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Transversality of Stable and Nehari Manifolds for a Semilinear Heat Equation

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This is a summary of [5], which is a joint work with F. Dickstein, Ph. Souplet and F. Weissler.

We are concerned with global existence and finite time blowup of solutions to the following semilinear parabolic initial value problem

$$(1)\qquad \begin{cases} u_t - \Delta u = |u|^{p-1}u, & x \in \Omega,\ t > 0, \\ u = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases}$$

where $p > 1$ and Ω is a bounded domain of $\mathbf{R}^n$ of class $C^{2+\eta}$ for some $\eta \in (0,1)$. This equation is a model problem for studying the competition between the dissipative effect of diffusion and the influence of an explosive source term. This specific problem has been the object of intense study over the past fourty years. The recent book [11] contains a detailed account of much of this literature.

We consider here the Sobolev sub-critical case

$$(2)\qquad 1 < p < p_S := (n+2)/(n-2)_+,$$

so that in particular, $H_0^1(\Omega) \subset L^{p+1}(\Omega)$. Under this assumption, it is well known that problem (1) is locally well-posed in $H_0^1(\Omega)$ (see [1],[4],[15]). More precisely, given $u_0 \in H_0^1(\Omega)$ there exists a maximal time $T(u_0) \in (0,\infty]$ and a unique solution $u \in C([0,T(u_0)); H_0^1(\Omega)) \cap C^1([0,T(u_0)); H^{-1}(\Omega))$ of (1). This solution is classical for $0 < t < T(u_0)$. Furthermore, if $T(u_0) < \infty$ then $\|u(t)\|_{H_0^1} \to \infty$ and $\|u(t)\|_\infty \to \infty$ as $t \to T(u_0)$. In this case, the solution is said to blow up in finite time. If $T(u_0) = \infty$, the solution is said to be global. A major question, which has motivated a substantial amount of research, is to determine criteria on the intial value u_0 which enable one to decide whether or not the resulting solution is global or blows up in finite time.

The point of view taken in this article is at the crossroads of two classical methods used in the study of nonlinear partial differential equations: variational methods and critical point theory, and the theory of dynamical systems. Historically, both points of view have contributed to the study of (1), often in complementary ways. At issue, in particular, is the existence and stability of stationary solutions of (1).

From the variational point of view, there are two natural functionals on $H_0^1(\Omega)$ associated with the problem (1), the energy functional and the Nehari functional, defined respectively by

$$E(\phi) = \frac{1}{2}\int_\Omega |\nabla \phi|^2 - \frac{1}{p+1}\int_\Omega |\phi|^{p+1},$$

$$I(\phi) = E'(\phi)\cdot\phi = \int_\Omega |\nabla \phi|^2 - \int_\Omega |\phi|^{p+1}.$$

Stationary solutions of (1) are precisely critical points of the energy functional E. In particular, they satisfy $I(\phi) = 0$. We recall certain well-known results about these critical points. Many of the proofs are based on the mountain pass theorem of Ambrosetti and Rabinowitz [2]; see the recent books [11], [13] . Of course these results depend on the condition (2) that the power be Sobolev sub-critical, which implies that the energy functional is well defined in $H_0^1(\Omega)$ and satisfies the Palais-Smale condition.

There exists a positive regular stationary solution of (1), and an infinite sequence of regular stationary solutions ϕ_k with $E(\phi_k) \to \infty$. A positive stationary solution can be obtained either by a direct application of the mountain pass theorem, or else by minimizing $\int_\Omega |\nabla\phi|^2$ (or equivalently the energy functional) subject to the constraint that $\int_\Omega |\phi|^{p+1} = 1$. In this latter case, the resulting function needs to be multiplied by a constant to compensate for the Lagrange multiplier.

The Nehari functional enters as follows. If $\phi \in H_0^1(\Omega)$, and $\phi \not\equiv 0$, then there is a unique $\lambda_0 > 0$ such that $\frac{d}{d\lambda}|_{\lambda=\lambda_0} E(\lambda\phi) = 0$, and this gives the maximum value of $E(\lambda\phi)$. One checks easily that $I(\lambda_0\phi) = 0$ and $I(\lambda\phi) > 0$ if and only if $0 < \lambda < \lambda_0$. Thus, $\lambda_0 = 1$ if ϕ is a critical point of E. The Nehari manifold is defined by

$$\mathcal{N} = \{\phi \in H_0^1(\Omega) : I(\phi) = 0, \phi \not\equiv 0\}.$$

Note that if $I(\phi) = 0, \phi \not\equiv 0$, then $I'(\phi) \neq 0$. The Sobolev embedding $H_0^1(\Omega) \subset L^{p+1}(\Omega)$ and the Poincaré inequality imply that $\mathcal{N}$ is bounded away from 0 in $H_0^1(\Omega)$. A positive stationary solution of (1) can also be found by minimizing the energy functional on $\mathcal{N}$. The energy of the stationary solution obtained by this method,

$$d := \inf\{E(\phi);\ \phi \in H_0^1(\Omega) \setminus \{0\},\ I(\phi) = 0\},$$

is the minimum energy for a nontrivial stationary solution, and is precisely the mountainpass energy

$$d = \inf_{\phi \in H_0^1(\Omega)\setminus\{0\}} \max_{\lambda \geq 0} E(\lambda\phi) > 0.$$

In addition, the energy and the Nehari functionals play an important role in the dynamics of (1). Indeed, the energy is a Lyapunov function for the flow induced by (1). More precisely, if $u = u(t)$ is a non-stationary solution of (1), then

$$\frac{d}{dt}E(u(t)) < 0.$$

Also

$$\frac{d}{dt}\frac{1}{2}\int_\Omega |u(t)|^2 = -I(u(t)).$$

A fundamental result about problem (1), due to Levine [9], is that the solution u blows up in finite time, i.e. $T(u_0) < 0$, whenever the initial value $u_0 \in H_0^1(\Omega)$ has negative energy $E(u_0) < 0$.

In the case when $E(u_0) \geq 0$, classical results about blowup and global existence were obtained by Tsutsumi [14] and Payne and Sattinger [10] (see also Ishii [8]), using ideas of potential well theory developed by Sattinger [12] in the context of hyperbolic equations.

- If $u_0 \in H_0^1(\Omega)$ satisfies $E(u_0) < d$ and $I(u_0) < 0$, then u blows up in finite time [10].

- If $E(u_0) < d$ and $I(u_0) > 0$, then u is global and decays uniformly to 0 as $t \to \infty$ [,].

Note that d can be interpreted as the depth of the following potential-well

$$W := \{\phi \in H_0^1(\Omega) : E(\phi) < d,\ I(\phi) > 0\} \cup \{0\}.$$

In fact, W is invariant under the semiflow associated with problem (1). Basically, if the initial value has energy below the minimal energy on the Nehari manifold, then since energy is decreasing, the resulting solution must be entirely either inside or outside the potential well W.

In the limiting case $E(u_0) = d$, if $I(u_0) \geq 0$ then u is also global. Indeed, if $I(u_0) = 0$, then u_0 minimizes E on $\mathcal{N}$ and so is a stationary solution. If $I(u_0) > 0$, then u_0 is not a stationary solution, so $E(u(t)) < d$ and $I(u(t)) > 0$ for small $t > 0$, and so the result of [14], [8] applies. By a similar argument, reducing to the result of [10] for small $t > 0$, if $E(u_0) = d$ and $I(u_0) < 0$, then u blows up in finite time. Therefore, when $E(u_0) \leq d$, the question of whether or not $T(u_0)$ is finite is entirely determined by means of $I(u_0)$.

In the case, $E(u_0) > d$, the potential well arguments do not apply in any obvious way. Gazzola and Weth [6] have shown that there exist initial values u_0 and v_0 with arbitrarily large energy, and $I(u_0) > 0$, $I(v_0) > 0$, such that $T(u_0) < \infty$, $T(v_0) = \infty$ and the solution starting from v_0 decays uniformly to 0.

In view of the above results, as noted by Gazzola and Weth [6], it is natural to ask whether or not the condition $I(u_0) < 0$ is still sufficient for finite time blowup when $E(u_0) > d$. One of the main results in this paper is that the answer is negative. (See Theorems 1 and 2 below.)

Our approach to this question relies on a study of the local stable manifold of a nontrivial stationary solution ϕ of (1). Let $\mathcal{L}$ be the linearized operator around ϕ

$$\mathcal{L}u = \Delta u + p|\phi|^{p-1}u.$$

$\mathcal{L}$ is a self-adjoint operator in $L^2(\Omega)$ with domain $H_0^1(\Omega) \cap H^2(\Omega)$ whose spectrum consists entirely of eigenvalues. Since $I(\phi) = 0$, it is easy to see that $\langle \mathcal{L}\phi, \phi \rangle > 0$, and so the first eigenvalue of $\mathcal{L}$ is positive. Moreover, only finitely many eigenvalues of $\mathcal{L}$ are positive. It follows (see []) that ϕ has a local stable manifold $\mathcal{M}$ of nontrivial finite codimension.

From a geometric point of view, the main result of this paper is that the local stable manifold $\mathcal{M}$ of any nontrivial stationary solution ϕ intersects the Nehari manifold $\mathcal{N}$ transversally at ϕ. (See Theorem 3 below.) Consequently, part of the stable manifold $\mathcal{M}$ lies *outside* $\mathcal{N}$, i.e. where $I < 0$. Thus, we may choose $u_0 \in \mathcal{M}$, arbitrarily close to ϕ, with $I(u_0) < 0$. Since $u_0 \in \mathcal{M}$, this produces an initial value u_0, whose energy is bigger than but arbitrarily close to $E(\phi)$, such that $I(u_0) < 0$ but $T(u_0) = \infty$. For the same reason, part of the stable manifold $\mathcal{M}$ lies where $I > 0$ and the above conclusion remains true with $I(u_0) > 0$.

The proofs of our main results require the use of another fundamental tool in the study of partial differential equations: elliptic regularity. The principal technical result in the paper, from which we deduce our main results, is that a nontrivial stationary solution ϕ is not orthogonal to its own local stable manifold. The proof proceeds by contradiction. If ϕ were orthogonal to its own stable manifold, then ϕ would be equal to a linear combination of the (finitely many) eigenvectors of $\mathcal{L}$ with nonnegative eigenvalues. It then follows from (1) that $|\phi|^{p-1}\phi$ must also be a linear combination of those same vectors. When p is not an integer this results in a mis-match of regularity where ϕ vanishes. The case p integer is more delicate, and we obtain a contradiction by analyzing $\mathcal{L}^k\phi$ for some appropriate k.

We now give precise statements of our main results. Theorem 1 is simply the response to the question asked by Gazzola and Weth [6].

Theorem 1. Suppose that the power p satisfies (2). Then there exist initial data $u_0 \in H_0^1(\Omega)$ with $I(u_0) < 0$ such that the solution u of (1) is global.

Theorem 1 is a consequence of the following more precise result, which provides information on where such u_0 can be found in the region $E > d$. As noted earlier, such an initial value can be found on the stable manifold, arbitrarily close to any nontrivial stationary solution, and can be chosen so that $I(u_0) > 0$ or $I(u_0) < 0$. Additionally, there exist positive initial values u_0 with $I(u_0)$ of either sign, for which the resulting solution is global and converges to 0.

Theorem 2. Suppose that the power p satisfies (2).

(i) There exists $u_0 \in H_0^1(\Omega)$ with $I(u_0) < 0$ (resp. $I(u_0) > 0$), such that the resulting solution u of (1) is global and converges uniformly to 0 as $t \to \infty$. Moreover, we may take $u_0 > 0$ and $E(u_0) > d$ arbitrarily close to d.

(ii) Let $\phi > 0$ be a mountain-pass stationary solution of problem (1). Then there exists u_0 on the local stable manifold of ϕ, and arbitrarly close to ϕ, such that $I(u_0) < 0$ (resp. $I(u_0) > 0$). Moreover, we may take $u_0 > 0$ and $E(u_0) > d$ arbitrarily close to d.

(iii) Let $\phi \in C^2(\overline{\Omega})$ be any nontrivial stationary solution of problem (1). If $\phi > 0$ or if $p \in$, assume in addition that Ω is of class $C^{m+\varepsilon}$ where m is the integral part of $(p+1)/2$ and $\varepsilon > 0$. Then there exists u_0 on the local stable manifold of ϕ, and arbitrarily close to ϕ, such that $I(u_0) < 0$ (resp. $I(u_0) > 0$).

Theorem 3. Suppose that the power p satisfies (2), and let ϕ be a stationary solution as in (ii) or (iii) of Theorem 2. It follows that the local stable manifold $\mathcal{M}$ of ϕ intersects the Nehari manifold $\mathcal{N}$ transversally at ϕ. In other words, the tangent space of $\mathcal{M}$ at ϕ (a subspace of $H_0^1(\Omega)$ of non-zero finite codimension) is *not* a subspace of the tangent space at ϕ of the Nehari manifold $\mathcal{N}$ (a subspace of $H_0^1(\Omega)$ of codimension 1).

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