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Kyoto University
Existence and non-existence results of the Fučík type spectrum for the generalized $p$-Laplace operators

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1 Introduction

In this paper, we consider the existence of $(\alpha, \beta) \in \mathbb{R}^2$ for which the following quasilinear elliptic equation has a non-trivial solution:

$$(F)_{(\alpha, \beta)} \left\{ \begin{array}{ll} -\text{div} \ A(x, \nabla u) = \alpha u_+^{p-1} - \beta u_-^{p-1} & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{array} \right.$$ 

where $\nu$ denotes the outward unit normal vector on $\partial\Omega$, $1 < p < \infty$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with $C^2$ boundary $\partial\Omega$. Here, $A: \bar{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ is a map which is strictly monotone in the second variable and satisfies certain regularity conditions (see the following assumption $(A)$). The equation $(F)_{(\alpha, \beta)}$ contains the corresponding $p$-Laplacian problem as a special case, and in this case, $(\alpha, \beta)$ admitting a non-trivial solution to $(F)_{(\alpha, \beta)}$ is said to belong to the Fučík spectrum of the $p$-Laplacian. Although the $p$-Laplace operator is $(p-1)$-homogeneous, the operator $A$ is not supposed generally to be $(p-1)$-homogeneous in the second variable.

Here, we say that $u \in W^{1,p}(\Omega)$ is a (weak) solution of $(F)_{(\alpha, \beta)}$ if

$$\int_{\Omega} A(x, \nabla u) \nabla \varphi \, dx = \int_{\Omega} \alpha u_+^{p-1} \varphi \, dx - \int_{\Omega} \beta u_-^{p-1} \varphi \, dx$$

for all $\varphi \in W^{1,p}(\Omega)$.

Throughout this paper, we assume that the operator $A$ satisfies the following assumption $(A)$:

$(A)$ $A(x, y) = a(x, |y|)y$, where $a(x, t) > 0$ for all $(x, t) \in \bar{\Omega} \times (0, +\infty)$ and

(i) $A \in C^0(\bar{\Omega} \times \mathbb{R}^N, \mathbb{R}^N) \cap C^1(\bar{\Omega} \times (\mathbb{R}^N \setminus \{0\}), \mathbb{R}^N)$;

(ii) there exists a $C_1 > 0$ such that

$$|D_y A(x, y)| \leq C_1 |y|^{p-2}$$

for every $x \in \bar{\Omega}$, and $y \in \mathbb{R}^N \setminus \{0\}$;

(iii) there exists a $C_0 > 0$ such that

$$D_y A(x, y) \xi \cdot \xi \geq C_0 |y|^{p-2} |\xi|^2$$

for every $x \in \bar{\Omega}$, $y \in \mathbb{R}^N \setminus \{0\}$ and $\xi \in \mathbb{R}^N$. 

(iv) there exists $C_2 > 0$ such that

$$|D_x A(x, y)| \leq C_2(1 + |y|^{p-1})\quad\text{for every } x \in \bar{\Omega}, \ y \in \mathbb{R}^N \setminus \{0\}.$$  

Throughout this paper, we assume $C_0 \leq p - 1 \leq C_1$ because we can take such desired $C_0$ and $C_1$ anew if necessary.

The hypothesis (A) has been considered in the study of the quasilinear elliptic problems (cf. [6], [12], [13]). For example, we can treat the operators like the $p$-Laplacian with the positive weight and

$$\text{div } \left( |\nabla u|^{p-2} \nabla u \right) \quad\text{for } 1 < p \leq q < \infty.$$  

Let us recall the known results in the special case of $A(x, y) = |y|^{p-2} y$ that is, $p$-Laplace problem and $C_0 = C_1 = p - 1$. The set of all points $(\alpha, \beta) \in \mathbb{R}^2$ for which the equation

$$-\Delta_p u = \alpha u_+^{p-1} - \beta u_-^{p-1} \quad\text{in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad\text{on } \partial \Omega \quad (1)$$

has a non-trivial solution is called the Fučík spectrum of the $p$-Laplacian under the Neumann boundary condition. In this paper, we denote the Fučík spectrum of $p$-Laplacian by $\Theta_p$. It is well known that the first eigenvalue $\mu_1 = 0$ of $-\Delta_p$ is simple and every eigenfunction corresponding to $\mu_1 = 0$ is a constant function. Therefore, $\Theta_p$ contains the lines $\{0\} \times \mathbb{R}$ and $\mathbb{R} \times \{0\}$ (we call these lines as “the trivial lines”). Furthermore, by the same argument as in [5], it can be proved that there exists a Lipschitz continuous curve contained in $\Theta_p$ which is called “the first nontrivial curve” $\mathcal{C}$ (see Section 2). In the $p$-Laplacian case, many authors have treated the Fučík spectrum (see [5], [7], [8], [10] under the Dirichlet boundary condition and [2], [3] for Neumann boundary condition).

Let us return to the general case. In [14], D. Motreanu and the present author treated the equation

$$-\text{div } A(x, \nabla u) = f(x, u) \quad\text{in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad\text{on } \partial \Omega \quad (2)$$

with the following nonlinearity:

$$f(x, u) = \begin{cases} \alpha_0 u_+^{p-1} - \beta_0 u_-^{p-1} + o(|u|^{p-1}) & \text{at } 0, \\ \alpha_0 u_+^{p-1} - \beta_0 u_-^{p-1} + o(|u|^{p-1}) & \text{at } \infty \end{cases}$$

for $(\alpha_0, \beta_0), (\alpha, \beta) \in \mathbb{R}^2$. Roughly speaking, by constructing two curves $\mathcal{C}$ and $\tilde{\mathcal{C}}$ related to the map $A$ (see section 3), it was shown that the equation (2) has a sign-changing solution in the case where $(\alpha, \beta)$ is below the curve $\mathcal{C}$ and $(\alpha_0, \beta_0)$ is above the curve $\tilde{\mathcal{C}}$. In the $p$-Laplacian case, we see that two curves $\mathcal{C}$ and $\tilde{\mathcal{C}}$ coincide with the first nontrivial curve $\mathcal{C}$. Moreover, if the first nontrivial curve lies between $(\alpha_0, \beta_0)$ and $(\alpha, \beta)$, then equation $-\Delta_p u = f(x, u)$ in $\Omega$ (under the Dirichlet boundary condition) has a non-trivial solution. Therefore, even for the general case of $A$, it seems reasonable to expect the existence of uncountably many Fučík type spectrum between $\mathcal{C}$ and $\tilde{\mathcal{C}}$.

Mainly, this paper consists of results in [14] and [15]. In the final section, we see further results and several questions concerning our problem.
2 The first nontrivial curve contained in $\Theta_p$

Here, we recall the result for the special case of $A(x, y) = |y|^{p-2}y$, that is, $p$-Laplacian problems (note that we can take $C_0 = C_1 = p - 1$ in (A)). The construction of the curve $\mathcal{C}$ contained in the Fučík spectrum is carried out by the same argument as in [5]: For $s \geq 0$, we define

$$J_s(u) := \int_{\Omega} |\nabla u|^p \, dx - s \int_{\Omega} u^p \, dx$$

for $u \in W^{1,p}(\Omega)$, $\tilde{J}_s := J_s|S$ and $S := \{u \in W^{1,p}(\Omega) : \int_{\Omega} |u|^p \, dx = 1\}$, where $\psi_1 = 1/|\Omega|^{1/p}$ (so $\|\psi_1\|_p = 1$). Here, the set $C([0,1], S)$ denotes the set of continuous functions from $[0,1]$ to $S$ with the topology induced by the $W^{1,p}(\Omega)$ norm. Finally, we set

$$c(s) := \inf_{\gamma \in \Sigma} \max_{t \in [0,1]} \tilde{J}_s(\gamma(t)).$$

Then, it can be proved that $c(s)$ is a positive critical value of $\tilde{J}_s$ with $c(0) = \mu_2$, where $\mu_2$ is the second eigenvalue of the $p$-Laplacian under the Neumann boundary condition. Moreover, we can see that $c(s)$ is continuous, strictly decreasing in $s \geq 0$ and $c(s) + s$ is strictly increasing in $s \geq 0$ (refer to [1, Lemma 2.2] and [5, Proposition 4.1]). Then, $\mathcal{C}$ is defined as follows:

$$\mathcal{C} := \{ (c(s) + s, c(s)) \ ; \ s \geq 0 \} \cup \{ (c(s), c(s) + s) \ ; \ s \geq 0 \}.$$  

Finally, we remark that in the case of $N \geq p$, it is shown in [3] that $c(s) \to 0$ as $s \to \infty$, whence the asymptotic lines of the first nontrivial curve are the trivial lines $\mathbb{R} \times \{0\}$ and $\{0\} \times \mathbb{R}$. However, if $N < p$, then $c(s) \to \bar{\lambda}$ as $s \to \infty$, where $\bar{\lambda}$ is a positive constant defined by

$$\bar{\lambda} = \inf_B \int_{\Omega} |\nabla u|^p \, dx,$$

where $B := \{u \in S \ ; \ u(x_0) = 0 \text{ for some } x_0 \in \Omega\}$. This yields that the trivial lines are not the asymptotic lines of the first nontrivial curve.

3 Existence and non-existence results

To state the results for $(F)_{(\alpha, \beta)}$, we define curves $\mathcal{C}$ and $\mathcal{C}$ by

$$\mathcal{C} := \frac{C_0}{p - 1} \mathcal{C} := \{ (aC_0/(p - 1), bC_0/(p - 1)) \ ; \ (a, b) \in \mathcal{C} \},$$

$$\mathcal{C} := \frac{C_1}{p - 1} \mathcal{C} = \{ (aC_1/(p - 1), bC_1/(p - 1)) \ ; \ (a, b) \in \mathcal{C} \},$$

where $C_0$ and $C_1$ are positive constants satisfying (A). First, we state the elementary results for the equation $(F)_{(\alpha, \beta)}$ which is shown in [14].

Proposition 1 ([14, Proposition 2]) The following assertions hold:
(i) if $\alpha \beta < 0$ or $\max\{\alpha, \beta\} < 0$ holds, then $(F)_{(\alpha, \beta)}$ has no non-trivial solutions;

(ii) if $u$ is a non-trivial solution of $(F)_{(\alpha, \beta)}$ with $\min\{\alpha, \beta\} > 0$, then $u$ changes sign;

(iii) if $u$ is a non-trivial solution of $(F)_{(\alpha, \beta)}$ with $\alpha \beta = 0$, then $u$ is a constant function;

(iv) if $0 < \alpha < \alpha'$ and $0 < \beta < \beta'$ for some $(\alpha', \beta') \in \mathcal{E}$, then $(F)_{(\alpha, \beta)}$ has no non-trivial solutions.

Define $\beta_0(s)$ and $\beta_1(s)$ for $s \geq 0$ by 

$$
\beta_0(s) := \frac{C_0}{p-1} c\left(\frac{p-1}{C_0} s\right), \quad \beta_1(s) := \frac{C_1}{p-1} c\left(\frac{p-1}{C_1} s\right),
$$

where $c(\cdot)$ is a function defined by (3) (see the following figure):

\[ \begin{align*}
\beta_0(s) &:= \frac{C_0}{p-1} c\left(\frac{p-1}{C_0} s\right), \\
\beta_1(s) &:= \frac{C_1}{p-1} c\left(\frac{p-1}{C_1} s\right),
\end{align*} \]

Now, we state existence results.

**Theorem 2** ([15]) *For every $s \geq 0$ and $R > 0$, there exists a $\beta \in [\beta_0(s), \beta_1(s)]$ such that $(F)_{(\beta+s, \beta)}$ and $(F)_{(\beta, \beta+s)}$ have at least one sign-changing solution $u \in C^1(\overline{\Omega})$ with $\int_{\Omega} |u|^{p} \, dx \leq R^p$.*

**Theorem 3** ([15]) *Let $s \geq 0$, $\epsilon > 0$ and $R_2 > R_1 > 0$ be constants satisfying

$$
R_2 > \max \left\{ \frac{\beta_1(s)+s+\epsilon}{\min\{\beta_0(s), \epsilon\}}, \frac{C_1(\beta_1(s)+s+\epsilon)^2}{C_0(\beta_1(s)+\epsilon)^2}, \frac{s(C_1-C_0)}{C_0(\beta_1(s)+\epsilon)} \right\}^{1/p} R_1.
$$

Then, there exists a $\beta \in [\beta_0(s), \beta_1(s)+\epsilon]$ such that $(F)_{(\beta+s, \beta)}$ and $(F)_{(\beta, \beta+s)}$ have at least one sign-changing solution $u \in C^1(\overline{\Omega})$ with $R_1^p \leq \int_{\Omega} |u|^{p} \, dx \leq R_2^p$.*
3.1 Variational setting and notations

In what follows, we define the norm of $W := W^{1,p}(\Omega)$ by $\|u\|_p := \|\nabla u\|_p + \|u\|_p$, where $\|u\|_q$ denotes the norm of $L^q(\Omega)$ for $u \in L^q(\Omega)$ ($1 \leq q \leq \infty$). Define $G(x, y) := \int_0^{|y|}a(x, t)dt$, then we can easily see that

$$\nabla_y G(x, y) = A(x, y)$$

and $G(x, 0) = 0$

for every $x \in \bar{\Omega}$.

Remark 4 The following assertions hold:

(i) for all $x \in \bar{\Omega}$, $A(x, y)$ is maximal monotone and strictly monotone in $y$;
(ii) $|A(x, y)| \leq \frac{C_1}{p-1}|y|^{p-1}$ for every $(x, y) \in \bar{\Omega} \times \mathbb{R}^N$;
(iii) $A(x, y)y \geq \frac{C_0}{p(p-1)}|y|^p$ for every $(x, y) \in \bar{\Omega} \times \mathbb{R}^N$;
(iv) $G(x, y)$ is convex in $y$ for all $x$ and satisfies the following inequalities:

$$A(x, y)y \geq G(x, y) \geq \frac{C_0}{p(p-1)}|y|^p \quad \text{and} \quad G(x, y) \leq \frac{C_1}{p(p-1)}|y|^p \quad (4)$$

for every $(x, y) \in \bar{\Omega} \times \mathbb{R}^N$,

where $C_0$ and $C_1$ are the positive constants described in (A).

For parameters $s \geq 0$ and $\beta \in \mathbb{R}$, we define the $C^1$ functionals $I_{\beta, s}$ and $I_{\beta, s}^+$ on $W^{1,p}(\Omega)$ by

$$I_{\beta, s}(u) := \int_{\Omega} G(x, \nabla u) dx - \frac{\beta + s}{p} \int_{\Omega} u_+^p dx - \frac{\beta}{p} \int_{\Omega} u_-^p dx$$

with

$$\langle I_{\beta, s}'(u), v \rangle = \int_{\Omega} A(x, \nabla u) \nabla v dx - (\beta + s) \int_{\Omega} u_+^{p-1} v dx + \beta \int_{\Omega} u_-^{p-1} v dx,$$

$$I_{\beta, s}^+(u) := \int_{\Omega} G(x, \nabla u) dx - \frac{\beta + s}{p} \int_{\Omega} u_+^p dx$$

for $u, v \in W^{1,p}(\Omega)$. In this paper, we use the following notations:

$$B(r) := \{u \in W; \|u\| \leq r \}, \quad B_p(r) := \{u \in W; \|u\|_p \leq r \},$$
$$D(r, r') := \{u \in W; r \leq \|u\| \leq r' \}, \quad D_p(r, r') := \{u \in W; r \leq \|u\|_p \leq r' \}$$
$$rS := \{u \in W; \|u\|_p = r \}, \quad rS_+ := \{u \in W; \|u_+\|_p = r \}$$

for $r' \geq r > 0$. Here, we note that the topology of all subsets above are induced by the $W^{1,p}(\Omega)$ norm. We set

$$K(I_{\beta, s}) := \{u \in W; I'_{\beta, s}(u) = 0 \} \quad \text{and} \quad I_{\beta, s}^c := \{u \in W; I_{\beta, s}(u) \leq c \}$$

for $c \in \mathbb{R}$. 


Remark 5 Let \( u \in W^{1,p}(\Omega) \) be a critical point of \( I_{\beta,s} \), namely, \( u \) satisfies the equality
\[
\int_{\Omega} A(x, \nabla u) \nabla \varphi \, dx = (\beta + s) \int_{\Omega} u^{p-1}_{+} \varphi \, dx - \beta \int_{\Omega} u^{p-2}_{-} \varphi \, dx
\]
for every \( \varphi \in W^{1,p}(\Omega) \). Then, because of \( u \in L^{\infty}(\Omega) \) (see Appendix in [14]), we see \( u \in C^{1,\gamma}(\bar{\Omega}) \) (0 < \( \gamma < 1 \)) by the regularity result (cf. [11]).

By Theorem 3 in [4], \( u \) satisfies \((F)_{(\beta+s,\beta)}\) in the distribution sense and the boundary condition
\[
0 = \frac{\partial u}{\partial \nu_{A}} := A(\cdot, \nabla u) \nu = a(\cdot, |\nabla u|) \frac{\partial u}{\partial \nu} \quad \text{in} \quad W^{-1/q,q}(\partial\Omega)
\]
for every \( 1 < q < \infty \) (see [4] for the definition of \( W^{-1/q,q}(\partial\Omega) \)). Since \( u \in C^{1,\gamma}(\bar{\Omega}) \) and \( a(x, y) > 0 \) for every \( y \neq 0 \), \( u \) satisfies the Neumann boundary condition, that is, \( \frac{\partial u}{\partial \nu}(x) = 0 \) for every \( x \in \partial\Omega \).

By Proposition 1 and the remark above (note also that \( A(x, y) \) is odd in \( y \)), it is sufficient to prove the following theorems for the proofs of Theorem 2 and 3.

Theorem 6 ([15]) For every \( s \geq 0 \) and \( R > 0 \), there exists a \( \beta \in [\beta_{0}(s), \beta_{1}(s)] \) such that \( K(I_{\beta,s}) \cap B_{p}(R) \setminus \{0\} \neq \emptyset \).

Theorem 7 ([15]) Let \( s \geq 0 \), \( \varepsilon > 0 \) and \( R_{2} > R_{1} > 0 \) be constants satisfying (3) as in Theorem 3. Then, there exists a \( \beta \in [\beta_{0}(s), \beta_{1}(s) + \varepsilon] \) such that \( K(I_{\beta,s}) \cap D_{p}(R_{1}, R_{2}) \neq \emptyset \).

Roughly speaking, to show the existence of a non-trivial critical point near zero of \( I_{\beta,s} \), we see the variation of the critical groups at 0 for \( I_{\beta,s} \) when a parameter \( \beta \) changes from \( \beta_{0}(s) \) to \( \beta_{1}(s) \). Moreover, it is necessary to construct a flow for which \( B_{p}(R) \) (or \( D_{p}(R_{1}, R_{2}) \)) is invariant. Furthermore, we shall produce suitable paths to see that 0-th reduced homology group is trivial. For this purpose, we need to consider the constrained variational problems. The key point of our proof is to introduce a Finsler manifold \( rS_{+} \).

Finally, we state the result characterizing \( c(s) \) by Morse theory.

Corollary 8 ([15]) Let \( C_{0} = C_{1} = p - 1 \) (that is, the case of \( p \)-Laplace operator). Then, for every \( s \geq 0 \)
\[
c(s) = \min \left\{ \beta > 0 ; \bar{H}_{0}(I_{\beta,s}^{0} \setminus \{0\}) = 0 \right\}
\]
holds, where \( c(s) \) is a function defined by (3) and \( \bar{H}_{*} \) denotes the reduced homology groups.

This corollary means that the mountain pass value \( c(s) \) is attained by some continuous path \( \gamma_{s} \in \Sigma \) for each \( s \geq 0 \).
4 The constrained variational problems

Throughout this section, we fix any $s \geq 0$. Thus, set $I_{\beta,s}(\cdot)=I_{\beta}(\cdot)$ for $\beta \in \mathbb{R}$ to simplify the notation. First, we define $C^1$ functionals $\Phi$ and $\Phi_+$ on $W$ by $\Phi(u):=\frac{1}{p}\|u\|^p_p$ and $\Phi_+(u):=\frac{1}{p}\|u_+\|^p_p$ for $u \in W$. Because $r^p/p$ is a regular value of $\Phi$ and $\Phi_+$ for each $r > 0$, it is well known that the norm of the derivative at $u \in (rS)$ or $u \in (rS_+)$ of the restriction of $I_{\beta}$ or $I_{\beta}^+$ to $rS$ or $rS_+$ is defined as follows:

$$
\|I_{\beta}^t(u)\|_*: = \min \left\{ \|I_{\beta}^t(u) - t\Phi'(u)\|_{W^*}; \ t \in \mathbb{R} \right\} 
= \sup \left\{ \langle I_{\beta}^t(u), v \rangle; \ v \in T_u(rS), \ |v| = 1 \right\},
$$

$$
\|(I_{\beta}^+)^t(u)\|_*: = \min \left\{ \|(I_{\beta}^+)^t(u) - t\Phi_+'(u)\|_{W^*}; \ t \in \mathbb{R} \right\},
$$

(5)

where $T_u(rS)$ denotes the tangent space of $rS$ at $u$, that is, $T_u(rS) = \{v \in W; \int_{\Omega} |u|^p u dx = 0 \}$ (cf. section 5.3 in [17] for (5)). It is known that $rS$ and $rS_+$ are $C^1$ Finsler manifolds (cf. section 27.4 and 27.5 in [9]). Hence, $rS$ and $rS_+$ are locally path connected. Concerning $rS_+$, the following result is proved.

Corollary 9 ([15]) $rS_+$ is path connected for each $r > 0$.

To state our results for constrained variational problems, we set the following open subsets of $rS$ or $rS_+$ as follows:

$$
\mathcal{O}(I_{\beta}, r, b) := \{u \in rS; I_{\beta}(u) < b \}, \quad \mathcal{O}^+(I_{\beta}^+, r, b) := \{u \in rS_+; I_{\beta}^+(u) < b \}
$$

for $r > 0$ and $\beta, b \in \mathbb{R}$. Then, we have the following existence result.

Lemma 10 ([15]) Let $\beta \in \mathbb{R}$, $r > 0$ and $b \in \mathbb{R}$. Then, any nonempty maximal open connected subset of $\mathcal{O}(I_{\beta}, r, b)$ or $\mathcal{O}^+(I_{\beta}^+, r, b)$ contains at least one critical point of $I_{\beta}|_{rS}$ or $I_{\beta}^+|_{rS_+}$, respectively.

The above lemma plays an important role for the proof of constructing a suitable path. It is the developed result from one as in [5] for the manifold $S$.

5 Further results and remaining questions

Finally, the present author would like to take up two questions. First one is "Is the set $\Theta_A$ closed?" where $\Theta_A$ denotes the set of all $(\alpha, \beta)$ such that $(F)_{(\alpha, \beta)}$ has a non-trivial solution. Of course, in the case where $A$ is $(p-1)$-homogeneous in the second variable, we know that the above question is true. Second is "When does $\Theta_A$ contain a similar curve to the first non-trivial curve $C$?" We state the following result related to the first question.

Proposition 11 For $R_2 \geq R_1 > 0$, we set

$$
\Theta_A(R_1, R_2) := \{ (\alpha, \beta) \in \mathbb{R}^2; (F)_{(\alpha, \beta)} \text{ has a solution in } D(R_1, R_2) \},
\Theta_A(R_1, R_2)_p := \{ (\alpha, \beta) \in \mathbb{R}^2; (F)_{(\alpha, \beta)} \text{ has a solution in } D_p(R_1, R_2) \}.
$$

Then, $\Theta_A(R_1, R_2)$ and $\Theta_A(R_1, R_2)_p$ are closed for any $R_2 \geq R_1 > 0$. 
Proof. Let \( \{ (\alpha_n, \beta_n) \} \subset \Theta_A(R_1, R_2)_p \) (resp. \( \Theta_A(R_1, R_2) \)) be a sequence satisfying \( \alpha_n \to \alpha_0 \) and \( \beta_n \to \beta_0 \) as \( n \to \infty \). Because of \( (\alpha_n, \beta_n) \in \Theta_A(R_1, R_2)_p \) (resp. \( \Theta_A(R_1, R_2) \)), there exists a \( u_n \in D_p(R_1, R_2) \) (resp. \( D(R_1, R_2) \)) being a solution of \( (F)_{(\alpha_n, \beta_n)} \), that is, \( -\text{div} A(x, \nabla u_n) = \alpha_n u_n^{p-1} - \beta_n u_n^{p-1} \) in \( \Omega \), \( \partial u_n / \partial \nu = 0 \) on \( \partial \Omega \). Then, we can see that \( \{ u_n \} \) is bounded in \( L^\infty(\Omega) \). Indeed, by taking \( u_n \) as test function, we have

\[
\frac{C_0}{p-1} \| \nabla u_n \|_p^p \leq \int_{\Omega} A(x, \nabla u_n) \nabla u_n \, dx \leq \max\{ |\alpha_n|, |\beta_n| \} \| u_n \|_p^p \leq \max\{ |\alpha_n|, |\beta_n| \} R_2^p
\]

by Remark 4 (iii). This implies the boundedness of \( \| u_n \| \). Moreover, it is known that there exists a positive constant \( C \) independent of \( n \) such that \( \| u_n \|_\infty \leq C \| u_n \| \) because \( u_n \) is a solution of \( (F)_{(\alpha_n, \beta_n)} \) and

\[
|\alpha_n t_+^{p-1} - \beta_n t_-^{p-1}| \leq \max\{ |\alpha_0| + 1, |\beta_0| + 1 \} |t|^{p-1}
\]  

(6)

for every \( t \in \mathbb{R} \) and sufficiently large \( n \) (see Appendix in [14]). Thus, our claim is shown.

Because of the boundedness of \( \| u_n \|_\infty \) and (6), the regularity result in [11] guarantees that there exist \( \gamma \in (0,1) \) and \( M > 0 \) independent of \( n \) such that \( u_n \in C^{1,\gamma}(\overline{\Omega}) \) and \( \| u_n \|_{C^{1,\gamma}(\overline{\Omega})} \leq M \). Since the inclusion of \( C^{1,\gamma}(\overline{\Omega}) \) to \( C^{1}(\overline{\Omega}) \) is compact, we may assume that \( u_n \) converges some \( u_0 \) in \( C^{1}(\overline{\Omega}) \) by choosing a subsequence. As a result, \( u_0 \) is a solution of \( (F)_{(\alpha_0, \beta_0)} \) and \( u_0 \in D_p(R_1, R_2) \) (resp. \( D(R_1, R_2) \)). Thus, \( (\alpha_0, \beta_0) \in \Theta_A(R_1, R_2)_p \) (resp. \( \Theta_A(R_1, R_2) \)) holds, whence our conclusion is shown.

For any \( s \geq 0 \) and \( R_0 \geq R_1 > 0 \) such that \( K(I_{\beta,s}) \cap D_p(R_1, R_2) \neq 0 \) for some \( \beta > 0 \), we can define \( c_A(s, R_1, R_2) \) by

\[
c_A(s, R_1, R_2) := \inf \{ \beta \geq \beta_0(s) ; K(I_{\beta,s}) \cap D_p(R_1, R_2) \neq \emptyset \}.
\]

It follows from Proposition 11 that the above infimum is attained, that is,

\[
c_A(s, R_1, R_2) = \min \{ \beta \geq \beta_0(s) ; K(I_{\beta,s}) \cap D_p(R_1, R_2) \neq \emptyset \}.
\]

Then, the present author would like to consider the problem “What properties does \( c_A(s, R_1, R_2) \) have?” to answer to the second question.

5.1 Asymptotically \((p - 1)\) homogeneous case

In this subsection, we deal with the special case where the map \( A(x, y) \) is asymptotically \((p - 1)\) homogeneous in the following sense:

- \((AH)\) there exist a positive function \( a_\infty \in C^{1}(\overline{\Omega}, \mathbb{R}) \) and a function \( \bar{a}(x, t) \) on \( \overline{\Omega} \times \mathbb{R} \) such that

\[
A(x, y) = a_\infty(x) |y|^{p-2} y + \bar{a}(x, |y|) y \quad \text{for every } x \in \Omega, \ y \in \mathbb{R}^N,
\]

and \( \lim_{t \to +\infty} \frac{\bar{a}(x, t)}{t^{p-2}} = 0 \) uniformly in \( x \in \overline{\Omega} \).
For this weight $a_{\infty}$, we can define the following mountain pass value $c_{a_{\infty}}(s)$ by the same argument as in $c(s)$, namely

$$c_{a_{\infty}}(s) := \inf_{\gamma \in \Sigma} \max_{t \in [0,1]} \tilde{J}_{a_{\infty},s}(\gamma(t)), \quad (7)$$

$$J_{a_{\infty},s}(u) := \int_{\Omega} a_{\infty}(x)|\nabla u|^p \, dx - s \int_{\Omega} u_+^p \, dx, \quad \tilde{J}_{a_{\infty},s} := J_{a_{\infty},s}|_{S}.$$  

It can be proved that the interval $(0, c_{a_{\infty}}(s))$ has no critical values of $\tilde{J}_{a_{\infty},s}$.

Under the hypothesis $(AH)$, we have the following result.

**Proposition 12** Assume $(AH)$. Let $s \geq 0$, $\beta > 0$ and $\{u_n\}$ be a sequence of a solution for $(F)_{(s+\beta,\beta)}$. If $\|u_n\|_p \to \infty$ as $n \to \infty$, then $\beta \geq c_{a_{\infty}}(s)$ holds, where $c_{a_{\infty}}(s)$ is the constant defined by (7).

**Proof.** Here, we give the sketch of the proof. Set $v_n := u_n/\|u_n\|_p$. Then, by the same argument as in [16, Proposition 36], we can prove that $\{v_n\}$ has a subsequence strongly convergent to a solution $v$ of

$$-\text{div}(a_{\infty}(x)|\nabla u|^{p-2}\nabla u) = (s + \beta)u_+^{p-1} - \beta u_-^{p-1} \quad \text{in} \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \partial \Omega,$$

where $a_{\infty}$ is the positive function as in $(AH)$. This means that $v$ is a critical point of $J_{a_{\infty},s}$ with $\beta = \tilde{J}_{a_{\infty},s}(v)$. Because $\beta > 0$ and $(0, c_{a_{\infty}}(s))$ contains no critical values of $\tilde{J}_{a_{\infty},s}$, we obtain $\beta \geq c_{a_{\infty}}(s)$. \(\blacksquare\)

**Corollary 13** Assume $(AH)$ and $s \geq 0$. Then, we have

$$\lim \inf_{R \to \infty} c_A(s, R, \infty) \geq c_{a_{\infty}}(s),$$

where $c_A(s, R, \infty) := \inf \{\beta \geq \beta_0(s); K(I_{\beta,s}) \cap D_p(R, \infty) \neq \emptyset\}$.

**Proof.** By way of contradiction, we prove our assertion. So, we assume that there exists $s \geq 0$ such that $(0 < \beta_0(s) \leq \beta := \lim \inf_{R \to \infty} c_A(s, R, \infty) < c_{a_{\infty}}(s))$. Then, by choosing a subsequence, we can take a sequence $\{u_n\}$ of a solution for $(F)_{(s_0 + \beta_n, \beta_n)}$ with $\|u_n\|_p \to \infty$ and $\beta_n \to \beta$. By the same argument as in [16, Proposition 36], we can show that $\beta$ is a critical value of $\tilde{J}_{a_{\infty},s}$. Therefore, we have a contradiction because of $0 < \beta < c_{a_{\infty}}(s)$. \(\blacksquare\)

The present author expect that in Theorem 3, we can choose $\beta$ close to $c_{a_{\infty}}(s)$ under the additional hypothesis $(AH)$.

**References**


