数理解析研究所講究録
第 1781 巻 2012 年 66-75

KILLING SOME S-SPACES BY A COHERENT SUSLIN TREE

静岡大学理学部 依岡 輝幸 (Yorioka Teruyuki)
Department of Mathematics, Shizuoka University

1. Introduction

A regular space (X,τ) is called hereditarily separable if every subspace is separable, i.e.

$$\forall Y \subseteq X \exists Z \in [Y]^{\leq\aleph_0} \forall U \in \tau \Big(U \cap Y \neq \emptyset \rightarrow U \cap Z \neq \emptyset \Big)$$

and is called hereditarily Lindelöf if every subspace is Lindelöf, i.e.

$$\forall \mathcal{U} \subseteq \tau \exists \mathcal{V} \in [\mathcal{U}]^{\leq\aleph_0} \forall x \in X \Big(x \in \bigcup \mathcal{U} \rightarrow x \in \bigcup \mathcal{V} \Big).$$

Their properties look like dual notions in the sense that points are switched with open sets in their definitions. It was one of famous open problems in general topology whether they coincide. A regular space is called an S-space ([1]) if it is hereditarily separable but not hereditarily Lindelöf, and is called an L-space if it is hereditarily Lindelöf but not hereditarily separable. Stevo Todorčević proved that PFA implies that there are no S-spaces, e.g. [16], and Justin Tatch Moore proved that there are L-spaces [7, 8]. Zoltán Szentmiklóssy proved that $\mathsf{MA}_{\aleph_1}$ implies that there are no compact S-spaces [14]. For the study of S and L spaces, see [16], and [1, 10, 13].

The P-ideal dichotomy is defined by Todorčević. The origin of the P-ideal dichotomy is an analysis of the problem whether every hereditarily separable regular space is Lindelöf (i.e. there are no S-spaces [18, §23], and he proved that PFA implies the P-ideal dichotomy (e.g. [17]) and if the P-ideal dichotomy holds and $\mathfrak{p} > \aleph_1$, then there are no S-spaces [18, §23]. According to [10, §7], Todorčević firstly proved that PFA implies no S-spaces directly, that is, he proved that for each right-separated

Supported by FY 2010 Researcher Exchange Program between JSPS and NSERC, and Grant-in-Aid for Scientific Research (C) 22540124 Japan Society for the Promotion of Science.

[1]Usually, an S-space is denoted by an 'S'-space. However, in this note, we always use S as a (particular) coherent Suslin tree. So we adopt notation an 'S'-space.

Sometime an S-space is defined as a hereditarily separable non-Lindelöf regular space. But our terminology allows us to consider e.g. compact S-space [1]. We note that every compact space is of course Lindelöf.

([2]) hereditarily separable regular space of type ω_1, there is a proper forcing which adds an uncountable discrete subspace. It follows that PFA implies no S-spaces, because every S-space has a right-separated subspace of type ω_1, and a right-separated regular space of type ω_1 is an S-space iff it has no uncountable discrete subspace (e.g. [10, §3]).

In [19], Stevo Todorčević introduced the forcing axiom $\mathsf{PFA}(S)$, which says that there exists a coherent Suslin tree S such that the forcing axiom holds for every proper forcing which preserves S to be Suslin, that is, for every proper forcing $\mathbb{P}$ which preserves S to be Suslin and $\aleph_1$-many dense subsets D_α, $\alpha \in \omega_1$, of $\mathbb{P}$, there exists a filter on $\mathbb{P}$ which intersects all D_α's. $\mathsf{PFA}(S)[S]$ denotes the forcing extension with the coherent Suslin tree S which is a witness of $\mathsf{PFA}(S)$. Since the preservation of a Suslin tree by the proper forcing is closed under countable support iteration (due to Tadatoshi Miyamoto [6]), it is consistent relative to some large cardinal assumption that $\mathsf{PFA}(S)$ holds.

The first appear of such a forcing axiom is in the paper [5] due to Paul B. Larson and Todorčević. In this paper, they introduced the weak version of $\mathsf{PFA}(S)$, called Souslin's Axiom (in which the properness is replaced by the cccness), and under this axiom, the coherent Suslin tree S, which is a witness of the axiom, forces a weak fragment of Martin's Axiom. In [19], it is also proved that under $\mathsf{PFA}(S)$, S forces the open graph dichotomy ([3]) and the P-ideal dichotomy. Namely, many consequences of PFA are satisfied in the extension with S under $\mathsf{PFA}(S)$. On the other hand, many people proved that some consequences from $\diamondsuit$ are satisfied in the extension with a Suslin tree (e.g. [9, Theorem 6.15.]). In particular, the pseudo-intersection number $\mathfrak{p}$ is $\aleph_1$ in the extension with a Suslin tree. In fact, the extension with S under $\mathsf{PFA}(S)$ is designed as a universe which satisfied some consequences of $\diamondsuit$ and PFA simultaneously. By the use of this model, Larson and Todorčević proved that the affirmative answer of Katětov problem is consistent [5].

It is not known whether under $\mathsf{PFA}(S)$, S (which is a witness of $\mathsf{PFA}(S)$) forces that there are no S-spaces. In [19], Todorčević proved that there are no compact S-spaces in the extension with S under $\mathsf{PFA}(S)$. To do this, he develops the theory of compact countably tight spaces in $\mathsf{PFA}(S)[S]$, and proved that under $\mathsf{PFA}(S)$, S forces that every non-Lindelöf subspace of a compact countably tight space has an uncountable discrete subspace [19, 8.6 Theorem]. If fact, he

[2] A space is called right-separated if the set of points can be well-ordered such that every initial segments is open. We note that an uncountable right-separated space is not Lindelöf, and a non-hereditarily Lindelöf space has an uncountable right-separated subspace [10, §3].

[3] This is so called the open coloring axiom [16, §8].

proved that for every S-name for a non-Lindelöf subspace of a compact countably tight space, there is a proper forcing which adds an S-name for an uncountable discrete subspace. In this note, we will show the following.

Proposition 1.1. *Let $\dot{\tau}$ be an S-name for a right-separated hereditarily separable regular topology of type ω_1, and suppose that $\dot{\tau}$ has the following property:*

- $\star$ *For any point $\delta \in \omega_1$, S-name $\dot{U}$ for an open neighborhood of δ, $\alpha \in \omega_1$, $t \in S_\alpha$ and $F \in [S_\alpha]^{<\aleph_0}$, there exists an S-name $\dot{U}'$ for an open neighborhood of δ such that $t \Vdash_S$ " $\dot{U}' \subseteq \dot{U}$ " and for every $s \in F$,*

$$s \Vdash_S \text{ " } \psi_{t,s}(\dot{U}') \text{ is open in } \dot{\tau} \text{ ".}$$

Then $\mathbb{P}$ is proper and preserves S to be Suslin.

It follows from this proposition that under $\mathsf{PFA}(S)$, S forces that every topology on ω_1 generated by a basis in the ground model is not an S-topology. In [11, 12], Mary Ellen Rudin proved that the negation of Suslin Hypothesis (i.e. there exists a Suslin tree) implies the existence of S-spaces, so under $\mathsf{PFA}(S)$, there are S-spaces. By the proposition, we notice that they cannot generate an S-topology in the extension with S.

At last in the introduction, we introduce a coherent Suslin tree. A coherent Suslin tree S consists of functions in $\omega^{<\omega_1}$ and closed under finite modifications. That is,

- for any s and t in S, $s \leq_S t$ iff $s \subseteq t$,
- S is closed under taking initial segments,
- for any s and t in S, the set

$$\{\alpha \in \min\{\mathsf{lv}(s)\,, \mathsf{lv}(t)\}; s(\alpha) \neq t(\alpha)\}$$

  is finite (here, $\mathsf{lv}(s)$ is the length of s, that is, the size of s), and
- for any $s \in S$ and $t \in \omega^{\mathsf{lv}(s)}$, if the set $\{\alpha \in \mathsf{lv}(s)\,; s(\alpha) \neq t(\alpha)\}$ is finite, then $t \in S$ also.

For a countable ordinal α, let S_α be the set of the α-th level nodes, that is, the set of all members of S of domain α, and let $S_{\leq\alpha} := \bigcup_{\beta\leq\alpha} S_\beta$. For $s \in S$, we let

$$S{\restriction}s := \{u \in S; s \leq_S u\}\,.$$

We note that $\diamondsuit$, or adding a Cohen real, builds a coherent Suslin tree. A coherent Suslin tree has canonical commutative isomorphisms.

Let s and t be nodes in S with the same level. Then we define a function $\psi_{s,t}$ from $S{\restriction}s$ into $S{\restriction}t$ such that for each $v \in S{\restriction}s$,

$$\psi_{s,t}(v) := t \cup (v{\restriction}[\mathsf{lv}(s)\,, \mathsf{lv}(v)))$$

(here, $v{\restriction}[\mathsf{lv}(s)\,, \mathsf{lv}(v))$ is the function v restricted to the domain $[\mathsf{lv}(s)\,, \mathsf{lv}(v))$). We note that $\psi_{s,t}$ is an isomorphism, and if s, t, u are nodes in S with the same level, then $\psi_{s,t}$, $\psi_{t,u}$ and $\psi_{s,u}$ commutes. (On a coherent Suslin tree, see e.g. [2, 4].)

Theorem 1.2 (Miyamoto, [6, (1.1) Proposition.]). *For a Suslin tree S and a proper forcing $\mathbb{P}$, $\mathbb{P}$ preserves S to be Suslin iff for any sufficiently large regular cardinal θ, any countable elementary substructure N of $H(\theta)$ which contains $\mathbb{P}$ and S as members, any $(\mathbb{P}, N)$-generic p and any $t \in S$ of level $\omega_1 \cap N$, the pair $\langle p, t\rangle$ is $(\mathbb{P} \times S, N)$-generic.*

2. A proof of Proposition 1.1

Let S be a coherent Suslin tree and $\dot{\tau}$ an S-name for a regular topology on ω_1 such that

$$\Vdash_S \text{“ } (\omega_1, \dot{\tau}, <) \text{ is right-separated and hereditarily separable”},$$

where $<$ denotes the usual order of ordinals. If there exists a proper forcing which preserves S to be Suslin and adds an S-name for an uncountable discrete subset of $(\omega_1, \dot{\tau})$, then under $\mathsf{PFA}(S)$, S (which is a witness of $\mathsf{PFA}(S)$) forces that there are no S-spaces.

We consider a plain forcing notion which adds an S-name for an uncountable discrete subset of $(\omega_1, \dot{\tau})$ as in [16, 8.9. Theorem]. To do this, for each $\alpha \in \omega_1$, since $(\omega_1, \dot{\tau})$ is an S-name for a right-separated regular space, we take an S-name $\dot{U}_\alpha$ such that

$$\Vdash_S \text{“ } \alpha \in \dot{U}_\alpha \in \dot{\tau} \text{ (, i.e. open) and } \mathrm{cl}_{\dot{\tau}}(\dot{U}_\alpha) \cap [\alpha + 1, \omega_1) = \emptyset \text{”}.$$

$\mathbb{P}$ consists of finite functions p such that

- $\mathrm{dom}(p)$ is a finite $\in$-chain of countable elementary submodels of $H(\aleph_2)$ with S, $\dot{\tau}$ and $\left\langle \dot{U}_\alpha ; \alpha \in \omega_1 \right\rangle$,
- for any $M \in \mathrm{dom}(p)$, $p(M) = \langle t_M, \alpha_M\rangle \in (S \setminus M) \times (\omega_1 \setminus M)$ (hence $p(M) \notin M$),
- for any $M \in \mathrm{dom}(p)$ and $\beta \in \omega_1 \cap M$, t_M decides whether $\beta \in \dot{U}_{\alpha_M}$ or not,
- for any $M, M' \in \mathrm{dom}(p)$, if $M \in M'$, then $t_M, \alpha_M \in M'$, and
- for any $M, M' \in \mathrm{dom}(p)$, if $t_M <_S t_{M'}$, then

$$t_{M'} \Vdash_S \text{“ } \alpha_M \notin \dot{U}_{\alpha_{M'}} \text{”},$$

ordered by extensions. If $\mathbb{P}$ is proper and preserves S to be Suslin, then this $\mathbb{P}$ is as desired for a proof of no S-spaces in $\mathsf{PFA}(S)[S]$. However, it is not known whether this is true in general. Now we assume the following property of the S-name $\dot{\tau}$:

- $\star$ For any point $\delta \in \omega_1$, S-name $\dot{U}$ for an open neighborhood of δ, $\alpha \in \omega_1$, $t \in S_\alpha$ and $F \in [S_\alpha]^{<\aleph_0}$, there exists an S-name $\dot{U}'$ for an open neighborhood of δ such that $t \Vdash_S$ " $\dot{U}' \subseteq \dot{U}$ " and for every $s \in F$,

$$s \Vdash_S \text{“ } \psi_{t,s}(\dot{U}') \text{ is open in } \dot{\tau} \text{ ”}.$$

We note that in this property, for an $s \in F$, $s = \psi_{t,s}(t)$, and so it is true that

$$s \Vdash_S \text{“ } \psi_{t,s}(\dot{U}') \text{ is open in } \psi_{t,s}(\dot{\tau}) \text{ ”},$$

but it may happen that

$$s \nVdash_S \text{“ } \psi_{t,s}(\dot{U}') \text{ is open in } \dot{\tau} \text{ ”}.$$

However, for example, if $\dot{\tau}$ is an S-name for a topology generated by an open basis in the ground model, then this is true. So it follows from this proposition that under $\mathsf{PFA}(S)$, S forces that every topology on ω_1 generated by a basis in the ground model is not an S-topology.

In the rest of this section, we prove that $\mathbb{P}$ is proper and preserves S to be Suslin under the assumption of $\dot{\tau}$ above.

Let θ be a large enough regular cardinal, N a countable elementary submodel of $H(\theta)$ such that N contains S, $\dot{\tau}$, $\left\langle \dot{U}_\alpha; \alpha \in \omega_1 \right\rangle$, $\mathbb{P}$ and $H(\aleph_2)$, and $p_0(M) \in \mathbb{P} \cap N$. For each $M \in \mathrm{dom}(p_0)$, we write $p_0 = \langle t^{p_0}_M, \alpha^{p_0}_M \rangle$. Let $N' := N \cap H(\aleph_2)$, which is a countable elementary submodel of $H(\aleph_2)$. We take (arbitrary) $\alpha^{p_1}_{N'} \in \omega_1 \setminus N$, and take $t^{p_1}_{N'} \in S \setminus N$ such that for every $M \in \mathrm{dom}(p_0)$, $t^{p_0}_M$ and $t^{p_1}_{N'} \restriction (\omega_1 \cap N)$ are incomparable in S, and $t^{p_1}_{N'}$ decides whether $\beta \in \dot{U}_{\alpha^{p_1}_{N'}}$ or not for every $\beta \in \omega_1 \cap N$ $(= \omega_1 \cap N')$. Then we define

$$p_1 := p_0 \cup \{ \langle N', \langle t^{p_1}_{N'}, \alpha^{p_1}_{N'} \rangle \rangle \},$$

which is a condition of $\mathbb{P}$ and moreover an extension of p_0. Let $s_1 \in S_{\omega_1 \cap N}$. We show that $\langle p_1, s_1 \rangle$ is $(N, \mathbb{P} \times S)$-generic, which finishes the proof.

Let $\mathcal{D} \in N$ be a dense open subset of $\mathbb{P} \times S$. Let $r \leq_{\mathbb{P}} p_1$ and $u \geq_S s_1$ be such that $\langle r, u \rangle \in \mathcal{D}$. By extending u if necessary, we may assume that for every $M \in \mathrm{dom}(r)$, $\mathsf{lv}(u) \geq \mathsf{lv}(t^r_M)$ holds (where we denote

$r(M) = \langle t^r_M, \alpha^r_M \rangle$). By the coherency of S, we can take $\gamma \in \omega_1 \cap N$ such that for every $M \in \mathrm{dom}(r)$,

$$\{\xi \in \mathsf{lv}(t^r_M) \cap \mathsf{lv}(s_1)\,; t^r_M(\xi) \neq s_1(\xi)\} \subseteq \gamma.$$

We note that

$$\begin{aligned}\{\xi \in \mathsf{lv}(t^r_M) \cap \mathsf{lv}(s_1)\,; t^r_M(\xi) \neq s_1(\xi)\}&\\ = \{\xi \in \mathsf{lv}(t^r_M) \cap \omega_1 \cap N; t^r_M(\xi) \neq u(\xi)\}&.\end{aligned}$$

We enumerate $\mathrm{dom}(r)$ by $\{M^r_i; i \in n\}$ with respect to $\in$-increasing.

For each $v \in S$, we define

$$T_v := \left\{ \langle \alpha^q_M; M \in \mathrm{dom}(q) \setminus \mathrm{dom}(r \cap N) \rangle\,; \right.$$

- $q \in \mathbb{P} \cap N$,
- q is an end-extension of $r \cap N$,
- $\langle q, v \rangle \in \mathcal{D}$,
- $|q| = |r| = n$, and say $\mathrm{dom}(q) = \{M^q_i; i \in n\}$ which is an $\in$-increasing enumeration,
- for every $M \in \mathrm{dom}(q)$, $\mathsf{lv}(t^q_M) \leq \mathsf{lv}(v)$,
- for every $i \in n$, $t^q_{M_i} \restriction \gamma = t^r_{M_i} \restriction \gamma$ and

$$t^q_{M^q_i} \restriction \left[\gamma, \mathsf{lv}\left(t^q_{M^q_i}\right)\right) = v \restriction \left[\gamma, \mathsf{lv}\left(t^q_{M^q_i}\right)\right)$$

$$\left. \text{iff } t^r_{M^r_i} \restriction \left[\gamma, \mathsf{lv}\left(t^r_{M^r_i}\right)\right) = u \restriction \left[\gamma, \mathsf{lv}\left(t^r_{M^r_i}\right)\right) \right\}$$

We note that $\langle T_v; v \in S \rangle$ belongs to the model N, and for any $v, v' \in S$, if $v \leq_S v'$, then $T_v \subseteq T_{v'}$. We consider each T_v as a tree which consists of all initial segments of its members, here we consider that $\langle \alpha^q_M; M \in \mathrm{dom}(q) \setminus \mathrm{dom}(r \cap N) \rangle$ is ordered by the usual order on ordinals. For each $v \in S$, we shrink the tree T_v to the set

$$T_v \setminus \left\{ \sigma \in T_v; \exists \sigma' \in T_v \text{ such that } |\sigma'| = n - |r \cap N| - 1,\ \sigma' \subseteq \sigma \text{ and} \right.$$

$$\left. v \not\Vdash_S \text{“} \left\{\beta \in \omega_1; \exists t \in \dot{G}\left(\sigma' {}^\frown \langle \beta \rangle \in T'_t\right)\right\} \text{ is uncountable”} \right\}.$$

By repeating such a procedure finitely many times for each $v \in S$, we get $T'_v \subseteq T_v$ such that for every $\sigma \in T'_v$ which is not a terminal,

$$v \Vdash_S \text{“} \left\{\beta \in \omega_1; \exists t \in \dot{G}\left(\sigma {}^\frown \langle \beta \rangle \in T'_t\right)\right\} \text{ is uncountable”}.$$

We note again that $\langle T'_v; v \in S\rangle$ also belongs to the model N.

Claim 2.1. *$\langle \alpha^r_M; M \in \mathrm{dom}(r \setminus N)\rangle$ is a cofinal path through T'_u.*

Proof of Claim 2.1. Suppose that $\sigma^\frown\langle\alpha\rangle$ is an initial segment of $\langle \alpha^r_M; M \in \mathrm{dom}(r \setminus N)\rangle$ and $\sigma^\frown\langle\alpha\rangle \in T'_u$. Show that $\sigma \in T'_u$, that is,

$$u \Vdash_S \text{“ } \left\{\beta \in \omega_1; \exists t \in \dot{G}\left(\sigma^\frown\langle\beta\rangle \in T'_t\right)\right\} \text{ is uncountable”.}$$

Let $M \in \mathrm{dom}(r \setminus N)$ be such that $\sigma \in M$ and $\alpha \notin M$. Since $\mathsf{lv}(u) \geq \mathsf{lv}(t^r_M) \geq \omega_1 \cap M$, u is (S, M)-generic.

Suppose that

$$u \nVdash_S \text{“ } \left\{\beta \in \omega_1; \exists t \in \dot{G}\left(\sigma^\frown\langle\beta\rangle \in T'_t\right)\right\} \text{ is uncountable”.}$$

Then some extension of u forces that $\left\{\beta \in \omega_1; \exists t \in \dot{G}\left(\sigma^\frown\langle\beta\rangle \in T'_t\right)\right\}$ is countable. Since such an extension is also (M, S)-generic and the phrase “the set $\left\{\beta \in \omega_1; \exists t \in \dot{G}\left(\sigma^\frown\langle\beta\rangle \in T'_t\right)\right\}$ is countable” can be described in M, there exists $w \in S \cap M$ such that $w \leq_S u$ and

$$w \Vdash_S \text{“ } \left\{\beta \in \omega_1; \exists t \in \dot{G}\left(\sigma^\frown\langle\beta\rangle \in T'_t\right)\right\} \text{ is countable”}$$

([4]). Since S is $\aleph_0$-distributive, there are a countable set Z in N and $w' \in S \cap M$ such that $w \leq_S w' \leq_S u$ and

$$w' \Vdash \text{“ } Z = \left\{\beta \in \omega_1; \exists t \in \dot{G}\left(\sigma^\frown\langle\beta\rangle \in T'_t\right)\right\} \text{ ”.}$$

This is a contradiction because $u \geq_S w'$ and

$$u \Vdash_S \text{“ } \alpha \in \left\{\beta \in \omega_1; \exists t \in \dot{G}\left(\sigma^\frown\langle\beta\rangle \in T'_t\right)\right\} \setminus Z \text{”.}$$

$\dashv$(Claim 2.1)

Therefore, letting $m := |r \setminus N|$, the set

$$\{v \in S; u{\restriction}\gamma \leq_S v \text{ and } T'_v \text{ is of height } m\}$$

is not empty, in particular, contains u as a member. We note that this set is in N, so since u is (S, N)-generic, there exists $s_2 \in S \cap N$ such that $s_2 \leq_S u$ and T'_{s_2} has a cofinal branch of height m. Let

$$a := \left\{i \in n \setminus m; t^r_{M^r_i}{\restriction}\left[\gamma, \mathsf{lv}\left(t^r_{M^r_i}\right)\right) = u{\restriction}\left[\gamma, \mathsf{lv}\left(t^r_{M^r_i}\right)\right)\right\}.$$

[4]If u is (M, S)-generic and $A \in M \cap \mathcal{P}(S)$ contains u as a member, then there exists $w \in A \cap M$ with $w \leq_S u$. Because the set $\{t \in S; (S{\restriction}t) \cap A = \emptyset \text{ or } t \in A\}$ is in M and dense in S. So there exists $w <_S u$ which belongs to this set (we should remember that the set $\{w \in S; w <_S u\}$ is an (S, M)-generic filter). Since $u \in A$, it have to be true that $w \in A$.

If a is empty, then for any cofinal path of T'_{s_2} in N and its witness $p_2 \in \mathbb{P} \cap N$, $\langle p_2, s_2 \rangle \in \mathcal{D} \cap N$ and by the choice of γ, $\langle r \cup p_2, u \rangle$ is a common extension of $\langle r, u \rangle$ and $\langle p_2, s_2 \rangle$ ([5]), so the proof is finished. Therefore the interesting case is that a is not empty.

Suppose that a is not empty. Let $\dot{X}_0$ be an S-name such that

$$\Vdash_S \text{“ } \dot{X}_0 := \Big\{ \beta \in \omega_1 ; \exists t \in \dot{G} \Big(\langle \beta \rangle \in T'_t \Big) \Big\} \text{ ”}.$$

We note that $\dot{X}_0 \in N$ and

$$s_2 \Vdash_S \text{“ } \dot{X}_0 \text{ is uncountable ”}.$$

Since s_1 is (S, N)-generic above s_2, S is $\aleph_0$-distributive and $(\omega_1, \dot{\tau})$ is an S-name for a hereditarily separable space, there are $s_2^0 \in S \cap N$ and a countable set Y_0 in N such that $s_2 \leq_S s_2^0 \leq_S s_1$ ($\leq_S u$) and

$$s_2^0 \Vdash_S \text{“ } Y_0 \subseteq \dot{X}_0 \text{ and } \mathrm{cl}_{\dot{\tau}}(\dot{X}_0) = \mathrm{cl}_{\dot{\tau}}(Y_0) \text{ ”}.$$

For each $i \in n \setminus m$, let

$$b_i := \Big\{ j \in a \setminus m ; t^t_{M^r_j} \restriction \gamma = t^t_{M^r_i} \restriction \gamma \Big\}.$$

We note that for each $j \in b_i$, by the choice of γ, $t^t_{M^r_j} \restriction (\omega_1 \cap N) = t^t_{M^r_i} \restriction (\omega_1 \cap N)$.

Claim 2.2. *There exists $\beta_0 \in Y_0$ such that for every $j \in b_m$,*

$$t^t_{M^r_j} \Vdash_S \text{“ } \beta_0 \notin \dot{U}_{\alpha_{M^r_j}} \text{ ”}.$$

Proof of Claim 2.2. Let $b_i = \{ j_\zeta ; \zeta \in k \}$ and take any $w_\zeta \in S$ such that $t^r_{M^r_{j_\zeta}} \leq_S w_\zeta$, all w_ζ has the same level, and for some $\delta \in \omega_1$ which is larger than $\max_{j \in b_i} \alpha^r_{M^r_j}$, $w_0 \Vdash_S$ “ $\delta \in \mathrm{cl}_{\dot{\tau}}(Y_0)$ ” (we notice that this closure operator is an S-name). By induction on $\zeta \in k$, we take an S-name $\dot{V}_\zeta$ such that

- $w_\zeta \Vdash_S$ “ $\mathrm{cl}_{\dot{\tau}}(\dot{U}_{\alpha^r_{M^r_{j_\zeta}}}) \cap \dot{V}_\zeta = \emptyset$ and $\delta \in \dot{V}_\zeta$ ”,
- for every $\zeta' \in k$, $w_{\zeta'} \Vdash_S$ “ $\psi_{w_\zeta, w_{\zeta'}}(\dot{V}_\zeta)$ is open in $\dot{\tau}$ ”, and
- $w_{\zeta+1} \Vdash_S$ “ $\dot{V}_{\zeta+1} \subseteq \psi_{w_\zeta, w_{\zeta+1}}(\dot{V}_\zeta)$ ”.

This can be done by our special property of $\dot{\tau}$ in the proposition ([6]).

---

[5]Because then for any $M \in \mathrm{dom}(p_2) \setminus \mathrm{dom}(r \cap N)$ and $M' \in \mathrm{dom}(r \setminus N)$, it is true that $\mathsf{lv}(t^{p_2}_M) \leq \mathsf{lv}(s_2) < \omega_1 \cap N \leq \mathsf{lv}(t^r_{M'})$, $t^{p_2}_M \restriction [\gamma, \mathsf{lv}(t^{p_2}_M)) \neq s_2 \restriction [\gamma, \mathsf{lv}(t^{p_2}_M))$, $t^r_{M'} \restriction [\gamma, \omega_1 \cap N) = u \restriction [\gamma, \omega_1 \cap N)$ and $s_2 \leq_S u$, hence it holds that $t^{p_2}_M \not\leq_S t^r_{M'}$.

[6]This is the only point in which the property of $\dot{\tau}$ is used in the proof.

We take $\beta_0 \in Y_0$ such that some extension of w_0 forces that "$\beta_0 \in \psi_{w_{k-1},w_0}(\dot{V}_{k-1})$". Then for every $\zeta \in k$,

$$w_\zeta \Vdash_S \text{"} \beta_0 \in \psi_{w_{k-1},w_\zeta}(\dot{V}_{k-1}) \subseteq \dot{V}_\zeta, \text{ hence } \beta_0 \notin \dot{U}_{\alpha^r_{M^r_{j_\zeta}}} \text{"}.$$

Since $\beta_0 \in Y_0 \subseteq \omega_1 \cap N \subseteq M^r_{j_\zeta}$ and $t^r_{M^r_{j_\zeta}} \leq_S w_\zeta$, by the definition of conditions of $\mathbb{P}$, for every $j \in b_i$,

$$t^r_{M^r_j} \Vdash_S \text{"} \beta_0 \notin \dot{U}_{\alpha^r_{M^r_j}} \text{"},$$

which is what we want. ⊣(Claim 2.2)

By repeating this procedure, we can take $s_3 \in S \cap N$ and a cofinal branch $\langle \beta_i; i \in n-m \rangle$ through T'_{s_3} such that $s_2 \leq_S s_3 \leq_S s_1$ ($\leq_S u$) and for every $i \in n-m$ and $j \in b_{m+i}$,

$$t^r_{M^r_j} \Vdash_S \text{"} \beta_i \notin \dot{U}_{\alpha^r_{M^r_j}} \text{"}.$$

Since $\langle \beta_i; i \in n-m \rangle \in T'_{s_3} \cap N \subseteq T_{s_3} \cap N$, there exists $p_3 \in \mathbb{P} \cap N$ which is its witness. Then $\langle p_3, s_3 \rangle \in \mathcal{D} \cap N$ and $\langle r \cup p_3, u \rangle$ is a common extension of $\langle r, u \rangle$ and $\langle p_3, s_3 \rangle$, which finishes the proof.

Acknowledgements

The author was participating in a Researcher Exchange Program between JSPS and NSERC in 2010 hosted by Franklin D. Tall form April to August 2010. Then the author take a personal lecture from Franklin D. Tall about his paper [15], and take a lecture from Paul B. Larson about his preprint [3]. I would like to thank them. And I would like to thank Stevo Todorčević for encouraging me about a study of $\mathsf{PFA}(S)[S]$.

I am thankful all the members of the Toronto set theory group, and to the staff of the Fields Institute. A special thanks to Franklin D. Tall for his hospitality during the visit.

References

[1] I. Juhász. *A survey of S- and L-spaces.* Topology, Vol. II (Proc. Fourth Colloq., Budapest, 1978), pp. 675–688, Colloq. Math. Soc. János Bolyai, 23, North-Holland, Amsterdam-New York, 1980.

[2] B. König. *Trees, Games and Reflections*, Ph.D. thesis (2002) at the Ludwig-Maximilians-Universität München.

[3] P. B. Larson. *Notes on Todorcevic's Erice lectures on forcing with a coherent Souslin tree.* Preprint.

[4] P. Larson and S. Todorčević. *Chain conditions in maximal models.* Fund. Math. 168 (2001), no. 1, 77–104.

[5] P. Larson and S. Todorčević. *Katětov's problem.* Trans. Amer. Math. Soc. 354 (2002), no. 5, 1783–1791.

[6] T. Miyamoto. *ω_1-Souslin trees under countable support iterations.* Fund. Math. 142 (1993), no. 3, 257–261.

[7] J. Moore. *A solution to the L space problem.* J. Amer. Math. Soc. 19 (2006), no. 3, 717–736.

[8] J. Moore. *An L space with a d-separable square.* Topology Appl. 155 (2008), no. 4, 304–307.

[9] J. Moore, M. Hrušák and M. Džamonja. *Parametrized $\diamondsuit$ principles.* Transactions of American Mathematical Society, 356 (2004), no. 6, 2281–2306.

[10] J. Roitman. *Basic S and L.* Handbook of set-theoretic topology, 295–326, North-Holland, Amsterdam, 1984.

[11] M. E. Rudin. *A normal hereditarily separable non-Lindelöf space.* Illinois J. Math. 16 (1972), 621–626.

[12] M. E. Rudin. *A non-normal hereditarily-separable space.* Illinois J. Math. 18 (1974), 481–483.

[13] M. E. Rudin. *S and L spaces.* Surveys in general topology, pp. 431–444, Academic Press, New York-London-Toronto, Ont., 1980.

[14] Z. Szentmiklóssy. *S-spaces and L-spaces under Martin's axiom.* Topology, Vol. II (Proc. Fourth Colloq., Budapest, 1978), pp. 1139–1145, Colloq. Math. Soc. János Bolyai, 23, North-Holland, Amsterdam-New York, 1980.

[15] F. D. Tall. *PFA(S)[S]: more mutually consistent topological consequences of PFA and $V = L$*, Canad. J. Math. to appear.

[16] S. Todorčević. *Partition Problems in Topology.* volume 84 of *Contemporary mathematics.* American Mathematical Society, Providence, Rhode Island, 1989.

[17] S. Todorčdvić. *A dichotomy for P-ideals of countable sets.* Fund. Math. 166 (2000), no. 3, 251–267.

[18] S. Todorčević. *Combinatorial dichotomies in set theory*, Bull. Symbolic Logic 17 (2011), no. 1, 1–72.

[19] S. Todorčević. *Forcing with a coherent Suslin tree*, preprint.

DEPARTMENT OF MATHEMATICS, SHIZUOKA UNIVERSITY, OHYA 836, SHIZUOKA, 422-8529, JAPAN.

E-mail address: `styorio@ipc.shizuoka.ac.jp`