leq X_0 \leq A_n^{q} \leq A_{n+1}^{q_{n+1}}$$

for $p_i' \geq p_i$ for $i = 1, \ldots, n$, where $\omega_{n} = \left( \frac{w_1}{p_1 - q_1}, \ldots, \frac{w_n}{p_n - q_n} \right)$.

Remark. (i) in Theorem 1.E, that is, log $A_1 + \cdots + \log A_n \leq 0$ holds if and only if

$$\frac{1}{p_1} \log A_1^{p_1} + \cdots + \frac{1}{p_n} \log A_n^{p_n} \leq 0 \quad \text{for every } p_i > 0 \text{ and } i = 1, \ldots, n.$$
Therefore we recognize that Theorem 2.3 implies (i) $\Rightarrow$ (iii) in Theorem 1.E by putting $q_1 = \cdots = q_n = 0$ and $w_1 = \cdots = w_n = 1$ since
\[
\frac{1}{p_i} \frac{1}{\|\omega\|_1} = \frac{1}{p_i} + \cdots + \frac{1}{p_n} = \frac{p_{\neq i}}{\sum_j p_{\neq j}} \quad \text{for } i = 1, \ldots, n
\]
ensures $\omega = \frac{\hat{\omega}}{\|\hat{\omega}\|_1} = \left( \frac{1}{p_1 + q}, \ldots, \frac{1}{p_{n-1} + q}, \frac{n-1}{p_n - q} \right)$.

It is well known that we have a variant from Theorem 1.A by replacing $A$, $B$, $p$, $r$ with $A^q$, $B^q$, and $p$, $r$ with $\frac{p}{q}$, $\frac{r}{q}$ in Theorem 1.A respectively.

**Theorem 2.D** ([8]). Let $A > 0$, $B \geq 0$ and $q > 0$. Then
\[
A^q \geq B^q \quad \text{implies} \quad \bar{A} = \left( A_1^{-p_1}, A_2^{-p_2}, \ldots, A_n^{-p_n} \right) \leq A^q \leq B^q \leq A^q \quad \text{for } p \geq q \quad \text{and} \quad r \geq 0.
\]

Here we show that Theorem 2.3 is a generalization of Furuta inequality via weighted Riemannian mean of $n$-matrices. Precisely, we show that Theorem 2.3 ensures the following Theorem 2.4 and Theorem 2.4 is a generalization of Theorem 2.D.

**Theorem 2.4.** Let $A_1, \ldots, A_n \in P_m(\mathbb{C})$ and $q > 0$. Then $A_i^q \geq A_n^q > 0$ for $i = 1, \ldots, n-1$ implies
\[
\mathcal{G}_\delta(\omega; A_1^{-p_1}, \ldots, A_{n-1}^{-p_{n-1}}, A_n^p) \leq A_n^q \leq A_i^q
\]
for all $p_i \geq 0$, $i = 1, \ldots, n-1$ and $p_n > q$, where $\hat{\omega} = \left( \frac{1}{p_1 + q}, \ldots, \frac{1}{p_{n-1} + q}, \frac{n-1}{p_n - q} \right)$ and $\omega = \frac{\hat{\omega}}{\|\hat{\omega}\|_1}$.

**Proof.** Assume that $A_i^q \geq A_n^q > 0$ for $q > 0$ and $i = 1, \ldots, n-1$. Then $A_i^q \geq A_n^q > 0$ implies $\log A_i \geq \log A_n$. By (i) $\Rightarrow$ (iii) in Theorem 1.C, $\log A_i \geq \log A_n$ implies $A_i^{-p_1} A_j^{-p_j} A_n^p \leq I$ for all $p_i \geq 0$. This is equivalent to $A_i^{-q_1} A_j^{-q_j} A_n^p \geq I$, that is, $(A_i^{-\frac{q}{q_i}} A_j^{-\frac{q}{q_j}} A_n^{-\frac{q}{q_n}}) \leq A_i^q$. By taking logarithm, we have $\frac{1}{p_i + q} \log A_i^q \geq \frac{1}{p_n - q} \log A_n^{-q}$, that is,
\[
\frac{1}{p_i + q} \log A_i^{-q} (A_i^{-1})^{p_i} A_i^{-\frac{q}{q_n}} + \frac{1}{p_n - q} \log A_n^{-q} \leq 0
\]
for all $p_i \geq 0$, $i = 1, \ldots, n-1$ and $p_n > q$. Summing up (2.8) for $i = 1, \ldots, n-1$, we have
\[
\frac{1}{p_1 + q} \log A_i^{-q} (A_1^{-1})^{p_1} A_i^{-\frac{q}{q_n}} + \cdots
\]
\[
+ \frac{1}{p_{n-1} + q} \log A_i^{-q} (A_n^{-1})^{p_{n-1}} A_i^{-\frac{q}{q_n}} + \frac{n-1}{p_n - q} \log A_n^{-q} \leq 0.
\]
By applying Theorem 2.3 to $(A_i^{-1})^{-q} \geq A_n^q > 0$ and (2.9), we can obtain
\[ \mathfrak{G}_\delta(\omega; A_1^{-p_1}, \ldots, A_{n-1}^{-p_{n-1}}, A_n^p) \leq A_n^q \leq A_i^q \]
for all $p_i \geq 0 > -q$, $i = 1, \ldots, n - 1$ and $p_n > q$. \hfill \square

Proof of Theorem 2.D. Put $n = 2$, $p_1 = r$ and $p_2 = p$ in Theorem 2.4. Then $\hat{\omega} = \left( \frac{1}{r+q}, \frac{1}{p-q} \right)$ and $\omega = \left( \frac{p-q}{p+r}, \frac{q+r}{p+r} \right)$. Therefore we obtain the desired result. \hfill \square

3 3-matrices case

In this section, for the sake of readers' convenience, we state 3-matrices case of Theorems 2.3 and 2.4.

Corollary 3.1. Let $A, B, C \in P_m(\mathbb{C})$ and $w_1, w_2, w_3 > 0$. If
\[ A^{q_1} \geq C^{q_3} > 0, \quad B^{q_2} \geq C^{q_3} > 0, \]
and
\[ \frac{w_1}{p_1 - q_1} \log C^{-\frac{q}{2}} A^{p_1} C^{-\frac{q}{2}} + \frac{w_2}{p_2 - q_2} \log C^{-\frac{q}{2}} B^{p_2} C^{-\frac{q}{2}} + \frac{w_3}{p_3 - q_3} \log C^{-\frac{q}{2}} C^{p_3} C^{-\frac{q}{2}} \leq 0 \]
hold for $q_i \in \mathbb{R}$, $p_i > q_i$ and $i = 1, 2, 3$, then
\[ \mathfrak{G}_\delta(\omega'; A^{-p'_1}, B^{-p'_2}, C^{p'_3}) \leq \mathfrak{G}_\delta(\omega; A^{p_1}, B^{p_2}, C^{p_3}) \leq C^{q_3} \]
for all $p'_i \geq p_i$ and $i = 1, 2, 3$, where $\hat{\omega} = \left( \frac{w_1}{p_1 - q_1}, \frac{w_2}{p_2 - q_2}, \frac{w_3}{p_3 - q_3} \right)$, $\omega = \frac{\hat{\omega}}{\|\omega\|_1}$ and $\omega' = \frac{\hat{\omega'}}{\|\omega'\|_1}$.

Corollary 3.2. Let $A, B, C \in P_m(\mathbb{C})$ and $q > 0$. Then $A^q \geq C^q > 0$ and $B^q \geq C^q > 0$ implies
\[ \mathfrak{G}_\delta(\omega; A^{-r}, B^{-s}, C^p) \leq C^{q} \leq A^{q} \] (or $B^q$)
for $r \geq 0$, $s \geq 0$ and $p > q$, where $\hat{\omega} = \left( \frac{1}{r+q}, \frac{1}{s+q}, \frac{2}{p-q} \right)$ and $\omega = \frac{\hat{\omega}}{\|\omega\|_1}$.
References


[10] T. Furuta, $A \geq B \geq 0$ assures $(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0$, $p \geq 0$, $q \geq 1$ with $(1+2r)q \geq p+2r$, Proc. Amer. Math. Soc., 101 (1987), 85–88.


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