数理解析研究所講究録
第 1779 巻 2012 年 69-87

REGULARITY THEORY AND ASYMPTOTIC BEHAVIORS IN THE INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, we are going to discuss the recent development in the nonlocal the integro-differential equations and their applications.

1. INTRODUCTION

In this paper, we are going to consider nonlocal equations. The linear *elliptic and parabolic integro-differential operators* are given as

$$\mathcal{L}u(x) = \text{p.v.} \int_{\mathbb{R}^n} \mu(u, x, y) K(y)\, dy \tag{1}$$

for $\mu(u,x,y) = u(x+y) - u(x) - (\nabla u(x) \cdot y)\chi_{B_1}(y)$, which describes the infinitesimal generator of given purely jump processes, i.e. processes without diffusion or drift part [CS]. We refer the detailed definitions of notations to [CS, KL1, KL2]. Then we see that $\mathcal{L}u(x)$ is well-defined provided that $u \in C^{1,1}_x(x) \cap B(\mathbb{R}^n)$ where $B(\mathbb{R}^n)$ denotes *the family of all real-valued bounded functions defined on* $\mathbb{R}^n$ and $C^{1,1}_x(x)$ means $C^{1,1}$-function at x. If K is symmetric (i.e. $K(-y) = K(y)$), then an odd function $\left[(\nabla u(x) \cdot y)\chi_{B_1}(y)\right]K(y)$ will be canceled in the integral, and so we have that

$$\mathcal{L}u(x) = \text{p.v.} \int_{\mathbb{R}^n} \left[u(x+y) + u(x-y) - 2u(x)\right]K(y)\, dy.$$

Nonlinear integro-differential operators come from the stochastic control theory related with

$$\mathcal{I}u(x) = \sup_{\alpha} \mathcal{L}_{\alpha}u(x),$$

or game theory associated with

$$\mathcal{I}u(x) = \inf_{\beta} \sup_{\alpha} \mathcal{L}_{\alpha\beta}u(x), \tag{2}$$

when the stochastic process is of Lèvy type allowing jumps; see [S, CS, KL1]. Also an operator like $\mathcal{I}u(x) = \sup_{\alpha} \inf_{\beta} \mathcal{L}_{\alpha\beta}u(x)$ can be considered. Characteristic properties of these operators can easily be derived as follows;

(3) $$\inf_{\alpha\beta} \mathcal{L}_{\alpha\beta}v(x) \leq \mathcal{I}[u+v](x) - \mathcal{I}u(x) \leq \sup_{\alpha\beta} \mathcal{L}_{\alpha\beta}v(x).$$

1.1. **Operators.** In this section, we introduce a class of operators. All notations and the concepts of viscosity solution follow [CS] with minor changes.

For parabolic setting and our purpose, we shall consider functions $u(x)$ defined on $\mathbb{R}^n \times [0,T]$ and restrict our attention to the operators $\mathcal{L}$ where the measure is given by a positive kernel K which is symmetric. That is to say, the operators $\mathcal{L}$ are given by

(4) $$\mathcal{L}u(x) = \text{p.v.} \int_{\mathbb{R}^n} \mu(u,x,y)K(y)\,dy$$

where $\mu(u,x,y) = u(x+y) + u(x-y) - 2u(x)$. And we consider the class $\mathfrak{L}$ of the operators $\mathcal{L}$ associated with positive kernels $K \in \mathcal{K}_0$ satisfying that

(5) $$(2-\sigma)\frac{\lambda}{|y|^{n+\sigma}} \leq K(y) \leq (2-\sigma)\frac{\Lambda}{|y|^{n+\sigma}}, \quad 0 < \sigma < 2.$$

The maximal operator and the minimal operator with respect to $\mathfrak{L}$ are defined by

(6) $$\mathcal{M}^+_{\mathfrak{L}}u(x) = \sup_{\mathcal{L}\in\mathfrak{L}} \mathcal{L}u(x) \quad \text{and} \quad \mathcal{M}^-_{\mathfrak{L}}u(x) = \inf_{\mathcal{L}\in\mathfrak{L}} \mathcal{L}u(x).$$

In what follows, we let $\Omega \subset \mathbb{R}^n$ be a bounded open domain. For $(x) \in \Omega$ and a function $u : \mathbb{R}^n$ which is semicontinuous on $\overline{\Omega}$, we say that φ belongs to the function class $C^2_\Omega(u;x)^+$ (resp. $C^2_\Omega(u;x)^-$) and we write $\varphi \in C^2_\Omega(u;x)^+$ (resp. $\varphi \in C^2_\Omega(u;x)^-$) if there exists a $U_{t,\delta}$ such that $\varphi(x) = u(x)$ and $\varphi > u$ (resp. $\varphi < u$) on $U \setminus \{(x)\}$ for some open neighborhood $U \subset \Omega$ of x, where U. We note that geometrically $u - \varphi$ having a local maximum at (x) in Ω is equivalent to $\varphi \in C^2_\Omega(u;x)^+$ and $u-\varphi$ having a local minimum at (x) in Ω is equivalent to $\varphi \in C^2_\Omega(u;x)^-$. And the expression for $\mathcal{L}_{\alpha\beta}\,u(x)$ and $\mathcal{I}u(x)$ may be written as

$$\mathcal{L}_{\alpha\beta}\,u(x) = \int_{\mathbb{R}^n} \mu(u,x,y)K_{\alpha\beta}(y)\,dy,$$
$$\mathcal{I}u(x) = \inf_{\beta}\sup_{\alpha} \mathcal{L}_{\alpha\beta}\,u(x),$$

where $K_{\alpha\beta} \in \mathcal{K}_0$. Then we see $\mathcal{M}^-_{\mathfrak{L}}u(x) \le \mathcal{I}u(x) \le \mathcal{M}^+_{\mathfrak{L}}u(x)$, and $\mathcal{M}^+_{\mathfrak{L}}u(x)$ and $\mathcal{M}^-_{\mathfrak{L}}u(x)$ have the following simple forms;

$$\mathcal{M}^+_{\mathfrak{L}}u(x) = (2-\sigma)\int_{\mathbb{R}^n} \frac{\Lambda\mu^+(u,x,y) - \lambda\mu^-(u,x,y)}{|y|^{n+\sigma}}\,dy, \tag{7}$$
$$\mathcal{M}^-_{\mathfrak{L}}u(x) = (2-\sigma)\int_{\mathbb{R}^n} \frac{\lambda\mu^+(u,x,y) - \Lambda\mu^-(u,x,y)}{|y|^{n+\sigma}}\,dy,$$

where μ^+ and μ^- are given by

$$\mu^{\pm}(u,x,y) = \max\{\pm\mu(u,x,y), 0\}.$$

A function $u : \mathbb{R}^n \to \mathbb{R}$ is said to be $C^{1,1}_{x,\pm}$ at $(x) \in \mathbb{R}^n$ (we write $u \in C^{1,1}_{x,\pm}(x)$), if there are $r_0 > 0$ and $M > 0$ (independent of s) such that

$$\pm\,(u(x+y) + u(x-y) - 2u(x)) \le M\,|y|^2 \tag{8}$$

for any $y \in B_{r_0}(0)$.

We write $u \in C^{1,1}_{x,\pm}(U)$ if $u \in C^{1,1}_{x,\pm}(x)$ for any $(x) \in U$ and the constant M in (8) is independent of (x), where $U \subset \mathbb{R}^n$ for some $\delta > 0$ for an open subset U of $\mathbb{R}^n$. And we denote $C^{1,1}_x(x) = C^{1,1}_{x,+}(x) \cap C^{1,1}_{x,-}(x)$,and $C^{1,1}_x(U) = C^{1,1}_{x,+}(U) \cap C^{1,1}_{x,-}(U)$, .

We note that if $u \in C^{1,1}_x(x)$, then $\mathcal{I}u(x)$ and $\mathcal{M}^{\pm}_{\mathfrak{L}}u(x)$ will be well-defined. We shall use these maximal and minimal operators to obtain regularity estimates.

Let $K(x) = \sup_\alpha K_\alpha(x)$ where K_α's are all the kernels of all operators in a class $\mathfrak{L}$. For any class $\mathfrak{L}$, we shall assume that

$$\int_{\mathbb{R}^n} (|y|^2 \wedge 1)\,K(y)\,dy < \infty. \tag{9}$$

The following is a kind of operators of which the regularity result shall be obtained in this paper.

Definition 1.1. *Let $\mathfrak{L}$ be a class of linear integro-differential operators. Assume that (9) holds for $\mathfrak{L}$. Then we say that an operator $\mathcal{J}$ is elliptic with respect to $\mathfrak{L}$, if it satisfies the following properties:*

(a) $\mathcal{J}u(x)$ is well-defined for any $u \in C^{1,1}_x(x) \cap B(\mathbb{R}^n)$.

(b) $\mathcal{J}u$ is continuous on an open $\Omega \subset \mathbb{R}^n$, whenever $u \in C^{1,1}_x(\Omega) \cap B(\mathbb{R}^n)$.

(c) If $u, v \in C^{1,1}_x(x) \cap B(\mathbb{R}^n)$, then we have that

$$\mathcal{M}^-_{\mathfrak{L}}[u-v](x) \le \mathcal{J}u(x) - \mathcal{J}v(x) \le \mathcal{M}^+_{\mathfrak{L}}[u-v](x). \tag{10}$$

And We denote by $\mathcal{S}^{\mathfrak{L}}$ the class of integro-differential operators which is elliptic with respect to $\mathfrak{L}$.

1.2. **Fourier Transformation and Dirichlet to Neumann Map.** In this subsection, we are going to discuss fractional Laplacian interms of Fourier transformation and Dirichlet to Neumann Map. Let us consider the following simple Dirichlet problem:

$$\begin{cases} (-\Delta)^{\sigma} v = 0 & \text{in } \Omega \\ v = 0 & \text{on } \mathbb{R}^n \setminus \Omega \\ v(x) = v_0(x) & \text{non-negative and } \dot{H}^{\sigma}_0\text{-bounded} \end{cases} \tag{11}$$

with $0 < \sigma < 1$. The fractional Laplacian of a function $f : \mathbb{R}^n \to \mathbb{R}$ is expressed by the formula

$$(-\Delta)^{\sigma} f(x) = C_{n,\sigma} \int_{\mathbb{R}^n} \frac{f(x) - f(y)}{|x-y|^{n+2\sigma}} dy = C_{n,\sigma} \int_{\mathbb{R}^n} \frac{f(x+y) + f(x-y) - 2f(x)}{|y|^{n+2\sigma}} dy$$

where $C_{n,\sigma}$ is some normalization constant. The Fourier transformation of $(-\Delta)^{\sigma}[u]$ is $|\xi|^{2\sigma}\hat{u}$ and we have

- $(-\Delta)^{\sigma_1}(-\Delta)^{\sigma_2}[u] = (-\Delta)^{\sigma_1+\sigma_2}[u]$ and $(-\Delta)^0[u] = u$.
- If $\sigma \to 1$, then $(-\Delta)^{\sigma}[u] = -\Delta u$.

In addition, the norm in $\dot{H}^{\sigma}$ is given precisely by

$$\|f\|_{\dot{H}^{\sigma}} = \sqrt{\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^2}{|x-y|^{n+2\sigma}} dx dy}. \tag{12}$$

That is equivalent to

$$\|f\|_{\dot{H}^{\sigma}} \cong \sqrt{\int_{\mathbb{R}^n} |\xi|^{2\sigma} |\hat{f}(\xi)|^2 d\xi}.$$

for Fourier transform of f in x. Note that the Sobolev embedding results say that $\dot{H}^{\sigma} \subset L^{2n/(n-2\sigma)}$ (Chap V in [St]). Indeed, $\dot{H}^{\sigma}$ is the space of $L^{2n/(n-2\sigma)}$ functions for which (12) is integrable.

$(-\Delta)^{\sigma} v$ can be also thought as the normal derivative of some extension of v (the Dirichlet to Neumann operator of v). See [CS] for a general discussion. We introduce first the corresponding extension v^* defined from $C_0^{\infty}(\mathbb{R}^n)$ to $C_0^{\infty}(\mathbb{R}^n \times \mathbb{R}^+)$ by:

$$\begin{aligned} -\nabla(y^a \nabla v^*) &= 0 && \text{in } \mathbb{R}^n \times (0, \infty) \\ v^*(x, 0) &= v(x) && \text{for } x \in \mathbb{R}^n \end{aligned}$$

for $a = 1 - 2\sigma$. Then the following result holds true: for v defined on $\mathbb{R}^n$, we have:

$$(-\Delta)^{\sigma} v(x) = \partial_{\nu} v^*(x, 0) = -\lim_{y \to 0} y^a v^*_y(x, y)$$

where we denote $\partial_\nu v^*$ the outward normal derivative of v^* on the boundary $\{y = 0\}$.

1.3. **Outline.** In this paper, we are going to summarize the recent developement on Nonlocal Nonlinear equations and its application to the geometric properties of the first eigen function of fractional Laplacian. In the section 2, we show the Hölder regularity and asymptotic behavior of the dolutions for singular nonlocal equations. And in the section 3, we show the the geometric properties of the first eigen function of fractional Laplacian. Finally, at section 4, we summarize the regularity theory on nonlocal fully nonlinear equations.

2. Singular Non-local Equations

In this section, we consider initial value problem with fractional fast diffusion, [KsL1]:

$$\begin{cases} (-\Delta)^\sigma u^m + u_t = 0 & \text{in } \Omega \\ u = 0 & \text{on } \mathbb{R}^n \setminus \Omega \\ u(x,0) = u_0(x) & \text{non-negative and } \dot{H}^\sigma_0\text{-bounded} \end{cases} \tag{13}$$

in the range of exponents $\frac{n-2\sigma}{n+2\sigma} < m < 1$, with $0 < \sigma < 1$. In this work, we will deal with a Hölder regularity of $v = u^m$, which is a solution of

$$\begin{cases} (-\Delta)^\sigma v + \left(v^{\frac{1}{m}}\right)_t = 0 & \text{in } \Omega \\ v = 0 & \text{on } \mathbb{R}^n \setminus \Omega \\ v(x,0) = v_0(x) = u_0^m(x) & \text{in } \Omega, \end{cases} \tag{M.P}$$

assuming that the initial value v_0 is strictly positive in the interior of Ω in $\mathbb{R}^n$. Main two Theorems state as follows:

Theorem 2.1. *(From $L^{\frac{n-2\sigma}{n+2\sigma}}$ to L^∞)*
Let $v(x,t)$ be a function in $L^\infty(0,T;L^{\frac{2n}{n-2\sigma}}(\Omega)) \cap L^2(0,T;\dot{H}^\sigma_0(\mathbb{R}^n))$, then

$$\sup_{x\in\Omega} |v(x,T)| \le C^* \frac{\|v_0\|_{L^{\frac{2n}{n-2\sigma}}(\Omega)}}{T^{\frac{mn}{2mn-(n-2\sigma)(1+m)}}}$$

for some constant $C^* > 0$.

For the second theorem, we need better control of v.

Theorem 2.2 (Hölder regularity of fractional FDE).
For $x_0 = (x_0^1, \cdots, x_0^n)$, we define $Q_r(x_0,t_0) = [x_0^i - r, x_0^i + r]^n \times [t_0 - r^{2\sigma}, t_0]$, for $t_0 > r^{2\sigma} > 0$. Assume now that $[x_0^i - r, x_0^i + r]^n \subset \Omega$ and $v(x,t)$ is bounded in $\mathbb{R}^n \times [t_0 - r^{2\sigma}, t_0]$, then there exist constants γ and β in $(0,1)$

that can be determined a priori only in terms of the data, such that v is C^β in $Q_{\gamma r}(x_0, t_0)$.

The details of the proofs can be found at [KsL1]. In order to develop the Hölder regularity method, it is necessary to localize the energy inequality by space and time truncation. Due to the non-locality of the diffusion, this appears complicated. On the other hand, $(-\Delta)^\sigma v$ can be thought as the normal derivative of some extension of v (the Dirichlet to Neumann operator of v. See [CS] for a general discussion). This allows us to realize the truncation as a standard local equation in one more dimension: we introduce first the corresponding extension v^* defined from $C_0^\infty(\mathbb{R}^n)$ to $C_0^\infty(\mathbb{R}^n \times \mathbb{R}^+)$ by:

$$\begin{aligned} -\nabla(y^a \nabla v^*) &= 0 && \text{in } \mathbb{R}^n \times (0,\infty) \\ v^*(x,0) &= v(x) && \text{for } x \in \mathbb{R}^n \end{aligned}$$

for $a = 1 - 2\sigma$. (This extension consists simply in convolving v with the Poisson kernel of the upper half space in one more variable.) Then the following result holds true: for v defined on $\mathbb{R}^n$, we have:

$$(-\Delta)^\sigma v(x) = \partial_\nu v^*(x,0) = -\lim_{y\to 0} y^a v_y^*(x,y)$$

where we denote $\partial_\nu v^*$ the outward normal derivative of v^* on the boundary $\{y = 0\}$. Hence, it is possible to consider the solution v of problem (M.P) as the boundary value of v^* which is solution of

$$\begin{cases} \nabla(y^a \nabla v^*) = 0 & \text{in } y > 0 \\ \lim_{y\to 0} y^a v_y^*(x,y,t) = (v^{\frac{1}{m}})_t(x,0,t) & x \in \Omega \\ v^*(x,0,t) = 0 & \text{on } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{14}$$

Thus, we can obtain the Hölder estimate of v immediately by showing the Hölder regularity of v^*.

Since the diffusion coefficients $D(v) = |v|^{1-\frac{1}{m}}$ goes to infinity as $v \to 0$, we need to control the oscillation of v from below. Hence, we consider the new function w^* derived from v^* such that $w^*(x,y,t) = M - v^*(x,y,t+t_0)$ with $M = M(t_0) = \sup_{t\geq t_0>0} v^*$. By Theorem 2.1, we know that the solution satisfies

$$v^*(\cdot,t) \leq M(t_0) < \infty \qquad (t \geq t_0).$$

From this, we get to a familiar situation:
(15)
$$\begin{cases} \nabla(y^a \nabla w^*) = 0 & \text{in } y > 0 \\ -\lim_{y\to 0^+} y^a \nabla_y w^*(x,y) = \left[(M - w^*)^{\frac{1}{m}}\right]_t (x,0) & x \in \Omega \\ w^*(x,0,t) = M & \text{on } \mathbb{R}^n \setminus \Omega. \end{cases}$$

The research is divided into four setps: at first step, we study several properties of the Fast Diffusion Equation (shortly, FDE) with fractional powers

$$(16) \qquad (-\triangle)^\sigma u^m + u_t = 0, \qquad \left(\frac{n-2\sigma}{n+2\sigma} < m < 1\right).$$

More precisely, we explain *Scale Invariance*, *L^1-Contraction* and *Extinction on Finite Time*. In second step, we show the existence of weak solution of the problem (M.P). Also, we investigate the boundedness of the solutions of problem (M.P) for positive times. Lastly in this section, we compute local energy inequality of $(w^* - k)_\pm$ which will be a key step in establishing local Hölder estimates. The proof of the Hölder regularity of problems is given in third step. In this step, we consider the extension v^* of v that solves (M.P). This allows us to treat non-linear problems, involving fractional Laplacians, as a local problems. In the last section, we study the existence of non-linear eigenvalue problem with fractional powers which is asymptotic profile of the parabolic flow (M.P) on extinction time.

2.1. **Properties of Fast Diffusion Equations with Fractional Powers.** Since the operator $(-\triangle)^\sigma$ converges to $(-\triangle)$ as the quantity σ goes to 1, it is natural to expect that the solutions of the equation (16) has a lot in common with those of the FDE (of course, not in complete accord). Hence, before coming to main issue, we will discuss such properties of FDE in this section.

2.1.1. *Scale Invariance.* Let us examine the application of *scaling transformations* to the fractional powers of the FDE in some detail. Let $u = u(x,t)$ be a solution of the fractional powers of the FDE,

$$(17) \qquad (-\triangle)^\sigma u^m + u_t = 0 \qquad (0 < m < 1).$$

Then

$$(18) \qquad (\tau u)(x,t) = L^{\frac{2\sigma}{m-1}} T^{-\frac{1}{m-1}} u\left(\frac{x}{L}, \frac{t}{T}\right).$$

is again a solution of the fractional powers of the FDE in the same class.

2.1.2. *L^1-contraction.* This is a very important estimate which has played a key role in the fractional powers of the FDE theory. It will allow us to develop existence, uniqueness and stability theory in the space L^1.

Lemma 2.3 (L^1-contraction). *Let Ω is a bounded domain of $\mathbb{R}^n$ with smooth boundary, and let u and $\tilde{u}$ be two smooth solutions of the fractional powers of the fast diffusion equation (FDE):*

$$\begin{cases} (-\Delta)^\sigma u^m + u_t = 0 & \text{in } Q_T = \Omega \times (0,T) \\ u = 0 & \text{on } \mathbb{R}^n \setminus \Omega \end{cases}$$

with initial date u_0, $\tilde{u}_0$ respectively. We have for every $t > \tau \geq 0$

$$\int_\Omega \left[u(x,t) - \tilde{u}(x,t)\right]_+ dx \leq \int_\Omega \left[u(x,\tau) - \tilde{u}(x,\tau)\right]_+ dx \tag{19}$$

As a consequence,

$$\|u(t) - \tilde{u}(t)\|_1 \leq \|u_0 - \tilde{u}_0\|_1. \tag{20}$$

2.1.3. *Extinction in Finite Time.* The main difference with porous medium equation is the finite time convergence of the solutions to the zero solution, which replaces the infinite time stabilization that holds for $m \geq 1$. This phenomenon is called *extinction in finite time* and read as follows.

Lemma 2.4. *If $u(x,t)$ is the $C^{2,1}$ solution of the fast diffusion equation with fractional powers :*

$$\begin{cases} (-\Delta)^\sigma u^m + u_t = 0 & \text{in } Q_\infty = \Omega \times (0,\infty) \\ u = 0 & \text{on } \mathbb{R}^n \setminus \Omega \\ u(x,0) = u_0(x) \in C^0(\Omega) \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^n$ with smooth boundary, then there exists $T^* > 0$ such that $u(\cdot,t) = 0$ for all $t \geq T^*$, i.e.,

$$\lim_{t \to T^*} \|u(\cdot,t)\|_\infty = 0$$

for some $T^* > 0$. The solution can be continued past the extinction time T^* in a weak sense as $u \equiv 0$.

2.1.4. *No waiting time.* The main difference between local and non-local Porous Medium Equations is that there is no waiting time in the nonlocal equation since $u_t(x_0, t_0)$ is equation to nonlocal integral which is positive when the value of $u(x_0, t_0)$ is zero.

2.2. **Weak solutions.** First, we study the problem (M.P) in the class of nonnegative weak solutions. For the remainder of this paper we assume that

$$\frac{n-2\sigma}{n+2\sigma} < m < 1$$

holds.

Definition 2.1. *A non-negative weak solution of equation (M.P) is a locally integrable function, $v \in L^1_{loc}(\mathbb{R}^n \times [0,\infty))$, such that $(-\Delta)^\sigma v \in L^1_{loc}(\mathbb{R}^n \times [0,\infty))$ and $v = 0$ on $\mathbb{R}^n \setminus \Omega$, and the identity*

$$\int_0^\infty \int_\Omega v^{\frac{1}{m}} \eta_t \, dxdt = \int_0^\infty \int_\Omega v[-(-\Delta)^\sigma \eta] \, dxdt \tag{21}$$

holds for any test function $\eta \in C_c^{2,1}(\Omega \times [0,\infty))$ with $\eta = 0$ on $\mathbb{R}^n \setminus \Omega$.

We show the existence and comparison result for weak solutions. the proof is similar to the proof of the Theorem (5.5) in [Va].

Lemma 2.5. *There exists a non-negative weak solution of (M.P). Moreover, the comparison principle holds for these solutions: v, $\hat{v}$ are weak solutions with initial data such that $v_0 \le \hat{v}_0$ a.e. in Ω, then $v \le \hat{v}$ a.e. for all $t > 0$.*

2.3. **Local Energy Estimate of w^*.** The key ingredients for Hölder regularity are the Sobolev and local energy inequalities for the extension $w^*(x,y,t) = M - v^*(x,y,t+t_0)$ with $M = \sup_{t \ge t_0 > 0} v^*$. The effect of the non-local part of $(-\Delta)^\sigma$ becomes encoded locally in the extra variable. The first result, Sobolev inequality, states as follows:

Lemma 2.6 (Sobolev Inequality). *For a cut-off function η compactly supported in B_r,*

$$\|\eta v\|_{L^{\frac{2n}{n-2\sigma}}(\mathbb{R}^n)} \le C\|\eta v\|_{\dot{H}^\sigma(\mathbb{R}^n)} \tag{22}$$

and

$$\begin{aligned} &\|\eta v\|^2_{L^2(t_1,t_2;L^2(\mathbb{R}^n))} \le \\ &C\Big(\sup_{t_1 \le t \le t_2} \|\eta v\|^2_{L^2(\mathbb{R}^n)} + \|\nabla(\eta v)^*\|^2_{L^2(t_1,t_2;L^2(B_r^*,y^a))}\Big) |\{\eta v > 0\}|^{\frac{2\sigma}{n+2\sigma}} \end{aligned} \tag{23}$$

for some $C > 0$.

Lemma 2.7 (Local Energy Estimate).
Let t_1, t_2 be such that $t_1 < t_2$ and let $v^* \in L^\infty(t_1,t_2;L^2(\mathbb{R}^n \times \mathbb{R}^+))$ be solution to (14) and let $w^*(x,y,t) = M - v^*(x,y,t+t_0)$ with $M = \sup_{t \ge t_0 > 0} v^*$. Then, there exists a constant λ such that for every $t_1 \le t \le t_2$ and cut-off function

η such that the restriction of $\eta(w^* - k)_\pm$ on B_r^* is compactly supported in $B_r \times (-r, r)$:

$$
\begin{aligned}
&\frac{1}{m}\int_{B_r\times\{t_2\}} \eta^2\left[\int_0^{(w-k)_\pm}(M-k\mp\xi)^{\frac{1}{m}-1}\xi\,d\xi\right]dx \\
&\quad + \int_{t_1}^{t_2}\int_{B_r^*}|\nabla(\eta(w^*-k)_\pm)|^2\,y^a dx\,dy\,dt \\
&\le \int_{t_1}^{t_2}\int_{B_r^*}|(\nabla\eta)(w^*-k)_\pm|^2\,y^a dx\,dy\,dt \\
&\quad + \frac{2}{m}\int_{t_1}^{t_2}\int_{B_r}\left[\int_0^{(w-k)_\pm}(M-k\mp\xi)^{\frac{1}{m}-1}\xi\,d\xi\right]|\eta\eta_t|\,dx\,dt \\
&\quad + \frac{1}{m}\int_{B_r\times\{t_1\}} \eta^2\left[\int_0^{(w-k)_\pm}(M-k\mp\xi)^{\frac{1}{m}-1}\xi\,d\xi\right]dx
\end{aligned}
\tag{24}
$$

2.4. Key Lemmas.

Lemma 2.8. *There exists positive numbers ρ and λ independent of $\mu^\pm$ and ω such that if*

$$
\left|\left\{(x,t)\in Q_R(\theta_0); w(x,t) > \mu^+ - \frac{\omega}{2}\right\}\right| < \rho|Q_R(\theta_0)| \tag{25}
$$

then

$$
w(x,t) < \mu^+ - \frac{\lambda\omega}{4}
$$

for all $(x,t)\in Q_{\frac{R}{2}}(\theta_0)$.

Idea of the proof Set, for any non-negative integer k,

$$
R_k = \frac{R}{2} + \frac{R}{2^{k+1}}, \quad \text{and} \quad l_k = \mu^+ - \lambda\left(\frac{\omega}{4} + \frac{\omega}{2^{k+2}}\right).
$$

We denote by $\tilde{B}^*_{R,\delta}$ the set $B_R \times (0,\delta)$ and introduce the cylinders

$$
Q_k(\theta_0) = B_{\overline{R}_k} \times \left(-\theta_0^{-\alpha}R_k^{2\sigma}, 0\right)
$$

and

$$
\tilde{Q}^*_{R_k,\frac{\delta k}{4}}(\theta_0) = \tilde{B}^*_{R_k,\frac{\delta k}{4}} \times \left(-\theta_0^{-\alpha}R_k^{2\sigma}, 0\right).
$$

Let $\overline{w}(x,\tau) = w(x,\theta_0^{-\alpha}\tau)$ and $\overline{w}^*(x,y,\tau) = w^*(x,y,\theta_0^{-\alpha}\tau)$. We also define the quantity $\overline{Z}_k$ to be

$$
\overline{Z}_k = \left|\{(x,t)\in Q_{R_k}(1) : \overline{w} > l_k\}\right|.
$$

Then, $\overline{Z}_k$ will be equal to Z_k. Through the local energy estimate, we show that there are $0 < \sigma$, $\delta < 1$ such that we have

$$\overline{Z}_{k+1} \leq C4^{2k}\left(\frac{4^{3-a}}{(1+a)R^2} + \frac{4^5\theta_0^\alpha}{2mM^\alpha R^{2\sigma}}\right)\overline{Z}_{k-1}^{1+\frac{2\sigma}{n+2\sigma}} = C'4^{2k}\overline{Z}_{k-1}^{1+\frac{2\sigma}{n+2\sigma}}.$$

If $\overline{Z}_1$ is small, then $\overline{Z}_k$ converges to zero, which implies the conclusion.

□

Lemma 2.9 (Oscillation Lemma). *There exist constants* $\lambda^* > 0$ *and* $\kappa \in (0,1)$ *such that if*

$$\mathrm{osc}_{Q^*_{2R}} w^* = \omega = \mu^+ - \mu^-,$$

then

$$\mathrm{osc}_{Q^*_{\frac{R}{4}}(\theta_0,0,1-\frac{\varrho}{2})} w^* \leq \omega - \lambda^* = \kappa\omega.$$

2.5. **Asymptotic behaviour for the FDE with fractional powers.** The asymptotic description is based on the existence of appropriate solutions that serve as model for the behavior near extinction: there is a self-similar solution of the form

$$U(x,t;T) = (T-t)^{1/(1-m)} f(x) \tag{26}$$

for a certain profile $f > 0$, where $\varphi = f^m$ is the solution of the superlinear elliptic equation

$$(-\Delta)^\sigma \varphi(x) = \frac{1}{1-m}\varphi(x)^p, \qquad p = \frac{1}{m}$$

such that $\varphi > 0$ in Ω with zero on $\mathbb{R}^n \setminus \Omega$. Hence, similarity means in this case the separate-variables form. The existence and regularity of this solution depends on the exponent p, indeed it exists for $p < (n+2\sigma)/(n-2\sigma)$, the Sobolev exponent. Since $p = 1/m$, this means that smooth separate-variables solutions exist for

$$\frac{n-2\sigma}{n+2\sigma} < m < 1$$

an assumption that will be kept in the sequel. Note that the family of solutions (26) has a free parameter $T > 0$.

The above family of solutions allows to describe the behavior of general solutions near their extinction time.

Theorem 2.10. *Under the above assumptions on* u_0 *and* m, *we have the following property near the extinction time of a solution* $u(x,t)$: *for any sequence* $\{u(x,t_n)\}$, *we have a subsequence* $t_{n_k} \to T^*$ *and a* $\varphi(x)$ *such that*

$$\lim_{k\to\infty}(T^* - t_{n_k})^{-1/(1-m)}\left|u(x,t_{n_k}) - U(x,t_{n_k};T^*)\right| \to 0$$

uniformly in compact subset of Ω *for* $U(x,t;T^*) = (T^* - t)^{1/(1-m)}\varphi^{1/m}(x)$ *where* φ *is a eigen-function of fully nonlinear equation*

$$\begin{cases} (-\Delta)^\sigma\varphi = \dfrac{1}{1-m}\varphi^{\frac{1}{m}} & \text{in } \Omega \\ \varphi = 0 & \text{on } \mathbb{R}^n \setminus \Omega \\ \varphi > 0 & \text{in } \Omega. \end{cases}$$

3. Goemetric Properties of first eigenfunction

In this section we study convexity and concavity properties of non-local parabolic flows and derive related geometric properties for the asymptotic limits of such evolutions. More precisely, we consider the nonnegative solution $u(x,t)$ of the Dirichlet problem for the fractional power of Heat operator

$$(27)\qquad \begin{cases} u_t + (-\Delta)^{\frac{1}{2}}u = 0 & \text{in } \Omega \\ u(x,t) = u_0(x) > 0 & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \mathbb{R}^n \setminus \Omega \end{cases}$$

posed on a strictly convex and bounded domain $\Omega \in \mathbb{R}^n$. In the renormalized limit these flows converge to solutions of the nonlinear eigen-value problems (28) below, especially $p = 1$. The method we present produces power convexity results for these positive nonlinear eigen-functions. As a second type of result, the evolution approach also proves eventual power convexity in space for the solutions of the parabolic problems. Eventual power convexity means that it will hold for large enough times even for data that are not initially power convex.

3.1. Elliptic Problems and History.

Let us present the problems and concepts to motivate our work. Let the function $\varphi(x)$ satisfy the following nonlinear eigenvalue problem:

$$(28)\qquad \begin{cases} (-\Delta)^{\frac{1}{2}}\varphi = \lambda\varphi^p & \text{in } \Omega \\ \varphi > 0 & \text{in } \Omega \\ \varphi = 0 & \text{on } \mathbb{R}^n \setminus \Omega. \end{cases}$$

The main question we address is motivated by the following conjecture:

Conjecture 3.1 (cf. Conjecture 1.1 in [1]). *Let* φ_1^σ *be the ground state eigenfunction for the symmetric stable processes of index* $0 < \sigma < 1$ *killed upon leaving the interval* $I = (-1,1)$. *Then* φ_1^σ *is concave on* I.

The function φ_1^σ is the first eigen function of (28) with $\frac{1}{2}$ being replaced by σ, $(0 < \sigma < 1)$. The only known case is when $\Omega = (-1, 1)$ and $\sigma = \frac{1}{2}$, where the question is answered in the affirmative in [BK]. More precisely, in [BK], Bañuelos and Kulczycki consider the Cauchy process in $\mathbb{R}^n$, $n \geq 1$, killed on leaving a bounded domain Ω, and investigate the eigenvalues λ_n and eigenfunctions φ_n of the corresponding generator in $L^2(\Omega)$. The method is quite different from the ones used for the Brownian motion which are not applicable in this case due to the non-locality of the generator. The most detailed information about the eigenvalues and eigenfunctions is found for the case where $n = 1$ and $\Omega = (-1, 1)$. In particular, it is proved that φ_1 is symmetric and concave on $(-1, 1)$, φ_2 is antisymmetric, concave on $(0, 1)$ and convex on $(-1, 0)$, each φ_n has at most $2n - 2$ zeros on $(-1, 1)$.

In the particular case $\sigma = \frac{1}{2}$ it is easy to see that the operator $(-\Delta)^{\frac{1}{2}}$ coincides with the Dirichlet to Neumann operator in the upper half space of $\mathbb{R}^{n+1}$. More precisely, given $u(x)$ defined in $\mathbb{R}^n$, extend it to $u^*(x, y)$ in $\mathbb{R}^{n+1}$ by convolving with the classical Poisson kernel.

3.2. Power-Convexity : Bounded Domain.

Lemma 3.2. *There exist a weak solution $\psi \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^n)$ of the eigen-value problem*

$$\begin{cases} (-\Delta)^{\frac{1}{2}}\psi = \lambda\psi^p & \text{in } \Omega \\ \psi > 0 & \text{in } \Omega \\ \psi = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{29}$$

for $0 < p < \frac{n+1}{n-1}$.

We address now the long-time geometrical properties of solutions of the initial-value problem for the fractional power of the Heat Equation

$$(-\Delta)^{\frac{1}{2}}u + u_t = 0, \tag{30}$$

posed in a bounded domain Ω with

$$u = 0 \quad \text{on } \mathbb{R}^n \setminus \Omega, \qquad u > 0 \quad \text{in } \Omega \tag{31}$$

and initial data

$$u(x, 0) = u_0(x) \in \dot{H}_0^{\frac{1}{2}}(\Omega) \tag{32}$$

Our geometrical results will be derived under the extra assumption that Ω is strictly convex. It is easy to show that the fractional power of the Laplace operator has a countable discrete set of eigenvalues $\Sigma =$

$\{\lambda_i | \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots\}$, whose eigen-functions $\{\phi_n\}$ span $\dot{H}_0^{\frac{1}{2}}(\Omega)$, where ϕ_n is a normalized eigen-function corresponding to λ_n. Then, $u_n(x,t) = e^{\lambda_n t}\phi_n(x)$ is the solution of the fractional powers of the Heat operator with initial data $\phi_n(x)$. On the other hand, for $u_0(x) \in \dot{H}_0^{\frac{1}{2}}(\Omega)$, there are coefficients $\{a_n\}$ such that $u_0(x) = \sum_{n=1}^{\infty} a_n\phi_n(x)$. Hence,

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n t}\phi_n = a_1 e^{-\lambda_1 t}\varphi + e^{-\lambda_2 t}\eta(x,t)$$

where $\|\eta(x,t)\|_{L^2_x(\mathbb{R}^n)} < C < \infty$. Then $\varphi(x)$ will be a solution of

$$\begin{cases} (-\Delta)^{\frac{1}{2}}\varphi(x) = \lambda_1\varphi(x) & \text{in } \Omega \\ \varphi(x) = 0 & \text{on } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{33}$$

We have the following results in the bounded domain, [KsL2].

Lemma 3.3 (Approximation Lemma). *For every $u_0 \in \dot{H}_0^{\frac{1}{2}}(\Omega)$, we have*

$$|e^{\lambda_1 t}u(x,t) - a_1\varphi(x)| \leq Ce^{-(\lambda_2-\lambda_1)t} \tag{34}$$

and

$$\|e^{\lambda_1 t}u(x,t) - a_1\varphi(x)\|_{C^k_x(\Omega)} \leq CKe^{-(\lambda_2-\lambda_1)t} \tag{35}$$

for $k = 1, 2, \cdots$.

Corollary 3.4. *If Ω is convex, then the solution $u(x,t)$ of* (30)-(32) *is power-convex, i.e., $D^2_x\,(u(x,t))^{-\frac{2}{n+1}} \geq 0$.*

Corollary 3.5. *If Ω is convex, then the stationary profile $\varphi(x)$ of $u(x,t)$ is power-convex, i.e., $D^2_x\,(\varphi(x))^{-\frac{2}{n+1}} \geq 0$.*

3.3. **Power-Convexity : Unbounded Domain.** We now examine the same geometrical questions for the Gauchy problem for the Fractional Heat Equation

$$\begin{cases} u_t + (-\Delta)^{\frac{1}{2}}u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = u_0(x) & \text{on } \mathbb{R}^n \end{cases} \tag{36}$$

with initial data u_0 nonnegative, bounded, integrable and compactly supported. Then, the solution u of (36) is given by

$$u(x,t) = \int_{\mathbb{R}^n} p(x,\xi,t)u_0(\xi)\,d\xi$$

where

$$p(x,\xi,t) = \frac{c_n t}{(|x-\xi|^2+t^2)^{\frac{n+1}{2}}}, \qquad t > 0,\ x,\xi \in \mathbb{R}^n,\ c_n = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}}.$$

The extension argument is also applied to Cauchy problem. Let $u^*(x,z,t)$ be the solution of the following extension problem:

$$(37)\qquad \begin{cases} \Delta_x u^* + u^*_{zz} = 0 & \text{in } \mathbb{R}^n \times \mathbb{R}^+ \times (0,\infty) \\ u^*_z(x,0,t) - u^*_t(x,0,t) = 0 & x \in \mathbb{R}^n \\ u^*(x,0,0) = u_0(x) & x \in \mathbb{R}^n. \end{cases}$$

The details of proof of the following lemmas can be found at [KsL2].

Lemma 3.6. *Let $u_0 \geq 0$ be a continuous and bounded initial function with a compact support. If $(u_0)^{-\frac{2}{n+1}}$ is strictly convex, then the solution u^* is power-convex in the space variable x for all $t > 0$, i.e., $D^2_x\left[(u^*)^{-\frac{2}{n+1}}\right] \geq 0$.*

The fact that solution u of (36) is the trace of u^* gives the following Corollary.

Lemma 3.7. *Let $u_0 \geq 0$ be a continuous and bounded initial function with a compact support. If $(u_0)^{-\frac{2}{n+1}}$ is strictly convex, then the solution $u(x,t)$ of (36) is power-convex, i.e., $D^2_x\,(u(x,t))^{-\frac{2}{n+1}} \geq 0$.*

4. Regularity Theory of Nonlocal Fully nonlinear Equations

In this section, we are going to discuss the recent development in the regularity theory of nonlocal fully nonlinear equations.

4.1. Nonlocal Equations with nonsymmetric kernels. The concept of viscosity solutions, its comparison principle and stability properties can be found in [CS] for symmetric kernels and in [KL1] for possibly nonsymmetric kernels. Kim and Lee [KL1] considered much larger class of operators but prove the regularity of viscosity solutions only for $1 < \sigma < 2$.

Now we are going to consider a subclass $\mathcal{S}^{\mathfrak{L}}_\eta$ of $\mathcal{S}^{\mathfrak{L}}$ where the drift effect created by the nonsymmetric kernel is controllable. For $x \in B_R$ and $\varphi \in C^2_{B_R}(u;x)^{\pm}$, we set

$$\mu_R(u,x,y;\nabla\varphi) = u(x+y) - u(x) - (\nabla\varphi(x)\cdot y)\,\chi_{B_R}(y).$$

For $u \in C^{1,1}[x]$, we write $\mu_R(u,x,y) = \mu_R(u,x,y;\nabla u)$. Then we define $\mu^{\pm}_R$ and $\mathcal{M}^{\pm}_{\mathfrak{L}_0,R}u(x;\nabla\varphi)$ by replacing μ by μ_R in the definition $\mathcal{M}^{\pm}_{\mathfrak{L}_0}u(x;\nabla\varphi)$. We note if $u \in C^{1,1}[x]$, then $\mathcal{M}^{\pm}_{\mathfrak{L}_0,R}u(x;\nabla\varphi) = \mathcal{M}^{\pm}_{\mathfrak{L}_0,R}u(x) \risingdotseq$ p.v. $\int_{\mathbb{R}^n} \mu_R(u,x,y)K(y)\,dy$. Key observations are the following:

- For the nonsymmetric case, $K(y)$ and $K(-y)$ can be chosen any of $\lambda/|y|^{n+\sigma}$ or $\Lambda/|y|^{n+\sigma}$. Therefore there could be an extra term

$$\int_{\mathbb{R}^n} \frac{\left|(\nabla u(x)\cdot y)\chi_{B_1}(y)\right|}{|y|^{n+\sigma}}\,dy.$$

• The equation is not scaling invariant due to $|\chi_{B_1}(y)|$.

• Somehow the equation has a drift term, not only the diffusion term. The case $1 < \sigma < 2$ and the case $0 < \sigma \leq 1$ require different technique due to the difference of the blow rate as $|y|$ approaches to zero and the decay rate as $|y|$ approaches to infinity. When $1 < \sigma < 2$, a controllable decay rate of kernel allows Hölder regularities in a larger class, which is invariant under an one-sided scaling i.e. if u is a solution of the homogeneous equation, then so is $u_\epsilon(x) = \epsilon^{-\sigma}u(\epsilon x)$ for $0 < \epsilon \leq 1$. Critical case ($\sigma = 1$) and supercritical case ($0 < \sigma < 1$) have been studied in [BBC] with different techniques due to the slow decay rate of the kernel as $|x| \to \infty$.

Definition 4.1. *Let $0 < \eta \leq 1$ and $\mathcal{I} \in \mathcal{S}^{\mathfrak{L}}$, where $\mathfrak{L}$ is a class of linear integro-differential operators. Then we say that $\mathcal{I} \in \mathcal{S}^{\mathfrak{L}}_\eta$ if, for $R \in (0,1]$, there are $\mathcal{B}^{\pm}_R : \mathbb{R}^n \to \mathbb{R}$ such that*

(1) *$\mathcal{B}^{\pm}_R(\cdot)$ is homogeneous of degree one, i.e. $\mathcal{B}^{\pm}_R(0) = 0$ and $\mathcal{B}^{\pm}_R(\nabla u) = \mathcal{B}^{+}_R\left(\frac{\nabla u}{|\nabla u|}\right)|\nabla u|$ for $|\nabla u| \neq 0$,*
(2) *$|\{\nu \in S^{n-1} : \mathcal{B}^{\pm}_R(\nu) < 0\}| \geq \eta |S^{n-1}| > 0$,*
(3) *$\mathcal{M}^{-}_{\mathfrak{L},R,\eta}u(x) \leq \mathcal{I}u(x) - \mathcal{I}0(x) \leq \mathcal{M}^{+}_{\mathfrak{L},R,\eta}u(x)$ whenever $u \in C^{1,1}[x] \cap B(\mathbb{R}^n)$ for $x \in B_R$,*

where $\mathcal{M}^{\pm}_{\mathfrak{L},R,\eta}u(x) := \mathcal{M}^{\pm}_{\mathfrak{L},R}u(x) \pm \mathcal{B}^{\pm}_R(\nabla u(x)) \pm (2-\sigma)CR^{1-\sigma}|\nabla u(x)|$.

Definition 4.2. *Let $\mathcal{L} \in \mathfrak{L}$ be a linear integro-differential operator with a kernel K. For $0 < R < 1$, the drift vector $\mathfrak{b}_{\mathcal{L},R}$ of $\mathcal{L}$ at R is defined by*

$$\mathfrak{b}_{\mathcal{L},R} = (2-\sigma)\int_{B_1\setminus B_R} y\,K(y)\,dy.$$

The details of proof of the following lemmas and theorems can be found at [KL2].

Lemma 4.1. *Let $0 < \sigma < 2$ and $0 < R < 1$. Let $\mathfrak{L}$ be a class of linear integro-differential operators. Then we have the following results:*

(1) *If $\mathcal{L}$ is a linear integro-differential operator which is in $\mathfrak{L}$, then $\mathcal{L} \in \mathcal{S}^{\mathfrak{L}}_\eta$ for $0 < \eta \leq \frac{1}{2}$.*
(2) *Let $\mathcal{L}_i$ be a linear integro-differential operator with a kernel K_i for $i = 1, \cdots, N$. Let $\mathfrak{b}_{\mathcal{L}_i,R}$ be the drift vectors of $\mathcal{L}_i$. Assume that there is a unit vector $\mathfrak{a}$ such that for any nonzero drift vector $\mathfrak{b}_{\mathcal{L}_i,R}$,*

$$\left\langle \mathfrak{a}, \frac{\mathfrak{b}_{\mathcal{L}_i,R}}{|\mathfrak{b}_{\mathcal{L}_i,R}|}\right\rangle > 0.$$

If $\mathcal{I} \in \mathcal{S}^{\mathfrak{L}}$ satisfies that

$$\min_{i=1,\cdots,N} \mathcal{L}_i u(x) \le \mathcal{I}u(x) - \mathcal{I}0(x) \le \max_{i=1,\cdots,N} \mathcal{L}_i u(x)$$

whenever $u \in \mathrm{C}^{1,1}[x] \cap \mathrm{B}(\mathbb{R}^n)$ for $x \in B_R$, then $\mathcal{I} \in \mathcal{S}^{\mathfrak{L}}_{\eta}$ for some $\eta > 0$.

(3) *Let $\mathcal{L}_\alpha$ be a linear integro-differential operator with a kernel K_α, $\alpha \in I$. Let $\mathfrak{b}_{\mathcal{L}_\alpha,R}$ be the drift vectors of $\mathcal{L}_\alpha$. Assume that there is a vector $\mathfrak{a} \in S^{n-1}$ and $\eta > 0$ such that for any nonzero drift vectors $\mathfrak{b}_{\mathcal{L}_\alpha,R}$,*

$$\left\langle \mathfrak{a}, \frac{\mathfrak{b}_{\mathcal{L}_\alpha,R}}{|\mathfrak{b}_{\mathcal{L}_\alpha,R}|} \right\rangle \ge \eta^{\frac{1}{n-1}} \quad \text{for any } \alpha \in I.$$

If $\mathcal{I} \in \mathcal{S}^{\mathfrak{L}}$ satisfies that

$$\min_{\alpha\in I} \mathcal{L}_\alpha u(x) \le \mathcal{I}u(x) - \mathcal{I}0(x) \le \max_{\alpha \in I} \mathcal{L}_\alpha u(x)$$

whenever $u \in \mathrm{C}^{1,1}[x] \cap \mathrm{B}(\mathbb{R}^n)$ for $x \in B_R$, then we have $\mathcal{I} \in \mathcal{S}^{\mathfrak{L}}_{\eta}$.

Lemma 4.2.

(1) *Let $0 < \sigma < 2$ and let $\mathfrak{L}$ be a class of linear integro-differential operators. If $\mathcal{L} \in \mathcal{S}^{\mathfrak{L}}$ with a symmetric kernel K, then $\mathcal{L} \in \mathcal{S}^{\mathfrak{L}}_{\eta}$ for some $\eta \in (0,1]$. In addition, if $\mathcal{L}_\alpha$ has symmetric kernel for all α, then $\sup_\alpha \mathcal{L}_\alpha, \inf_\alpha \mathcal{L}_\alpha \in S^{\mathfrak{L}}_{\eta}$ for $1 \ge \eta > 0$.*

(2) *If $1 < \sigma < 2$, then $\mathcal{S}^{\mathfrak{L}_0} = \mathcal{S}^{\mathfrak{L}_0}_{\eta}$ for some $\eta \in (0,1]$.*

At [KL2]. we have the following Harnack inequality and Hölder regularity.

Theorem 4.3. *Let $\sigma_0 \in (1,2)$ and assume that $\sigma \in (0,1]$ or $\sigma \in (\sigma_0, 2]$ an that $R \in (0, R_0]$. If $u \in \mathrm{B}(\mathbb{R}^n)$ is a positive function such that*

$$\mathcal{M}^-_{\mathfrak{L}_0} u \le \frac{C_0}{R^\sigma} \quad \text{and} \quad \mathcal{M}^+_{\mathfrak{L}_0} u \ge -\frac{C_0}{R^\sigma} \quad \text{with } \sigma_0 < \sigma < 2 \text{ on } B_{2R}$$

or

$$\mathcal{M}^-_{\mathfrak{L}_0,R,\eta} u \le \frac{C_0}{R^\sigma} \quad \text{and} \quad \mathcal{M}^+_{\mathfrak{L}_0,R,\eta} u \ge -\frac{C_0}{R^\sigma} \quad \text{with } 0 < \sigma \le 1 \text{ on } B_{2R}$$

in the viscosity sense, then there is uniform constant $C > 0$ such that

$$\sup_{B_{R/2}} u \le C\left(\inf_{B_{R/2}} u + C_0\right).$$

For $\sigma \in (\sigma_0, 2)$, C depends only on λ, Λ, the dimension n, and σ_0. And for $\sigma \in (0,1]$, C depends only on λ, Λ, n, σ, and η.

Theorem 4.4. *Let $\sigma_0 \in (1,2)$ and assume that $\sigma \in (0,1]$ or $\sigma \in (\sigma_0, 2]$ and that $R \in (0, R_0]$. If u is a bounded function on $\mathbb{R}^n$ such that*

$$\mathcal{M}^{-}_{\mathfrak{L}_0} u \leq \frac{C_0}{R^\sigma} \quad \text{and} \quad \mathcal{M}^{+}_{\mathfrak{L}_0} u \geq -\frac{C_0}{R^\sigma} \quad \text{with } \sigma_0 < \sigma < 2 \text{ on } B_{2R}$$

or

$$\mathcal{M}^{-}_{\mathfrak{L}_0,R,\eta} u \leq \frac{C_0}{R^\sigma} \quad \text{and} \quad \mathcal{M}^{+}_{\mathfrak{L}_0,R,\eta} u \geq -\frac{C_0}{R^\sigma} \quad \text{with } 0 < \sigma \leq 1 \text{ on } B_{2R}$$

in the viscosity sense, then there is some constant $\alpha > 0$ such that

$$\|u\|_{C^\alpha(B_{R/2})} \leq \frac{C}{R^\alpha}\left(\|u\|_{L^\infty(\mathbb{R}^n)} + C_0\right)$$

where $C > 0$ is some universal constant. For $\sigma \in (\sigma_0, 2)$, α and C depend only on λ, Λ, the dimension n, and σ_0. And for $\sigma \in (0,1]$, α and C depend only on λ, Λ, n, σ, and η.

4.2. **Nonlocal Nonlinear Parabolic Equations with symmetric kernels.** Similar regularity theory have been discussed on parabolic nonlocal nonlinear equation at [KL3]. Key observations are the following:

- The equation is local in time while it is nonlocal in the space variable. Caffarelli and Silvestre considered a sequence of dyadic rings in space at A-B-P estimate to find the balance of quantities in the integral. But a simple generalization of the ring in space to one in space-time fails since the equation is local in the time variable. Such unbalance between local and nonlocal terms in the equation requires more fine analysis to find a parabolic version of A-B-P estimate.
- There is a time delay to control the lower bound in a small neighborhood of a point by the current value at the point, which is a main difference between elliptic and parabolic equations. The details of results can be found at [KL3]

Acknowledgement. Ki-Ahm Lee was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) grant funded by the Korea government(MEST)(2010-0001985).

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