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Kyoto University
Matrix inequalities including Furuta inequality via Riemannian mean of $n$-matrices

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Abstract

In this report, we shall obtain a generalization of Furuta inequality via weighted Riemannian mean, a kind of geometric mean, of $n$-matrices. This result is related to Yamazaki’s recent results which is a kind of generalizations of Ando-Hiai inequality and Furuta inequality for chaotic order.

1 Introduction

The weighted geometric mean of two positive definite matrices $A$ and $B$ defined by $A \#_\alpha B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}$ for $\alpha \in [0,1]$. In particular, we call $A \#_1 B$ (denoted by $A \# B$ simply) the geometric mean of $A$ and $B$. The weighted geometric mean sometimes appears in famous matrix inequalities, for example, Furuta inequality [10] (see also [6, 11, 13, 17, 20]) and Ando-Hiai inequality [1]. We remark that these inequalities hold even in the case of bounded linear operators on a complex Hilbert space. In what follows, we denote $A \geq 0$ if $A$ is a positive semidefinite matrix (or operator), and we denote $A > 0$ if $A$ is a positive definite matrix (or operator).

Theorem 1.A (Satellite form of Furuta inequality [10, 17]).

\[ A \geq B \geq 0 \text{ with } A > 0 \implies A^{-r} \#_{\frac{1+r}{p+r}} B^p \leq B \leq A \text{ for } p \geq 1 \text{ and } r \geq 0. \]

Theorem 1.B (Ando-Hiai inequality [1]). For $A, B > 0$,

\[ A \#_\alpha B \leq I \text{ for } \alpha \in (0,1) \implies A^r \#_\alpha B^r \leq I \text{ for } r \geq 1. \]

For $A, B > 0$, it is well known that chaotic order $\log A \geq \log B$ is weaker than usual order $A \geq B$ since $\log t$ is a matrix (or operator) monotone function. The following result is known as the Furuta inequality for chaotic order.

Theorem 1.C (Furuta inequality for chaotic order [7, 12]). Let $A, B > 0$. Then the following assertions are mutually equivalent;

(i) $\log A \geq \log B$,

(ii) $A^{-p} \#^p B^p \leq I$ for all $p \geq 0$,

(iii) $A^{-r} \#^r \frac{r}{p+r} B^p \leq I$ for all $p \geq 0$ and $r \geq 0$.

It has been a longstanding problem to extend the (weighted) geometric mean for three or more positive definite matrices. Many authors attempt to find a natural extension, for example, Ando-Li-Mathias' mean and its refinement [2, 5, 15, 16] and Riemannian mean (or the least squares mean) [4, 18, 19]. We remark that Ando-Li-Mathias [2] originally proposed the following ten properties (P1)–(P10) which should be required for a reasonable geometric mean $\mathfrak{G}$ of positive definite matrices. We note that, in [2], they require continuity from above as (P5).

Let $P_m(\mathbb{C})$ be the set of $m \times m$ positive definite matrices on $\mathbb{C}$. Let $A_i, A'_i, B_i \in P_m(\mathbb{C})$ for $i = 1, \ldots, n$ and let $\omega = (w_1, \ldots, w_n)$ be a probability vector. Then

(P1) Consistency with scalars. If $A_1, \ldots, A_n$ commute with each other, then

$$\mathfrak{G}(\omega; A_1, \ldots, A_n) = A_1^{w_1} \cdots A_n^{w_n}.$$

(P2) Joint homogeneity. For positive numbers $a_i > 0$ ($i = 1, \ldots, n$),

$$\mathfrak{G}(\omega; a_1 A_1, \ldots, a_n A_n) = a_1^{w_1} \cdots a_n^{w_n} \mathfrak{G}(\omega; A_1, \ldots, A_n).$$

(P3) Permutation invariance. For any permutation $\pi$ on $\{1, \ldots, n\}$,

$$\mathfrak{G}(\omega; A_1, \ldots, A_n) = \mathfrak{G}(\pi(\omega); A_{\pi(1)}, \ldots, A_{\pi(n)}),$$

where $\pi(\omega) = (w_{\pi(1)}, \cdots, w_{\pi(n)})$.

(P4) Monotonicity. If $B_i \leq A_i$ for each $i = 1, \ldots, n$, then

$$\mathfrak{G}(\omega; B_1, \ldots, B_n) \leq \mathfrak{G}(\omega; A_1, \ldots, A_n).$$

(P5) Continuity. For each $i = 1, \ldots, n$, let $\{A_i^{(k)}\}_{k=1}^{\infty}$ be positive definite matrix sequences such that $A_i^{(k)} \rightharpoonup A_i$ as $k \to \infty$. Then

$$\mathfrak{G}(\omega; A_1^{(k)}, \ldots, A_n^{(k)}) \rightharpoonup \mathfrak{G}(\omega; A_1, \ldots, A_n) \quad \text{as} \quad k \to \infty.$$

(P6) Congruence invariance. For any invertible matrix $S$,

$$\mathfrak{G}(\omega; S^* A_1 S, \ldots, S^* A_n S) = S^* \mathfrak{G}(\omega; A_1, \ldots, A_n) S.$$

(P7) Joint concavity.

\[ \mathcal{G}(\omega; \lambda A_1 + (1 - \lambda)A_1', \ldots, \lambda A_n + (1 - \lambda)A_n') \geq \lambda \mathcal{G}(\omega; A_1, \ldots, A_n) + (1 - \lambda)\mathcal{G}(\omega; A_1', \ldots, A_n') \quad \text{for } 0 \leq \lambda \leq 1. \]

(P8) Self-duality. \( \mathcal{G}(\omega; A_1^{-1}, \ldots, A_n^{-1})^{-1} = \mathcal{G}(\omega; A_1, \ldots, A_n). \)

(P9) Determinant identity. \[ \det \mathcal{G}(\omega; A_1, \ldots, A_n) = \prod_{i=1}^{n} (\det A_i)^{w_i}. \]

(P10) The arithmetic-geometric-harmonic mean inequality.

\[ \left( \sum_{i=1}^{n} w_i A_i^{-1} \right)^{-1} \leq \mathcal{G}(\omega; A_1, \ldots, A_n) \leq \sum_{i=1}^{n} w_i A_i. \]

For \( A, B \in P_m(\mathbb{C}) \), Riemannian metric between \( A \) and \( B \) is defined as \( \delta_2(A, B) = \| \log A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \|_2 \), where \( \| X \|_2 = (\text{tr} X^* X)^{\frac{1}{2}} \) (details are in [3]). By using Riemannian metric, Riemannian mean is defined as follows:

**Definition 1** ([3, 4, 18, 19]). Let \( A_1, \ldots, A_n \in P_m(\mathbb{C}) \) and \( \omega = (w_1, \ldots, w_n) \) be a probability vector. Then weighted Riemannian mean \( \mathfrak{G}_{\delta}(\omega; A_1, \ldots, A_n) \in P_m(\mathbb{C}) \) is defined by

\[ \mathfrak{G}_{\delta}(\omega; A_1, \ldots, A_n) = \arg \min_{X \in P_m(\mathbb{C})} \sum_{i=1}^{n} w_i \delta_2^2(A_i, X), \]

where \( \arg \min f(X) \) means the point \( X_0 \) which attains minimum value of the function \( f(X) \). In particular, we call \( \mathfrak{G}_{\delta}(\omega; A_1, \ldots, A_n) \) (denoted by \( \mathfrak{G}_{\delta}(A_1, \ldots, A_n) \) simply) Riemannian mean if \( \omega = \left( \frac{1}{n}, \ldots, \frac{1}{n} \right) \).

We remark that \( \mathfrak{G}_{\delta}(\omega; A, B) = A \#_{\alpha} B \) for \( \alpha \in [0, 1] \) and \( \omega = (1 - \alpha, \alpha) \) since the property \( \delta_2(A, A \#_{\alpha} B) = \alpha \delta_2(A, B) \) holds.

It is shown in [3, 4, 18, 19] that weighted Riemannian mean satisfies (P1)–(P10) (see also [21]). We remark that Riemannian mean has a stronger property (P5') than (P5).

(P5') Non-expansive.

\[ \delta_2(\mathfrak{G}_{\delta}(\omega; A_1, \ldots, A_n), \mathfrak{G}_{\delta}(\omega; B_1, \ldots, B_n)) \leq \sum_{i=1}^{n} w_i \delta_2(A_i, B_i). \]

Very recently, Yamazaki [21] has obtained an excellent generalization of Theorems 1.B and 1.C via weighted Riemannian mean \( \mathfrak{G}_{\delta} \) of n-matrices. We recall that \( \omega = (w_1, \ldots, w_n) \) is a probability vector if the components satisfy \( \sum_i w_i = 1 \) and \( w_i > 0 \) for \( i = 1, \ldots, n. \)
Theorem 1.D ([21]). Let $A_1, \ldots, A_n \in P_m(\mathbb{C})$ and $\omega = (w_1, \ldots, w_n)$ be a probability vector. Then
\[ \mathfrak{G}_\delta(\omega; A_1, \ldots, A_n) \leq I \quad \text{implies} \quad \mathfrak{G}_\delta(\omega; A_1^p, \ldots, A_n^p) \leq I \quad \text{for } p \geq 1. \]

Theorem 1.E ([21]). Let $A_1, \ldots, A_n \in P_m(\mathbb{C})$. Then the following assertions are mutually equivalent;

(i) $\log A_1 + \cdots + \log A_n \leq 0$,

(ii) $\mathfrak{G}_\delta(A_1^p, \ldots, A_n^p) \leq I$ for all $p > 0$,

(iii) $\mathfrak{G}_\delta(\omega; A_1^{p_1}, \ldots, A_n^{p_n}) \leq I$ for all $p_1, \ldots, p_n > 0$, where $p_{\neq i} = \prod_{j \neq i} p_j$ and
\[ \omega = \left( \frac{p_{\neq 1}}{\sum_i p_{\neq i}}, \ldots, \frac{p_{\neq n}}{\sum_i p_{\neq i}} \right). \]

Theorems 1.D and 1.E imply Theorems 1.B and 1.C, respectively, since $\mathfrak{G}_\delta(\omega; A, B) = A \#_{\alpha} B$ for $\omega = (1 - \alpha, \alpha)$. Moreover, it has been shown in [21] that Theorem 1.D does not hold for other geometric means satisfying (P1)-(P10).

In this report, corresponding to Theorem 1.E, we shall obtain a generalization of Furuta inequality (Theorem 1.A) via weighted Riemannian mean of $n$-matrices. Moreover we shall show an extension of Theorem 1.D.

2 Results

Firstly, we show an extension of Theorem 1.D. Theorem 1.D follows from Theorem 2.1 by putting $p_1 = \cdots = p_n = p$.

Theorem 2.1. Let $A_1, \ldots, A_n \in P_m(\mathbb{C})$ and $\omega = (w_1, \ldots, w_n)$ be a probability vector. If $\mathfrak{G}_\delta(\omega; A_1, \ldots, A_n) \leq I$, then
\[ \mathfrak{G}_\delta(\omega'; A_1^{p_1}, \ldots, A_n^{p_n}) \leq \mathfrak{G}_\delta(\omega; A_1, \ldots, A_n) \leq I \quad \text{for } p_1, \ldots, p_n \geq 1, \]
where $\omega' = (\frac{w_{1}}{p_1}, \ldots, \frac{w_{n}}{p_n})$ and $\omega' = \frac{\omega'}{||\omega'||_1}$.

We remark that $\| \cdot \|_1$ means 1-norm, that is, $\|x\|_1 = \sum_i |x_i|$ for $x = (x_1, \ldots, x_n)$. In order to prove Theorem 2.1, we use the following results.
Theorem 2.1 ([18, 19]). Let \( \omega = (w_1, \ldots, w_n) \) be a probability vector. Then \( X = \mathfrak{G}_\delta(\omega; A_1, \ldots, A_n) \) is the unique positive solution of the following matrix equation:

\[
w_1 \log X^{\frac{-1}{2}} A_1 X^{\frac{-1}{2}} + \cdots + w_n \log X^{\frac{-1}{2}} A_n X^{\frac{-1}{2}} = 0.
\]

Theorem 2.2 ([21]). Let \( \omega = (w_1, \ldots, w_n) \) be a probability vector. Then

\[
w_1 \log A_1 + \cdots + w_n \log A_n \leq 0 \quad \text{implies} \quad \mathfrak{G}_\delta(\omega; A_1, \ldots, A_n) \leq I.
\]

Proof of Theorem 2.1. Let \( X = \mathfrak{G}_\delta(\omega; A_1, \ldots, A_n) \leq I \). Then for each \( p_1, \ldots, p_n \in [1, 2] \), by Theorem 2.1 and Hansen's inequality [14],

\[
0 = \frac{1}{\|\omega\|_1} \sum_{i=1}^{n} w_i \log X^{\frac{1}{2}} A_i^{-1} X^{\frac{1}{2}} = \frac{1}{\|\omega\|_1} \sum_{i=1}^{n} \frac{w_i}{p_i} \log (X^{\frac{1}{2}} A_i^{-1} X^{\frac{1}{2}})^{p_i} \leq \frac{1}{\|\omega\|_1} \sum_{i=1}^{n} \frac{w_i}{p_i} \log X^{\frac{1}{2}} A_i^{-p_i} X^{\frac{1}{2}},
\]

that is,

\[
\sum_{i=1}^{n} \frac{w_i}{p_i} \log X^{\frac{1}{2}} A_i^{p_i} X^{\frac{1}{2}} \leq 0.
\]

By applying Theorem 2.2,

\[
\mathfrak{G}_\delta(\omega'; A_1^{p_1}, \ldots, A_n^{p_n}) \leq X \leq I
\]

where \( \omega' = (\frac{w_1}{p_1}, \ldots, \frac{w_n}{p_n}) \) and \( \omega = \frac{\omega'}{\|\omega\|_1} \). Therefore we have that

\[
X \leq I \quad \text{implies} \quad \mathfrak{G}_\delta(\omega'; A_1^{p_1}, \ldots, A_n^{p_n}) \leq X \leq I \quad \text{for} \quad p_1, \ldots, p_n \in [1, 2]. \tag{2.1}
\]

Put \( Y = \mathfrak{G}_\delta(\omega'; A_1^{p_1}, \ldots, A_n^{p_n}) \leq I \). Then by (2.1), we get

\[
\mathfrak{G}_\delta(\omega''; A_1^{q_1}, \ldots, A_n^{q_n}) \leq Y \leq X \leq I
\]

for \( q_1, \ldots, q_n \in [1, 2] \), where \( \omega'' = (\frac{w_1}{q_1}, \ldots, \frac{w_n}{q_n}) \) and \( \omega' = \frac{\omega''}{\|\omega''\|_1} \). Therefore, by putting \( q_i = p_i p_i' \) for \( i = 1, \ldots, n \), we have that

\[
X \leq I \quad \text{implies} \quad \mathfrak{G}_\delta(\omega''; A_1^{q_1}, \ldots, A_n^{q_n}) \leq X \leq I \quad \text{for} \quad q_1, \ldots, q_n \in [1, 4]. \tag{2.2}
\]

where \( \omega'' = (\frac{w_1}{q_1}, \ldots, \frac{w_n}{q_n}) \) and \( \omega' = \frac{\omega''}{\|\omega''\|_1} \).

By repeating the same way from (2.1) to (2.2), we have the conclusion. \( \square \)

Theorem 2.1 also implies generalized Ando-Hiai inequality [9] since \( \mathfrak{G}_\delta(\omega; A, B) = A \#_{\alpha} B \) for \( \omega = (1-\alpha, \alpha) \) and \( \omega' = \left( \frac{1-\alpha}{1-\alpha + \alpha}, \frac{\alpha}{1-\alpha + \alpha} \right) = \left( \frac{(1-\alpha)s}{(1-\alpha)s + \alpha r}, \frac{\alpha r}{(1-\alpha)s + \alpha r} \right) \).
Theorem 2.C (Generalized Ando-Hiai inequality [9]). Let $A, B > 0$. If $A \#_{\alpha} B \leq I$ for $\alpha \in (0, 1)$, then

$$A^r \#_{(1-s)\alpha+s} B^s \leq A \#_{\alpha} B \leq I \text{ for } s \geq 1 \text{ and } r \geq 1.$$  

The following Theorem 2.2 is a variant from Theorem 2.1.

Theorem 2.2. Let $A_1, \ldots, A_n \in P_m(C)$ and $\omega = (w_1, \ldots, w_n)$ be a probability vector. For each $i = 1, \ldots, n$ and $q \in \mathbb{R}$, if

$$\mathcal{G}_\delta(\omega; A_1^{p_1}, \ldots, A_i^{p_i}, \ldots, A_n^{p_n}) \leq A_i^q \text{ for } p_1, \ldots, p_n \in \mathbb{R} \text{ with } p_i > q,$$

then

$$\begin{align*}
\mathcal{G}_\delta(\omega; A_1^{p_1}, \ldots, A_i^{p_i-1}, A_i^{p_i}, A_{i+1}^{p_{i+1}}, \ldots, A_n^{p_n}) & \leq \mathcal{G}_\delta(\omega; A_1^{p_1}, \ldots, A_i^{p_i}, A_{i+1}^{p_{i+1}}, \ldots, A_n^{p_n}) \\
& \leq A_i^q
\end{align*}$$

for $p_i' \geq p_i$, where $\tilde{\omega}' = (w_1, \ldots, w_{i-1}, \frac{p_i-q}{p_i-q}w_i, w_{i+1}, \ldots, w_n)$ and $\omega' = \frac{\omega'}{\|\omega\|_1}$.

Proof. We may assume $i = 1$ by permutation invariance of $\mathcal{G}_\delta$.

For $p_1, \ldots, p_n \in \mathbb{R}$ with $p_1 \geq q$, $\mathcal{G}_\delta(\omega; A_1^{p_1}, A_2^{p_2}, \ldots, A_n^{p_n}) \leq A_1^q$ if and only if

$$\mathcal{G}_\delta(\omega; A_1^{p_1-q}, A_1^{\frac{q}{p_1-q}} A_2^{p_2} A_1^{\frac{q}{p_1-q}}, \ldots, A_1^{\frac{q}{p_1-q}} A_n^{p_n} A_1^{\frac{q}{p_1-q}}) \leq I.$$  

By applying Theorem 2.1,

$$\begin{align*}
\mathcal{G}_\delta(\omega'; A_1^{p_1-q}, A_1^{\frac{q}{p_1-q}} A_2^{p_2} A_1^{\frac{q}{p_1-q}}, \ldots, A_1^{\frac{q}{p_1-q}} A_n^{p_n} A_1^{\frac{q}{p_1-q}}) & \leq \mathcal{G}_\delta(\omega; A_1^{p_1-q}, A_1^{\frac{q}{p_1-q}} A_2^{p_2} A_1^{\frac{q}{p_1-q}}, \ldots, A_1^{\frac{q}{p_1-q}} A_n^{p_n} A_1^{\frac{q}{p_1-q}}) \\
& \leq I,
\end{align*}$$

holds for $\frac{p_1-q}{p_1-q} \geq 1$, where $\tilde{\omega}' = (\frac{p_1-q}{p_1-q}w_1, w_2, \ldots, w_n)$. Therefore

$$\mathcal{G}_\delta(\omega'; A_1^{p_1}, A_2^{p_2}, \ldots, A_n^{p_n}) \leq \mathcal{G}_\delta(\omega; A_1^{p_1}, A_2^{p_2}, \ldots, A_n^{p_n}) \leq A_1^q$$

holds for $p_i' \geq p_i$. \hfill \Box

Next, we show our main result. The following Theorem 2.3 is a generalization of Theorem 1.A, and also a parallel result to (i) \implies (iii) in Theorem 1.E.
Theorem 2.3. Let $A_1, \ldots, A_n \in P_m(\mathbb{C})$ and $w_1, \ldots, w_n > 0$. If
\[ A_1^{w_1} \geq A_n^{w_n} > 0 \tag{2.3} \]
and
\[ \frac{w_1}{p_1 - q_1} \log A_1^{-q_1} A_{1}^{p_1} A_n^{-q_n} + \cdots + \frac{w_{n-1}}{p_{n-1} - q_{n-1}} \log A_{n-1}^{-q_{n-1}} A_n^{-q_n} + \frac{w_n}{p_n - q_n} \log A_n^{-q_n} \leq 0 \tag{2.4} \]
hold for $q_i \in \mathbb{R}$, $p_i > q_i$ and $i = 1, \ldots, n$, then
\[ \hat{\omega} = \left( \frac{w_1}{p_1 - q_1}, \ldots, \frac{w_n}{p_n - q_n} \right), \quad \omega = \frac{\hat{\omega}}{\|\hat{\omega}\|_1} \text{ and } \omega' = \frac{\hat{\omega}'}{\|\omega\|_1}. \]

Proof. Applying Theorem 2.B to (2.4), we have
\[ \mathfrak{G}_\delta(\omega; A_1^{\frac{-q_1}{p_1}} A_{1}^{p_1} A_n^{\frac{-q_n}{p_n}}, \ldots, A_n^{\frac{-q_n}{p_n}} A_{n-1}^{p_{n-1}} A_{n}^{\frac{-q_n}{p_n}}, A_{n}^{p_n}) \leq I \]
so that by (2.3),
\[ X_0 = \mathfrak{G}_\delta(\omega; A_1^{p_1}, \ldots, A_{n-1}^{p_{n-1}}, A_{n}^{p_n}) \leq \log A_1 + \cdots + \log A_n = 0. \tag{2.5} \]
By applying Theorem 2.2 to (2.5) and by (2.3),
\[ X_1 = \mathfrak{G}_\delta(\omega_1; A_1^{p_1}, A_2^{p_2}, \ldots, A_n^{p_n}) \leq X_0 \leq A_n^{q_n} \leq A_2^{q_2} \tag{2.6} \]
for $p_1' \geq p_1$, where $\hat{\omega}_1 = \left( \frac{w_1}{p_1' - q_1}, \ldots, \frac{w_n}{p_n' - q_n} \right)$ and $\omega_1 = \frac{\hat{\omega}_1}{\|\hat{\omega}_1\|_1}$. By applying Theorem 2.2 to (2.6) and by (2.3),
\[ X_2 = \mathfrak{G}_\delta(\omega_2; A_1^{p_1}, A_2^{p_2}, A_3^{p_3}, \ldots, A_n^{p_n}) \leq X_1 \leq X_0 \leq A_n^{q_n} \leq A_3^{q_3} \]
for $p_1' \geq p_1$ and $p_2' \geq p_2$, where $\hat{\omega}_2 = \left( \frac{w_1}{p_1' - q_1}, \frac{w_2}{p_2' - q_2}, \frac{w_3}{p_3' - q_3}, \ldots, \frac{w_n}{p_n' - q_n} \right)$ and $\omega_2 = \frac{\hat{\omega}_2}{\|\hat{\omega}_2\|_1}$. By repeating this argument, we can get
\[ X_n = \mathfrak{G}_\delta(\omega'; A_1^{p_1}, \ldots, A_n^{p_n}) \leq X_{n-1} \leq X_0 \leq A_n^{q_n} \]
for $p_i' \geq p_i$ for $i = 1, \ldots, n$, where $\hat{\omega}' = \hat{\omega}_n = \left( \frac{w_1}{p_1' - q_1}, \ldots, \frac{w_n}{p_n' - q_n} \right)$. \(\square\)

Remark. (i) in Theorem 1.E, that is, log $A_1 + \cdots + \log A_n \leq 0$ holds if and only if
\[ \frac{1}{p_1} \log A_1^{p_1} + \cdots + \frac{1}{p_n} \log A_n^{p_n} \leq 0 \quad \text{for every } p_i > 0 \text{ and } i = 1, \ldots, n. \]
Therefore we recognize that Theorem 2.3 implies (i) \( \implies \) (iii) in Theorem 1.E by putting \( q_1 = \ldots = q_n = 0 \) and \( w_1 = \ldots = w_n = 1 \) since
\[
\frac{1}{p_i} = \frac{1}{p_i} + \ldots + \frac{1}{p_n} = \frac{p_{\neq i}}{\sum_j p_{\neq j}} \quad \text{for } i = 1, \ldots, n
\]
ensures \( \omega = \frac{\hat{\omega}}{\|\hat{\omega}\|_1} = \left( \frac{1}{p_1+q}, \ldots, \frac{1}{p_{n-1}+q}, \frac{n-1}{p_n-q} \right) \).

It is well known that we have a variant from Theorem 1.A by replacing \( A, B \) with \( A^q, B^q \) and \( p, r \) with \( \frac{p}{q}, \frac{r}{q} \) in Theorem 1. respectively.

**Theorem 2.D** ([8]). Let \( A > 0, B \geq 0 \) and \( q > 0 \). Then
\[
A^q \geq B^q \quad \text{implies} \quad A^{-r} \#_p \frac{q}{p} B^p \leq B^q \leq A^q \quad \text{for } p \geq q \quad \text{and} \quad r \geq 0.
\]

Here we show that Theorem 2.3 is a generalization of Furuta inequality via weighted Riemannian mean of \( n \)-matrices. Precisely, we show that Theorem 2.3 ensures the following Theorem 2.4 and Theorem 2.4 is a generalization of Theorem 2.D.

**Theorem 2.4.** Let \( A_1, \ldots, A_n \in P_m(\mathbb{C}) \) and \( q > 0 \). Then \( A_i^q \geq A_n^q > 0 \) for \( i = 1, \ldots, n-1 \) implies
\[
\mathfrak{G}_\delta(\omega; A_1^{-p_1}, \ldots, A_{n-1}^{-p_{n-1}}, A_n^p) \leq A_n^q \leq A_i^q
\]
for all \( p_i \geq 0, i = 1, \ldots, n-1 \) and \( p_n > q \), where \( \omega = \frac{\hat{\omega}}{\|\hat{\omega}\|_1} = \left( \frac{1}{p_1+q}, \ldots, \frac{1}{p_{n-1}+q}, \frac{n-1}{p_n-q} \right) \) and
\[
\hat{\omega} = \left( \frac{1}{\|\hat{\omega}\|_1}, \ldots, \frac{1}{\|\hat{\omega}\|_1} \right) = \left( \frac{p_{\neq 1}}{\sum_i p_{\neq i}}, \ldots, \frac{p_{\neq n}}{\sum_i p_{\neq i}} \right).
\]

**Proof.** Assume that \( A_i^q \geq A_n^q > 0 \) for \( q > 0 \) and \( i = 1, \ldots, n-1 \). Then \( A_i^q \geq A_n^q > 0 \) implies \( \log A_i \geq \log A_n \). By (i) \( \implies \) (iii) in Theorem 1.C, \( \log A_i \geq \log A_n \) implies \( A_i^{-p_i} \#_p \frac{q}{p+q} A_n^p \leq I \) for all \( p_i \geq 0 \). This is equivalent to \( A_n^{-q} \#_{p+q} A_i^p \geq I \), that is, \( (A_n^q A_i^p A_n^q A_i^p)_{p+q} \geq A_n^q \). By taking logarithm, we have \( \frac{1}{p_i+q} \log A_n^q A_i^p A_n^q \geq \frac{1}{p_n-q} \log A_n^{p-q}, \) that is,
\[
\frac{1}{p_i+q} \log A_n^q A_i^{p-1} A_n^q + \frac{1}{p_n-q} \log A_n^{p-q} \leq 0
\]
for all \( p_i \geq 0, i = 1, \ldots, n-1 \) and \( p_n > q \). Summing up (2.8) for \( i = 1, \ldots, n-1 \), we have
\[
\frac{1}{p_1+q} \log A_n^q A_1^{p_1} A_n^q + \ldots + \frac{1}{p_{n-1}+q} \log A_n^q A_{n-1}^{p_{n-1}} A_n^q + \frac{n-1}{p_n-q} \log A_n^{p-q} \leq 0.
\]
By applying Theorem 2.3 to \((A_i^{-1})^{-q} \geq A_n^q > 0\) and (2.9), we can obtain

\[ \mathfrak{G}_\delta(\omega; A_1^{-p_1}, \ldots, A_{n-1}^{-p_{n-1}}, A_n^{p_n}) \leq A_n^q \leq A_i^q \]

for all \(p_i \geq 0 > -q, i = 1, \ldots, n - 1\) and \(p_n > q\).

\[ \square \]

**Proof of Theorem 2.D.** Put \(n = 2, p_1 = r\) and \(p_2 = p\) in Theorem 2.4. Then \(\hat{\omega} = \left(\frac{1}{r+q}, \frac{1}{p-q}\right)\) and \(\omega = \left(\frac{p-q}{r+p}, \frac{q+r}{r+p}\right)\). Therefore we obtain the desired result.

\[ \square \]

### 3 3-matrices case

In this section, for the sake of readers' convenience, we state 3-matrices case of Theorems 2.3 and 2.4.

**Corollary 3.1.** Let \(A, B, C \in P_m(\mathbb{C})\) and \(w_1, w_2, w_3 > 0\). If

\[ A^{q_1} \geq C^{q_3} > 0, \quad B^{q_2} \geq C^{q_3} > 0, \]

and

\[ \frac{w_1}{p_1-q_1} \log C^{-\frac{q_3}{2}} A^{p_1} C^{-\frac{q_3}{2}} + \frac{w_2}{p_2-q_2} \log C^{-\frac{q_3}{2}} B^{p_2} C^{-\frac{q_3}{2}} + \frac{w_3}{p_3-q_3} \log C^{-\frac{q_3}{2}} C^{p_3} C^{-\frac{q_3}{2}} \leq 0 \]

hold for \(q_i \in \mathbb{R}, p_i > q_i\) and \(i = 1, 2, 3\), then

\[ \mathfrak{G}_\delta(\omega'; A^{p_1'}, B^{p_2'}, C^{p_3'}) \leq \mathfrak{G}_\delta(\omega; A^{p_1}, B^{p_2}, C^{p_3}) \leq C^{q_3} \]

for all \(p_i' \geq p_i\) and \(i = 1, 2, 3\), where \(\hat{\omega} = \left(\frac{w_1}{p_1-q_1}, \frac{w_2}{p_2-q_2}, \frac{w_3}{p_3-q_3}\right)\), \(\hat{\omega}' = \left(\frac{w_1}{p_1'-q_1}, \frac{w_2}{p_2'-q_2}, \frac{w_3}{p_3'-q_3}\right)\), \(\omega = \frac{\omega}{\|\omega\|_1}\) and \(\omega' = \frac{\omega'}{\|\omega'\|_1}\).

**Corollary 3.2.** Let \(A, B, C \in P_m(\mathbb{C})\) and \(q > 0\). Then \(A^q \geq C^q > 0\) and \(B^q \geq C^q > 0\) implies

\[ \mathfrak{G}_\delta(\omega; A^{-r}, B^{-s}, C^p) \leq C^q \leq A^q \quad \text{(or } B^q) \]

for \(r \geq 0, s \geq 0\) and \(p > q\), where \(\hat{\omega} = \left(\frac{1}{r+q}, \frac{1}{s+q}, \frac{2}{p-q}\right)\) and \(\omega = \frac{\omega}{\|\omega\|_1}\).
References


[10] T. Furuta, $A \geq B \geq 0$ assures $(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0$, $p \geq 0$, $q \geq 1$ with $(1+2r)q \geq p+2r$, Proc. Amer. Math. Soc., 101 (1987), 85–88.


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