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SOME CONTROL PROBLEM VIA MOMENT PROBLEM

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Abstract

In this report, we introduce moment theory application to some control problem. First, we consider the linear time optimal control, and then admissible control problem, finally we introduce estimation the domain of contraction for SIR epidemic model.

1 Introduction of MP

1.1. What is MP ?

In mechanics, we consider infinitely long thin bar that rotate along an axis, if the density function $\psi(x)$ is given, then

$$\int_0^\infty d\psi(x), \int_0^\infty xd\psi(x), \int_0^\infty x^2d\psi(x), \cdots, \int_0^\infty x^n d\psi(x)$$

is the distributed over $[0,\infty)$, the first (statical) moment, the second (inertia) moment, ..., and the n-th moment, respectively.

In mathematical statistics, if the probability distribution function $p(x)$ is given, then $\langle x\rangle=\int p(x)xdx$ is the mean (expectation) value, and $\langle x^n\rangle=\int p(x)x^n dx$ is the n-th moment.

In above, we first know the density function or the probability distribution function, then we obtain some data. Conversely, in many usual cases, first some data are given, we want find the density function or the probability distribution function, this is moment problem.

In 1894-95, Stieltjes first posed the moment problem: Given a moment sequence $\{\mu_n\}_0^\infty$. Find a bounded non-decreasing function $\psi(x)$ in the interval $[0,\infty)$ such that

$$\int_0^\infty x^n d\psi(x) = \mu_n, \qquad n = 0, 1, 2, \cdots .$$

1.2. Power moment problem

Power moment problem. Given

$$\gamma = \{\gamma_0, \gamma_1, ...,\} \text{ and } K \subseteq \mathbb{C}.$$

Find a positive Borel measure μ on $\mathbb{C}$ such that

$$\int t^i d\mu(t) = \gamma_i \ (i \geq 0) \text{ and } \operatorname{supp} \mu \subseteq K.$$

In particular, the classical moment problems are the problems of Stieltjes, Hamburger, Hausdorff and Toeplitz for $K = [0, +\infty), K = \mathbb{R}, K = [a, b]\,(a, b \in \mathbb{R})$, and $K = \mathbb{T} := \{t \in \mathbb{C} : |t| = 1\}$, respectively.

Truncated power moment problem. For $0 \leq m < \infty$, let $\gamma = (\gamma_0, ..., \gamma_m) \in \mathbb{C}^{m+1}$. Find a positive Borel measure μ on $\mathbb{C}$ such that

$$\int t^i d\mu(t) = \gamma_i \ (0 \leq i \leq m).$$

The classical truncated moment problems were solved completely by Curto-Fialkow([3, 1991]).

2 The Curto and Fialkow's TCMP

2.1. The Curto and Fialkow's TCMP

Given a closed subset $K \subseteq \mathbb{C}$ and a doubly indexed finite sequence of complex numbers

$$\gamma : \gamma_{00},\ \gamma_{01},\ \gamma_{10},\ \gamma_{02},\ \gamma_{11},\ \gamma_{20}, \cdots, \gamma_{0,2n},\ \cdots,\ \gamma_{2n,0}, \tag{2.1}$$

with $\gamma_{00} > 0$ and $\gamma_{ji} = \overline{\gamma}_{ij}$, the truncated K-moment problem entails finding a positive Borel measure μ such that

$$\gamma_{ij} = \int \bar{z}^i z^j d\mu \quad (0 \leq i + j \leq 2n) \text{ and } \operatorname{supp} \mu \subseteq K; \tag{2.2}$$

γ is called a truncated moment sequence (of order $2n$) and μ is called a representing measure for γ.

2.2. The moment matrix $M(n)$.

For $n \geq 1$, let $m \equiv m(n) := (n+1)(n+2)/2$. For $A \in M_m(\mathbb{C})$ (the set of $m \times m$ complex matrices), we denote the successive rows and columns according to the following lexicographic-functional ordering:

$$\underbrace{\mathit{1}}_{(1)}, \underbrace{Z, \bar{Z}}_{(2)}, \underbrace{Z^2, \bar{Z}Z, \bar{Z}^2}_{(3)}, \cdots, \underbrace{Z^n, \bar{Z}Z^{n-1}, \cdots, \bar{Z}^{n-1}Z, \bar{Z}^n}_{(n+1)}.$$

We denote the entry in row $\bar{Z}^k Z^l$ $(0 \leq k+l \leq n)$ and column $\bar{Z}^i Z^j$ $(0 \leq i+j \leq n)$ of A by $A_{(k,l)(i,j)}$. For γ as in (2.1), we define $M(n)(\gamma) \in M_m(\mathbb{C})$ as follows: for $0 \leq i+j \leq n, 0 \leq k+l \leq n$,

$$M(n)_{(k,l)(i,j)} = \gamma_{l+i,k+j}.$$

- Quadratic moment problem

For $n = 1$, i.e.,

$$\gamma : \gamma_{00}, \gamma_{01}, \gamma_{10}, \gamma_{02}, \gamma_{11}, \gamma_{20},$$

we have

$$M(1) = \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} \\ \gamma_{10} & \gamma_{11} & \gamma_{20} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} \end{pmatrix}. \tag{2.3}$$

Theorem 1. *γ admits a representing measure if and only if the moment matrix $M(1)$ in (2.3) is positive.*

- Quartic moment problem

For $n = 2$, i.e.,

$$\gamma : \gamma_{00}, \gamma_{01}, \gamma_{10}, \gamma_{02}, \gamma_{11}, \gamma_{20}, \gamma_{03}, \gamma_{12}, \gamma_{21}, \gamma_{30},$$
$$\gamma_{04}, \gamma_{13}, \gamma_{22}, \gamma_{31}, \gamma_{40},$$

we obtain

$$M(2) = \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} & \gamma_{02} & \gamma_{11} & \gamma_{20} \\ \gamma_{10} & \gamma_{11} & \gamma_{20} & \gamma_{12} & \gamma_{21} & \gamma_{30} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} & \gamma_{03} & \gamma_{12} & \gamma_{21} \\ \gamma_{20} & \gamma_{21} & \gamma_{30} & \gamma_{22} & \gamma_{31} & \gamma_{40} \\ \gamma_{11} & \gamma_{12} & \gamma_{21} & \gamma_{13} & \gamma_{22} & \gamma_{31} \\ \gamma_{02} & \gamma_{03} & \gamma_{12} & \gamma_{04} & \gamma_{13} & \gamma_{22} \end{pmatrix}. \tag{2.4}$$

2.3. Necessary conditions

Condition 1. If γ admits a representing measure, then $M(n) \geq 0$.

Definition. We say that $M(n)$ is recursively generated if

$$p, q, pq \in \mathcal{P}_n, \quad p(Z, \bar{Z}) = \mathbf{0} \implies (pq)(Z, \bar{Z}) = \mathbf{0}. \tag{RG}$$

Condition 2. If γ admits a representing measure, then $M(n)$ is recursively generated.

Condition 3. ([1, 1996]) If γ admits a representing measure, then

$$\text{card supp } \mu \geq \text{rank } M(n).$$

Remark. Condition 1 and Condition 2 are not sufficient ([5, 2005]).

2.4. Flat extension of positive moment matrix

For $k, l \in Z_+$, let $A \in M_k(\mathbb{C}), A = A^*, B \in M_{k,l}(\mathbb{C}), C \in M_l(\mathbb{C})$; we refer to any matrix of the form

$$\tilde{A} = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$$

as an extension of A. If rank $\tilde{A} =$ rank A, then we call $\tilde{A}$ is a flat extension of A.

Lemma 2. *For $A \geq 0$, the following statements are equivalent:*

(i) $\tilde{A} \geq 0$*;*

(ii) *There exists W such that $AW = B$ and $C \geq W^*AW$.*

2.5. Curto and Fialkow's main theorem

Theorem 3. ([1, 1996]) *γ has a rank $M(n)$-atomic representing measure if and only if $M(n) \geq 0$ and $M(n)$ admits a flat extension $M(n+1)$.*

3 Linear time optimal control problem

3.1. Linear time optimal control problem

We consider

$$\dot{x} = A(t)x + B(t)u(t), \quad x \in R^n, u \in R, |u| \leq 1, \tag{3.1}$$

$$x(0) = x^0 = (x_1^0, x_2^0, \cdots, x_n^0)^T, \; x(\theta) = 0, \; \theta \to \min \tag{3.2}$$

where $A(t), B(t)$ are analytic $(n \times n)$ and $(n \times 1)$ matrices on some interval $[0, \tau]$. The pair $(\theta\left(x^0\right), u\left(t; x^0\right))$ is called the solution of the time-optimal control problem (3.1), (3.2), where $\theta\left(x^0\right)$ is the optimal time and $u\left(t; x^0\right)$ is the optimal control.

3.2. The time control problem without restrictions

Theorem 4. *Given*

$$\dot{x} = A(t)x + B(t)u(t), \tag{3.3}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}$, and $x(0) = x^0$, $x(\theta) = 0$. Then

$$x_k^0 = \langle g_k, u \rangle = \int_0^\theta g_k(t)\, u(t)\, dt, \qquad k = 1, ..., n, \tag{3.4}$$

where

$$(g_1(t), g_2(t), \cdots, g_n(t))^T = -\phi^{-1}(t)B(t),$$

and $\phi(t)$ is the fundamental matrix of the system

$$\dot{x} = A(t)x, \quad \textit{such that } \phi(0) = I.$$

Controllability problem. Given a sequence of real numbers $s_0, s_1, ..., s_n$ and a sequence of functions $\{g_k(t)\}_{k=1}^n$, continuous in $\mathbb{R}$, find $u(t)$ such that

$$s_k = \int g_k(t)\, u(t)\, dt, \qquad k = 1, ..., n. \tag{3.5}$$

The truncated Hamburger moment problem. Given a sequence of real numbers $\gamma_0, \gamma_1, ..., \gamma_n$, find a positive Borel measure μ on $\mathbb{R}$ such that

$$\gamma_j = \int t^j d\mu(t), \qquad j = 0, 1, ..., n. \tag{3.6}$$

3.3. The solution of the truncated Hamburger moment problem

Define

$$A(k) = \begin{pmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_k \\ \gamma_1 & \gamma_2 & \cdots & \gamma_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_k & \gamma_{k+1} & \cdots & \gamma_{2k} \end{pmatrix}, \qquad \mathbb{V}_i = \begin{pmatrix} \gamma_i \\ \gamma_{i+1} \\ \vdots \\ \gamma_{i+k} \end{pmatrix}.$$

Set $\gamma = (\gamma_0, \gamma_1, ..., \gamma_{2k+1})$ and $r = \text{rank}\,\gamma$. Thus $\mathbb{V}_0, ..., \mathbb{V}_{r-1}$ is linearly independent, and $\exists 1 \varphi_0, ..., \varphi_{r-1}$ in $\mathbb{R}$ such that

$$\mathbb{V}_r = \varphi_0 \mathbb{V}_0 + \cdots + \varphi_{r-1} \mathbb{V}_{r-1}.$$

We call the polynomial

$$P_\gamma(t) = t^r - (\varphi_0 + \cdots + \varphi_{r-1} t^{r-1}) \qquad (t \in \mathbb{R}) \tag{3.7}$$

is the generating function of γ.

Theorem 5. (Odd case) *The truncated Hamburger moment problem is solvable for $n = 2k+1$ if and only if $A = A(k) \geq 0$ and $\mathbb{V}_{k+1} \in$ Ran A.*

(i) *If $r \leq k$, then the unique solution is of the following*

$$\mu := \rho_0 \delta_{t_0} + \cdots + \rho_{r-1} \delta_{t_{r-1}}, \tag{3.8}$$

where $t_0, ..., t_{r-1}$ are distinct and the zeros of $P_\gamma(t)$, and

$$\begin{pmatrix} \rho_0 \\ \rho_1 \\ \vdots \\ \rho_{r-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ t_0 & t_1 & \cdots & t_{r-1} \\ \vdots & \vdots & \ddots & \vdots \\ t_0^{r-1} & t_1^{r-1} & \cdots & t_{r-1}^{r-1} \end{pmatrix}^{-1} \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \vdots \\ \gamma_{r-1} \end{pmatrix}.$$

(ii) *If $r = k+1$, then it has infinitely many solutions.*

Theorem 6. (Even case) *The truncated Hamburger moment problem is solvable for $n = 2k$ if and only if $A = A(k) \geq 0$ and rank $A(k) =$ rank γ.*

(i) *Suppose* $r := rank\,\gamma \leq k$. *Let* $\Phi := \Phi(\gamma)$ *and let*

$$\gamma_{2k+1} := \varphi_0\gamma_{2k+1-r} + \cdots + \varphi_{r-1}\gamma_{2k}.$$

Then $\tilde{\gamma} := (\gamma_0, \gamma_1, ..., \gamma_{2k+1})$ *has the unique solution is of (3.8), which is also the unique representing measure of* γ.

(ii) *Suppose* $r = k+1$. *For each* $\gamma_{2k+1} \in \mathbb{R}$, *let* $\tilde{\gamma} := (\gamma_0, \gamma_1, ..., \gamma_{2k+1})$. *Then* $\tilde{\gamma}$ *has infinitely many representing measure and each is a representing measure of* γ.

4 Admissible control problem

Consider the following completely controllable system

$$\dot{x} = Ax + b\tilde{u}, \quad x(0) = x_0, \tag{4.1}$$

where $x \in R^n, A \in M_{n\times n}, b \in R^n$.

Admissible Control Problem (ACP). For given x_0 and θ, find $\tilde{u} = \tilde{u}_{x_0,\theta}$ such that

(1) $|\tilde{u}| \leq 1$,

(2) The solution $x(t)$ of (4.1) with $\tilde{u}$ satisfies $x(\theta) = 0$.

Let $C_{0,L}$ be the set of all measurable functions on $[0,\theta]$ such that $0 \leq f(\tau) \leq L$, $\forall \tau \in [0,\theta]$; and $\mathcal{M}[0,\theta]$ be the set of all nonnegative measures on $[0,\theta]$.

L-Markov MP (MMP on $[0,\theta]$). For given a sequence of real numbers $(c_j)_{j=0}^k$, find $f \in C_{0,L}$ such that

$$c_j = \int_0^\theta \tau^j f(\tau)d\tau, \qquad j = 1, ..., k.$$

Hausdorff MP (HMP on $[0,\theta]$). For given a sequence of real numbers $(s_j)_{j=0}^k$, find $\sigma \in \mathcal{M}[0,\theta]$ such that

$$s_j = \int_0^\theta \tau^j d\sigma(\tau), \qquad j = 1, ..., k.$$

Theorem 7. *There is a bijection between the set $\mathcal{C}_{0,L}$ and measures $\sigma \in \mathcal{M}[0,\theta]$ satisfying $\int_0^\theta d\sigma(\tau) = 1$ given by*

$$\int_0^\theta \frac{d\sigma(\tau)}{\tau - z} = -\frac{1}{z} \exp\left(\frac{1}{L}\int_0^\theta \frac{f(\tau)\, d\tau}{\tau - z}\right).$$

Let

$$\begin{aligned} s_0 &= 1, \\ s_1 &= c_0, \\ s_{j(\geq 2)} &= \frac{1}{j! L^j} \begin{vmatrix} c_0 & -L & \cdots & 0 \\ 2c_1 & c_0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ (j-1)\, c_{j-2} & (j-2)\, c_{j-3} & \cdots & -(j-1)\, L \\ jc_{j-1} & (j-1)\, c_{j-2} & \cdots & c_0 \end{vmatrix}. \end{aligned}$$

Then we have

Theorem 8. *The L-Markov MP with $c_{j-1}(\theta, x_0), j \in \{1, ..., n\}$ is solvable if and only if the HMP on $[0,\theta]$ with $s_{j-1}(\theta, x_0), j \in \{1, ..., n\}$ is solvable.*

Assume that

$$A = \begin{bmatrix} 0 & & & & & \\ 1 & 0 & & & & \\ & 1 & 0 & & & \\ & & 1 & \ddots & & \\ & & & \ddots & 0 & \\ & & & & 1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Then we obtain

$$c_{j-1}(x_0, \theta) = \frac{\theta^j + (-1)^j j! x_0^{(j)}}{2j}, \quad j = 1, 2, ..., n.$$

Theorem 9. ([8, 2010]) *If the HMP on $[0,\theta]$ with $s_{j-1}(\theta, x_0), j \in \{1, ..., n\}$ is solvable and $\sigma(\tau)$ is a solution, let $S(z) = \int_0^\theta \frac{d\sigma(\tau)}{\tau - z}$, then*

$$\tilde{u}(t) = -\frac{2}{\pi} \lim_{\epsilon \to +0} \arg\left(-(t + i\epsilon)\, S(t + i\epsilon)\right) - 1, \quad t \in [0, \theta].$$

5 The problem of DOA

Given the autonomous system

$$\dot{x} = f(x), \qquad x(0) = x^0, \tag{5.1}$$

where $x \in \mathbb{R}^n$ and $f(0) = 0$.

Definition. (DOA) The domain of attraction of $x = 0$ is

$$S = \left\{ x^0 \in \mathbb{R}^n | \lim_{t \to \infty} x(t, x^0) = 0 \right\}, \tag{5.2}$$

where $x(\cdot, x^0)$ denotes the solution of (5.1) corresponding to the initial condition $x(0) = x^0$.

5.1. Lyapunov function method

Definition. Let $V(x)$ be a continuously differentiable real-valued function defined on a domain $D \subset \mathbb{R}^n$ containing the origin. The function $V(x)$ is called a **Lyapunov function** for the system (5.1) if the following conditions are satisfied:

(i) $V(x)$ is positive definite on D,

(ii) $\dot{V}(x) = \left(\frac{\partial V}{\partial x}\right)^{\mathrm{T}} f(x)$ is negative semidefinite on D.

Theorem 10. *Let $V(x)$ be a Lyapunov function for the system (5.1) in the domain*

$$\Omega_c = \{x \in \mathbb{R}^n | V(x) \leq c\}, \qquad c > 0. \tag{5.3}$$

Assume that Ω_c is bounded and $0 \in \Omega_c$. If $\dot{V}(x)$ is negative definite in Ω_c, then $\Omega_c \subset S$.

5.2. The estimation of the DOA as an optimization problem

Let

$$V(x) = x^{\mathrm{T}} P x, \qquad P = P^{\mathrm{T}} \in \mathbb{R}^{n \times n}, \qquad P \succ 0. \tag{5.4}$$

Here, $(\succ 0) \Leftrightarrow$ positive definite. Our *objective* is to find the maximum value c^* of c such that $\dot{V}(x)$ is negative definite in Ω_c. This c^* is defined by the following optimization problem

$$\begin{cases} \text{find } c^* = \min V(x) \\ \text{subject to the constraints:} \\ \dot{V}(x) = 0, \qquad x \neq 0. \end{cases} \tag{5.5}$$

5.3. The estimation of the DOA via moment matrix

In [4, 2001] Lasserre considers the following two classical problems

- Global minimization

$$\mathbb{P} \mapsto p^* := \min_{x \in \mathbb{R}^n} p(x) \tag{5.6}$$

- Constrained optimization

$$\mathbb{P}_K \mapsto p_K^* := \min_{x \in K} p(x) \tag{5.7}$$

where $p(x)$ is a real-valued polynomial and K is a compact set defined by polynomial inequalities $g_i(x) \geq 0, i = 1, \ldots, r$.

Let $1, x_1, x_2, \ldots, x_n, x_1^2, x_1x_2, \cdots, x_1x_n, x_2x_3, \ldots, x_1^m, \ldots, x_n^m$ be a basis for the m-degree real-valued polynomials $p(x)$, and let $s(2m)$ be its dimension, where $s(m) := \begin{pmatrix} n+m \\ n \end{pmatrix} = \frac{(n+m)!}{n!m!}$. Let $p(x) = \sum_\alpha p_\alpha x^\alpha$, where

$$x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}, \qquad \sum_{i=1}^n \alpha_i \leq m.$$

Given an $s(2m)$-vector $y := \{y_\alpha\}$ with first element $y_{0,\ldots,0} = 1$, let $M_m(y)$ be the moment matrix of dimension $s(m)$. Let $g(x) : \mathbb{R}^n \to \mathbb{R}$ be a real valued polynomial of degree w with coefficient vector $g \in \mathbb{R}^{s(w)}$. If the entry (i,j) of the matrix $M_m(y)$ is y_β, let $\beta(i,j)$ denote the subscript β of y_β. Then $M_m(gy)$ is defined by

$$M_m(gy)(i,j) = \sum_\alpha g_\alpha y_{\{\beta(i,j)+\alpha\}}.$$

Let $\deg g_i(x) = w_i$, and define

$$\tilde{w}_i = \left\lceil \frac{w_i}{2} \right\rceil$$

which is *the smallest integer larger than* $\frac{w_i}{2}$. Then (5.7) is equivalent to the following problem.

Problem.

$$\mathbb{Q}_K^N \mapsto \begin{cases} \inf_y \sum_\alpha p_\alpha y_\alpha \\ \text{subject to the constraints:} \\ M_N(y) \succeq 0 \\ M_{N-\tilde{w}_i}(g_i y) \succeq 0, \qquad i = 1, \ldots, r. \end{cases} \tag{5.8}$$

The number N has to be chosen according to the following conditions

$$N \geq \left\lceil \frac{m}{2} \right\rceil \quad \text{and} \quad N \geq \max_i \tilde{w}_i.$$

Theorem 11. ([4, 2001]) *Let $K \subseteq \{x : \|x\| \leq a\}$ for sufficiently large $a > 0$. Then*

(1) *as $N \to \infty$, one has* $\inf \mathbb{Q}_K^N \uparrow p_K^*$,

(2) *for N sufficiently large, there is no duality gap between $\mathbb{Q}_K^N$ and its dual $\left(\mathbb{Q}_K^N\right)^*$, if K has nonempty interior.*

5.4. The Estimation of DOA of SIR Epidemic Model

We consider the following SIR epidemic model ([7, 2009])

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} -\frac{13}{10} & -\frac{3}{2} & \frac{1}{2} \\ \frac{4}{5} & 0 & 0 \\ 0 & \frac{1}{2} & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} -\frac{1}{2}xy \\ \frac{1}{2}xy \\ 0 \end{pmatrix}, \tag{5.9}$$

where $x \geq -4, y \geq -\frac{8}{5}, z \geq -\frac{4}{5}$. We know that $P(0,0,0)$ is local asymptotically stable. Let

$$V = \frac{299}{475}x^2 + \frac{6457}{3800}y^2 + \frac{286}{475}z^2 + \frac{378}{475}xy + \frac{184}{475}yz + \frac{194}{475}xz. .$$

Then V is positive definite, and

$$\left.\frac{dV}{dt}\right|_{(5.9)} = -x^2 - y^2 - z^2 + \frac{989}{760}xy^2 - \frac{1}{95}xyz - \frac{22}{95}x^2y.$$

Now we use moment matrix to obtain the DOA of (5.9). Indeed, we must solve the following optimization problem

$$\begin{cases} \min\left(\frac{299}{475}x^2 + \frac{6457}{3800}y^2 + \frac{286}{475}z^2 + \frac{378}{475}xy + \frac{184}{475}yz + \frac{194}{475}xz\right) \\ \text{s.t.} \begin{cases} g_1 = -x^2 - y^2 - z^2 + \frac{989}{760}xy^2 - \frac{1}{95}xyz - \frac{22}{95}x^2y \geq 0 \\ g_2 = \left(x^2 + y^2 + z^2\right) - 1 \geq 0 \\ g_3 = 4 - \left(x^2 + y^2 + z^2\right) \geq 0. \end{cases} \end{cases}$$

It is equivalent to solve the following optimization problem

$$\left\{\begin{array}{l} c^* = \min\left(\begin{array}{c} \frac{299}{475}y_{2,0,0} + \frac{6457}{3800}y_{0,2,0} + \frac{286}{475}y_{0,0,2} \\ +\frac{378}{475}y_{1,1,0} + \frac{184}{475}y_{0,1,1} + \frac{194}{475}y_{1,0,1} \end{array}\right) \\ \text{s.t.} \left\{\begin{array}{l} M_N(y) \geq 0 \\ M_{N-\tilde{w}_i}(g_i y) \geq 0, i = 1,2,3. \end{array}\right. \end{array}\right. \tag{5.10}$$

By using the YALMIP-yet another LMI package of Matlab, we can solve the optimization problem (5.10), and obtain

N	3	4	5
$M_N(y)$	$\mathbb{R}^{20\times 20}$	$\mathbb{R}^{35\times 35}$	$\mathbb{R}^{56\times 56}$
c^*	1.5590	2.6094	2.6094

Therefore, the set

$$\left\{\begin{array}{c} (x,y,z) : x \geq -4, y \geq -\frac{8}{5}, z \geq -\frac{4}{5}, \text{ and } \frac{299}{475}x^2 + \frac{6457}{3800}y^2 \\ +\frac{286}{475}z^2 + \frac{378}{475}xy + \frac{184}{475}yz + \frac{194}{475}xz \leq 2.6094 \end{array}\right\}$$

is a subset of the domain of attraction for SIR epidemic model (5.9).

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