Title
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Citation
数理解析研究所講究録 (2012), 1779: 98-110

Issue Date
2012-02

URL
http://hdl.handle.net/2433/171804

Type
Departmental Bulletin Paper

Textversion
publisher

Kyoto University
Collapsing behaviour of the logarithmic diffusion equation

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June 11, 2011

Abstract
I will report my result on the collapsing behaviour of the maximal solution of the equation $u_t = \Delta \log u$ in $\mathbb{R}^2 \times (0, T)$, $u(x, 0) = u_0(x)$ in $\mathbb{R}^2$, near its extinction time $T = \int_{\mathbb{R}^2} u_0 dx / 4\pi$ without using the Hamilton-Yau Harnack inequality. This result extends the recent result of P. Daskalopoulos, M.A. del Pino and N. Sesum [DP2], [DS2].

In this report I will discuss my recent result [Hu5] on the collapsing behaviour of the maximal solution of the logarithmic diffusion equation:

\[
\begin{cases}
    u_t = \Delta \log u, u > 0, & \text{in } \mathbb{R}^2 \times (0, T) \\
    u(x, 0) = u_0 & \text{in } \mathbb{R}^2.
\end{cases}
\]

(0.1) arises in many physical and geometric models. For example P.L. Lions and Toscani [LT] have shown that (0.1) arises as the diffusive limit for finite velocity Boltzmann kinetic models and T.G. Kurtz [Ku] has proved that (0.1) is the limiting density distribution of two gases moving against each other and obeying the Boltzmann equation. In [G] P.G. de Gennes showed that (0.1) also appears in the model of viscous liquid film lying on a solid surface and subjecting to long range Van der Waals interactions with fourth order term neglected.

Recently K.M. Hui [Hu3], [Hu4] (for the case $m > 0$ and $m < 0$), and J.R. Esteban, A. Rodriguez, J.L. Vazquez [ERV] (for the case $m > 0$) have shown that the solution of the porous medium equation

\[ u_t = \Delta \left( \frac{u^m}{m} \right) \]
converges to the maximal solution of (0.1) as \( m \to 0 \). In [W1], [W2], [H], Angenent, L. Wu and R. Hamilton showed that the equation also arises in the study of the conformally equivalent metric \( g_{ij} = u\delta_{ij} \) on \( \mathbb{R}^2 \) under the Ricci flow

\[
\frac{\partial}{\partial t} g_{ij} = -2R_{ij}
\]

(0.2)

where \( R_{ij} \) is the Ricci curvature of the metric \( g_{ij} \). Note that in \( \mathbb{R}^2 \) the scalar curvature \( R \) is given by

\[
R = -\frac{\Delta \log u}{u}
\]

(0.3)

and the Ricci curvature is given by

\[
R_{ij} = \frac{1}{2} R g_{ij}.
\]

(0.4)

By (0.3) and (0.4) the Ricci flow equation (0.2) is equivalent to the logarithmic diffusion equation:

\[
u_t = \Delta \log u.
\]

1 Existence and properties of solutions

The equation (0.1) has many properties different from the heat equation such as existence of infinite many solutions for any given initial \( L^1 \) data. There also does not exist any fundamental solution for (0.1) [Hu1] which suggests that conservation of mass does not hold. K.M. Hui [Hu2] by using approximation by Neumann solutions in bounded domains and P. Daskalaopoulos and M.A. del Pino [DP1] by using approximation by Dirichlet solutions in bounded domains proved independently that corresponding to each

\[
0 \leq u_0 \in L^p(\mathbb{R}^2) \cap L^1(\mathbb{R}^2), p > 1, 2 \leq f \in L^1(0,\infty),
\]

there exists a classical solution \( u \) of (0.1) in \( \mathbb{R}^n \times (0, T) \) satisfying the mass loss equation,

\[
\int_{\mathbb{R}^2} u(x, t) \, dx = \int_{\mathbb{R}^2} u_0 \, dx - 2\pi \int_0^t f(s) \, ds \quad \forall 0 \leq t < T
\]

where \( T = T(u_0, f) > 0 \) given by

\[
\int_{\mathbb{R}^2} u_0 \, dx = 2\pi \int_0^T f(s) \, ds
\]

is the extinction time for the solution \( u \). Hence the solution with flux \( f \) vanishes identically to zero at time \( T \).
Note that the maximal solution of (0.1) is the solution of (0.1) that corresponds to flux $f = 2$ which satisfies

$$
\int_{\mathbb{R}^2} u(x, t) \, dx = \int_{\mathbb{R}^2} u_0 \, dx - 4\pi t \quad \forall 0 \leq t < T
$$

with

$$
T = \frac{1}{4\pi} \int_{\mathbb{R}^2} u_0 \, dx.
$$

For any $2 < f \in C(0, T)$ the solution $u$ with flux $f$ satisfies the following decay condition at infinity:

$$
\lim_{|x| \to \infty} \frac{\log u}{\log |x|} = -f \quad \text{uniformly on } [a, b] \quad \forall 0 < a < b < T
$$

or equivalently

$$
\lim_{|x| \to \infty} \frac{ru}{u} = -f \quad \text{uniformly on } [a, b] \quad \forall 0 < a < b < T.
$$

One would like to ask what is the asymptotic behaviour of the solution with constant flux $f = \gamma \geq 2$? When $\gamma > 2$, S.Y. Hsu [Hs1], [Hs2], by using the lap number method of Matano [M], V.A. Galaktionov and L.A. Peletier [GP], proved that if the initial value is radially symmetric and monotone decreasing and $u$ is the solution with flux $\gamma > 2$, then there exist unique constants $\alpha > 0$, $\beta > -1/2$, $\alpha = 2\beta + 1$, depending only on $\gamma$ such that the rescaled function

$$
v(y, s) = \frac{u(y/(T-t)^{\beta}, t)}{(T-t)^{2\beta + 1}}, \quad s = -\log (T-t),
$$

converges uniformly on every compact subset of $\mathbb{R}^2$ to $\phi_{\lambda, \beta}(y)$ for some constant $\lambda > 0$ as $s \to \infty$ where $\phi_{\lambda, \beta}(y) = \phi_{\lambda, \beta}(|y|)$ is radially symmetric and satisfies the following O.D.E.:

$$
\frac{1}{r} \left( \frac{r\phi'}{\phi} \right)' + \alpha \phi + \beta r\phi' = 0 \quad \text{in } (0, \infty)
$$

with

$$
\phi(0) = 1/\lambda, \phi'(0) = 0.
$$

In particular for $\gamma = 4$, the rescaled solution

$$
v(x, s) = \frac{u(x, t)}{T-t}, \quad s = -\log(T-t),
$$

converges uniformly on every compact subsets of $\mathbb{R}^2$ to the function

$$
\frac{8\lambda}{(\lambda + |x|^2)^2}
$$
as $s \to \infty$ for some constant $\lambda > 0$. What this said is that for solution with flux $f = 4$,

$$u(x,t) \approx \frac{8\lambda(T-t)}{(\lambda + |x|^2)^2} \quad \text{as} \quad t \nearrow T$$

which corresponds to contracting spheres Ricci flow solution on $S^2$.

What about the asymptotic behaviour of the solution with flux $f = 2$? J.R. King [K] by using inner and outer asymptotic expansion and matching asymptotic method showed that if $u$ is the solution of the logarithmic diffusion equation (0.1) with flux $f = 2$ then as $t$ approaches the extinction time $T$ the vanishing behaviour for solution is very different from the vanishing behaviour for the case $f \equiv \gamma > 2$. J.R. King found that for compactly supportly finite mass initial value the maximal solution behaves like

$$\frac{(T-t)^2}{\frac{1}{2}\lambda + e^{\frac{2T}{(T-t)}}|x|^2}$$

in the inner region $(T-t)\log |x| \leq T$ and behaves like

$$\frac{2t}{|x|^2(\log |x|)^2}$$

in the outer region $(T-t)\log |x| \geq T$ as $t \nearrow T$. In [DP2] P. Daskalopoulos and M.A. del Pino gave a rigorous proof of an extension of this formal result for radially symmetric initial value $u_0(r) \geq 0$ satisfying the conditions,

$$u_0(x) = u_0(|x|) \text{ is decreasing on } r = |x| \geq r_1 \text{ for some constant } r_1 > 0,$$

$$u_0(x) = \frac{2\mu}{|x|^2(\log |x|)^2}(1 + o(1)) \quad \text{as} \quad |x| \to \infty,$$

for some constant $\mu > 0$ and

$$R_0(x) := -\frac{\Delta\log u_0}{u_0} \geq -\frac{1}{\mu} \quad \text{on } \mathbb{R}^2.$$

Later P. Daskalopoulos and N. Sesum [DS2] extended this result to the case of compactly supported $0 \leq u_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. However their proof of the behaviour of the maximal solution in the outer region near the extinction time is very difficult and uses the Hamilton-Yau Hamack inequality. Recently in [Hu5] I extended their result to the case of initial value

$$0 \leq u_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$$

that satisfies

$$u_0(x) = u_0(|x|) \text{ is decreasing on } r = |x| \geq r_1 \text{ for some constant } r_1 > 0,$$
\[ u_0(x) = \frac{2\mu}{|x|^2(\log |x|)^2}(1 + o(1)) \quad \text{as } |x| \to \infty, \]

and

\[ R_0(x) := -\frac{\Delta \log u_0}{u_0} \geq -\frac{1}{\mu} \quad \text{on } \mathbb{R}^2 \]

for some constant \( \mu \geq 0 \) with the right hand side being replaced by \(-\infty\) if \( \mu = 0 \) and

\[ \text{ess inf}_{\overline{B}_{r_1}(0)} u_0 \geq \text{ess sup}_{\mathbb{R}^2 \setminus B_{r_2}(0)} u_0 \quad \text{for some constant } r_2 > r_1. \quad (1.1) \]

Note that (1.1) is automatically satisfied if \( u_0 \) has compact support in \( \mathbb{R}^2 \). In [Hu5] I proved the behaviour of the maximal solution in the outer region near the extinction time by elementary method without using the difficult Hamilton-Yau Harnack inequality for surfaces. I also obtained the behaviour of the maximal solution in the inner region as the extinction time is approached.

I will now assume that \( u_0 \) satisfies the above structural conditions and \( u \) is the maximal solution of (0.1) in \( \mathbb{R}^2 \times (0, T) \) with flux \( f \equiv 2 \) and

\[ T = \frac{1}{4\pi} \int_{\mathbb{R}^2} u_0 \, dx. \]

I will sketch of proof of [Hu5] here.

## 2 Inner region behaviour

By using the reflection method of D.G. Aronson and L.A. Caffarelli [AC] one has the following lemma:

**Lemma 2.1.** (Lemma 1.1 of [Hu5]) The solution \( u \) satisfies

\[ u(x, t) \geq u(y, t) \]

for any \( t \in (0, T) \) and \( x, y \in \mathbb{R}^2 \) such that \( |y| \geq |x| + 2r_2 \).

Then for any \( 0 < t < T \) there exists \( x_i \in \overline{B}_{2r_2} \) such that

\[ u(x_i, t) = \max_{x \in \mathbb{R}^2} u(x, t). \]

That is the maximum of \( u(\cdot, t) \) is attained on the compact set \( \overline{B}_{2r_2} \). We will now perform a rescaling of the solution of (0.1). Let

\[ \overline{u}(x, \tau) = \tau^2 u(x, t), \quad \tau = \frac{1}{T-t}, \tau > 1/T. \]
Then \( \overline{u} \) satisfies
\[
\overline{u}_\tau = \Delta \log \overline{u} + \frac{2\overline{u}}{\tau} \quad \text{in } \mathbb{R}^2 \times (1/T, \infty).
\]

Let
\[
R_{\max}(t) = \max_{x \in \mathbb{R}^2} R(x, t)
\]
and let \( W(t) \) be the width function with respect to the metric \( g_{ij}(t) \) as defined by P. Daskalopoulos and R. Hamilton [DH]. We now recall a result of P. Daskalopoulos and R. Hamilton [DH]:

**Theorem 2.2.** ([DH]) There exist positive constants \( c > 0 \) and \( C > 0 \) such that

1. \( c(T - t) \leq W(t) \leq C(T - t) \)
2. \( \frac{c}{(T - t)^2} \leq R_{\max}(t) \leq \frac{c}{(T - t)^2} \)

hold for any \( 0 < t < T \).

By Theorem 2.2,
\[
c \leq \lim_{t \nearrow T} \sup (T - t)^2 R_{\max}(t) \leq C.
\]

Hence the singularity is a type II singularity. Note that \( u \) satisfies the Aronson-Bénilan inequality,
\[
\frac{u_t}{u} \leq \frac{1}{t} \quad \text{in } \mathbb{R}^2 \times (0, T).
\]

As
\[
R = -\frac{\Delta \log u}{u} = -\frac{u_t}{u},
\]
the Aronson-Bénilan inequality is equivalent to
\[
R \geq -\frac{1}{t}.
\]

So if we let
\[
\overline{R}(x, \tau) = -\frac{\Delta \log \overline{u}}{\overline{u}}.
\]

Then
\[
\frac{2}{\tau} + \frac{2}{\tau^2 T} \geq \frac{\overline{u}_\tau}{\overline{u}} \geq -C \quad \text{in } \mathbb{R}^2 \times (2/T, \infty).
\]

**Theorem 2.3.** (Theorem 1.3 of [Hu5]) For any sequence \( \{\tau_k\}_{k=1}^{\infty}, \tau_k \to \infty \) as \( k \to \infty \), let
\[
\overline{u}_k(y, \tau) = \alpha_k \overline{u}(\alpha_k^{\frac{1}{3}} y + x_{t_k}, \tau + \tau_k), \quad y \in \mathbb{R}^2, \tau > -\tau_k + T^{-1}
\]
where
\[
t_k = T - \tau_k^{-1} \quad \forall k \in \mathbb{Z}^+.
\]
\[ \alpha_k = 1/\overline{u}(x_{t_k}, \tau_k). \]

Then \( \{\overline{u}_k\}_{k=1}^\infty \) has a subsequence \( \{\overline{u}_{k_i}\}_{i=1}^\infty \) that converges uniformly on \( C^\infty(K) \) for any compact set \( K \subset \mathbb{R}^2 \times (-\infty, \infty) \) as \( i \to \infty \) to a positive solution

\[ U(y, \tau) = \frac{1}{\lambda |y|^2 + e^{4\lambda\tau}}. \]

of equation

\[ U_\tau = \Delta \log U \quad \text{in} \quad \mathbb{R}^2 \times (-\infty, \infty) \]

with uniformly bounded non-negative scalar curvature and uniformly bounded width on \( \mathbb{R}^2 \times (-\infty, \infty) \) with respect to the metric \( \overline{g}_{ij}(t) = U(\cdot, t)\delta_{ij} \) where \( \lambda > 0 \) is some constant.

Proof: (Sketch) By definition,

\[ \overline{u}_k(0,0) = 1 \quad \text{and} \quad \overline{u}_k(y,0) \leq 1 \quad \forall y \in \mathbb{R}^2. \]

Let \( a < b \). By (2.1) there exist constants \( M_1 > 0 \) and \( k_0 \in \mathbb{Z}^+ \) such that

\[ \overline{u}_k(x, \tau) \leq M_1 \quad \text{and} \quad |\overline{u}_{k,\tau}(x, \tau)| \leq CM_1 \quad x \in \mathbb{R}^2, a \leq \tau \leq b, k \geq k_0. \]  

By (2.2) one can deduce the following Harnack inequality:

For any \( R_1 > 0, a < b \), there exists a constant \( C_2 > 0 \) such that

\[ \sup_{|y| \leq R_1} \inf_{a \leq \tau_1 \leq b} \overline{u}_k(y, \tau_1) \leq C_2 \inf_{|y| \leq R_1} \sup_{a \leq \tau_2 \leq b} \overline{u}_k(x, \tau_2) \quad \forall k \geq k_0. \]

Hence the sequence \( \overline{u}_k \) is uniformly parabolic on \( \overline{B}_{R_1} \times [a, b] \) and are uniformly H"older continuous in \( C^{2\gamma,1/\gamma}(\overline{B}_{R_1} \times [a, b]) \) for any \( \gamma \in \mathbb{Z}^+ \). Then the sequence \( \{\overline{u}_k\}_{k=1}^\infty \) has a subsequence which we may assume without loss of generality to be the sequence itself that converges uniformly in \( C^\infty(K) \) as \( k \to \infty \) for any compact set \( K \subset \mathbb{R}^2 \times (-\infty, \infty) \) to some positive function \( U \) that satisfies the logarithmic diffusion equation. Let

\[ \overline{R}_k = -\frac{\Delta \log \overline{u}_k}{\overline{u}_k}. \]

Then \( \overline{R}_k \) converges uniformly on every compact subset of \( \mathbb{R}^2 \times (-\infty, \infty) \) as \( k \to \infty \) to the scalar curvature

\[ \overline{R} = -\frac{\Delta \log U}{U} \]

of the metric \( \overline{g}_{ij}(\tau) = U(\cdot, \tau)\delta_{ij} \). Moreover

\[ 0 \leq \overline{R}(y, \tau) \leq C \quad \forall (y, \tau) \in \mathbb{R}^2 \times (-\infty, \infty). \]
By an approximation argument the width function with respect to the metric 
\( \overline{g}_{ij}(\tau) = U(\cdot, \tau)\delta_{ij} \) is uniformly bounded on \( \mathbb{R}^{2} \times (-\infty, \infty) \). Then by the result of P. Daskalopoulos and N. Sesum [DS1],

\[
U(y, \tau) = \frac{2}{\beta(|y - y_0|^2 + \delta e^{2\beta \tau})}
\]

for some \( y_0 \in \mathbb{R}^{2} \) and constants \( \beta > 0, \delta > 0 \). Since \( \overline{u}_k(y, 0) \) attains its maximum at \( y = 0, U(y, 0) \) will attain its maximum at \( y = 0 \). Hence \( y_0 = 0 \).

\[
U(0, 0) = 1 \quad \Rightarrow \quad 1 = \frac{2}{\beta \delta}.
\]

Thus

\[
U(y, \tau) = \frac{1}{\lambda |y|^2 + e^{4\lambda \tau}}.
\]

for some constant \( \lambda > 0 \).

It can be proved that 
\( \alpha_k \to \infty \) as \( k \to \infty \)

Hence we can change the origin in rescaling and have the following result:

**Lemma 2.4.** (Lemma 1.10 of [Hu5]) Let

\[
q_k(y, \tau) = \alpha_k \overline{u}(\alpha_k^{-\frac{1}{2}} y, \tau + \tau_k).
\]

Then \( q_k(y, \tau) \) converges uniformly in \( C^\infty(K) \) for every compact set \( K \subset \mathbb{R}^{2} \) to the function \( U(y, \tau) \) as \( \tau \to \infty \).

We will now perform a change of scaling. Let

\[
\beta(\tau) = 1/\overline{u}(0, \tau),
\]

\( \beta_k = \beta(\tau_k) \), and

\[
\overline{q}_k(y, \tau) = \beta_k \overline{u}(\beta_k^{-\frac{1}{2}} y, \tau + \tau_k).
\]

Then

\[
\frac{\alpha_k}{\beta_k} = q_k(0, 0) \to U(0, 0) = 1 \quad \text{as} \quad k \to \infty.
\]

Hence there exists \( k_0 \in \mathbb{Z}^+ \) and constants \( c_2 > c_1 > 0 \) such that

\[
c_1 \leq \frac{\beta_k}{\alpha_k} \leq c_2 \quad \forall k \geq k_0.
\]

Thus we have the following result.
Theorem 2.5. (Theorem 1.11 of [Hu5]) \( \overline{q}_k \) has a subsequence \( \overline{q}_{k_l} \) that converges uniformly on \( C^\infty(K) \) for any compact set \( K \subset \mathbb{R}^2 \times (-\infty, \infty) \) to \( U(y, \tau) \) as \( \tau \to \infty \). Moreover \( \beta_k \to \infty \) as \( k \to \infty \).

Lemma 2.6. (Proposition 1.17 and Proposition 1.18 of [Hu5])

\[
\lim_{\tau \to \infty} \frac{\beta'(\tau)}{\beta(\tau)} = \lim_{\tau \to \infty} \frac{\log \beta(\tau)}{\tau} = 2(T + \mu).
\]

Let

\[
\tilde{R}_k = -\Delta \log \overline{q}_k.
\]

Since

\[
\tilde{R}_k(0,0) = \frac{\beta'(\tau_k)}{\beta(\tau_k)} + \frac{2}{\tau_k},
\]

then

\[
4\lambda = \lim_{\tau \to \infty} \tilde{R}_k(0,0) = 2(T + \mu).
\]

Let

\[
\overline{q}(y, \tau) = \beta(\tau) \overline{u}(\beta(\tau)^{\frac{1}{2}}y, \tau).
\]

We then have the following main theorem for inner region.

Theorem 2.7. (Theorem 1.21 of [Hu5]) \( \overline{q}(y, \tau) \) converges uniformly on \( C^\infty(K) \) for any compact set \( K \subset \mathbb{R}^2 \) to the function

\[
U_{\mu}(y) = \frac{1}{\frac{(T+\mu)}{2}|y|^2 + 1}
\]

as \( \tau \to \infty \).

Corollary 2.8. (cf. [Hu5]) For any \( \epsilon > 0 \) and \( M > 0 \) there exist \( \tau_0 > 1/T \) and \( C > 0 \) such that

\[
\left\{ \begin{array}{ll}
|u(x,t) - \frac{(T-t)^2}{\lambda|x|^2 + \beta(\tau)}| < u(0,t)\epsilon & \forall |x| \leq \beta(\tau)^{\frac{1}{2}}M, \tau > \tau_0 \\
u(0,t) \leq C(T-t)^2 & \forall t > T - \tau_0^{-1}.
\end{array} \right.
\]

where \( \tau = 1/(T-t) \).

As in [DP2], [DS2], [Hu5], we now consider the cylindrical change of variables,

\[
v(\zeta, \theta, t) = r^2 u(r, \theta, t), \quad \zeta = \log r, r = |x|
\]

and let

\[
\overline{v}(\xi, \theta, t) = \tau^2 v(\tau \xi, \theta, t), \quad \tau = 1/(T-t), \tau \geq 1/T.
\]

Then \( \overline{v} \) satisfies

\[
\tau \overline{v}_\tau = \frac{1}{\tau}(\log \overline{v})_{\xi\xi} + \tau(\log \overline{v})_{\theta\theta} + \overline{v}_\xi + 2\overline{v} \quad \text{in } \mathbb{R} \times [0, 2\pi] \times (1/T, \infty).
\]
Corollary 2.9. (Lemma 1.23 of [Hu5]) For any $\epsilon > 0$ there exists $\tau_0 > 1/T$ such that
\[
|\overline{v}(\xi, \theta, \tau) - \frac{e^{2\tau \xi}}{\frac{T+\mu}{2}e^{2\tau \xi} + \beta(\tau)}| < \frac{e^{2\tau \xi}}{\beta(\tau)} \epsilon \quad \forall \xi \leq \frac{\log \beta(\tau)}{2\tau}, \theta \in [0,2\pi], \tau \geq \tau_0.
\]

Corollary 2.10. (Corollary 1.24 of [Hu5])
\[
\int_{-\infty}^{\xi^-} \int_{0}^{2\pi} \overline{v}(\xi, \theta, \tau) \, d\theta \, d\xi \to 0 \quad \text{as} \quad \tau \to \infty
\]
and
\[
\lim_{\tau \to \infty} \overline{v}(\xi, \theta, \tau) = 0 \quad \text{uniformly on} \quad (-\infty, \xi^-] \times [0,2\pi]
\]
for any $\xi^- < T + \mu$.

3 Outer region behaviour

Let
\[
\xi(\tau) = (\log \beta(\tau))/2\tau.
\]

Lemma 3.1. (Lemma 2.1 of [Hu5]) There exists constants $C_1 > 0$, $C_2 > 0$, $C_3 > 0$ and $\tau_0 > 1/T$ such that the following holds.

(i) $\overline{v}(\xi, \theta, \tau) \leq C_1$ $\forall \xi \in \mathbb{R}, \theta \in [0,2\pi], \tau \geq 1/T$

(ii) $\overline{v}(\xi, \theta, \tau) \geq \frac{C_2}{\xi \tau}$ $\forall \xi \geq \xi(\tau), \theta \in [0,2\pi], \tau \geq \tau_0$

(iii) $\overline{v}(\xi, \theta, \tau) \leq \frac{C_3}{\xi^2}$ $\forall \xi > 0, \theta \in [0,2\pi], \tau \geq \tau_0$.

Moreover
\[
\xi(\tau) = T + \mu + o(1) \quad \text{as} \quad \tau \to \infty.
\]

We now let
\[
w(\xi, \theta, s) = \overline{v}(\xi, \theta, \tau)
\]
with
\[
s = \log \tau = -\log(T-t).
\]

Then
\[
w_s = e^s(\log w)_{\xi \xi} + e^s(\log w)_{\theta \theta} + \xi w_\xi + 2w \quad \text{in} \quad \mathbb{R} \times [0,2\pi] \times (-\log T, \infty).
\]

The following is the main theorem for outer region.
Theorem 3.2. (Theorem 2.3 of [Hu5]) As $\tau \to \infty$, the function $\overline{v}$ converges to the function

$$
V(\xi) = \begin{cases}
\frac{2(T + \mu)}{\xi^2} & \forall \xi > T + \mu \\
0 & \forall \xi < T + \mu.
\end{cases}
$$

Moreover the convergence is uniform on $(-\infty, a]$ for any $a < T + \mu$ and on $[\xi_0, \xi'_0]$ for any $\xi'_0 > \xi_0 > T + \mu$.

Proof: (Sketch) Let $\{s_k\}_{k=1}^\infty$ be a sequence such that $s_k \to \infty$ as $k \to \infty$ and

$$
w_k(\xi, \theta, s) = w(\xi, \theta, s + s_k) \quad \forall \xi \in \mathbb{R}, 0 \leq \theta \leq 2\pi, s \geq -\log T - s_k.
$$

Let

$$
W_k^b(\eta, s) = \int_{\eta}^{b} \int_{0}^{2\pi} w_k(\xi, \theta, s) d\theta d\xi \quad \forall b \geq \eta > T + \mu, s > -\log T - s_k, k \in \mathbb{Z}^+,
$$

$$
W_k(\eta, s) = \int_{\eta}^{\infty} \int_{0}^{2\pi} w_k(\xi, \theta, s) d\theta d\xi \quad \forall \eta > T + \mu, s > -\log T - s_k, k \in \mathbb{Z}^+.
$$

and let $\{b_i\}_{i=1}^\infty$ be a monotonically increasing sequence such that $b_i > T + \mu$ for any $i \in \mathbb{Z}^+$ and $b_i \to \infty$ as $i \to \infty$.

Since

$$
\int_{\mathbb{R}^2} u(x, t) dx = 4\pi(T - t) \quad \forall 0 < t < T,
$$

$$
\int_{-\infty}^{\infty} \int_{0}^{2\pi} w_k(\xi, \theta, s) d\theta d\xi = 4\pi \quad \forall s > -\log T - s_k, k \in \mathbb{Z}^+.
$$

One can prove that there exists a function $\overline{w}$ and a subsequence of $\mathbb{Z}^+$ which we may assume without loss of generality to be $\mathbb{Z}^+$ itself such that

$$
W_k^i \to W^i \quad \text{uniformly on } [a, b] \times [c, d] \quad b > a > T + \mu, d > c \quad \text{as } k \to \infty
$$

for any $i \in \mathbb{Z}^+$ and

$$
W_k \to W \quad \text{uniformly on } [a, b] \times [c, d] \quad b > a > T + \mu, d > c \quad \text{as } k \to \infty
$$

where

$$
W^i(\eta, s) = \int_{\eta}^{b} \int_{0}^{2\pi} \overline{w}(\xi, \theta, s) d\theta d\xi, \quad W(\eta, s) = \int_{\eta}^{\infty} \int_{0}^{2\pi} \overline{w}(\xi, \theta, s) d\theta d\xi.
$$

By elementary argument one can show that

$$
\eta W(\eta, s) = \overline{\eta} W(\overline{\eta}, \overline{s}) \quad \forall \eta, \overline{\eta} > T + \mu, s, \overline{s} \in \mathbb{R}
$$
and
\[ W(T + \mu, s) = 4\pi \quad \forall s \in \mathbb{R}. \]

Letting $\bar{\eta} \to T + \mu$,
\[ W(\eta, s) = \frac{4\pi(T + \mu)}{\eta} \quad \forall \eta > T + \mu, s \in \mathbb{R} \]

which will then imply the theorem after some elementary computation. \(\Box\)

References


[Hu3] K.M. Hui, Singular limit of solutions of the equation $u_t = \Delta (u^m/m)$ as $m \to 0$, Pacific J. Math. 187 (1999), no. 2, 297–316.


[Hs1] S.Y. Hsu, Asymptotic behaviour of solutions of the equation $u_t = \Delta \log u$ near the extinction time, Advances in Differential Equations 8 (2003), no. 2, 161–187.


