On local connectivity of boundaries of CAT(0) spaces

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We introduce (non-)local connectivity of boundaries of CAT(0) spaces and hyperbolic CAT(0) spaces.

Definitions and basic properties of CAT(0) spaces, hyperbolic spaces and their boundaries are found in [3], [10] and [11].

A metric space $X$ is said to be proper if every closed metric ball is compact. A group $G$ is called a \textit{CAT(0) group} if $G$ acts geometrically (i.e. properly and cocompactly by isometries) on some CAT(0) space. It is known that a CAT(0) space on which a CAT(0) group acts geometrically is proper. A boundary $\partial X$ of a CAT(0) space $X$ on which a CAT(0) group $G$ acts geometrically is called a \textit{boundary} of the CAT(0) group $G$. It is known that in general a CAT(0) group $G$ does not determine its boundary [5]. If $G$ is a hyperbolic group then $G$ determines its boundary up to homeomorphisms (cf. [3], [10] and [11]).

The following problems are open.

\textbf{Problem.} When is a boundary of a CAT(0) group (non-)locally connected?

\textbf{Problem.} If $G$ is a hyperbolic CAT(0) group whose boundary is connected then is the boundary locally connected?

\textbf{Problem.} For a CAT(0) group $G$ and CAT(0) spaces $X$ and $Y$ on which $G$ acts geometrically, is it the case that the boundary $\partial X$ is locally connected if and only if the boundary $\partial Y$ is locally connected?

There is a research on (local) $n$-connectivity of boundaries of hyperbolic Coxeter groups by A. N. Dranishnikov in [8], and there are some research on (non-)local connectivity of boundaries of CAT(0) groups and Coxeter groups by M. Mihalik, K. Ruane and S. Tschantz in [17] and [18].
The purpose of this paper is to introduce sufficient conditions of

(i) a hyperbolic CAT(0) group whose boundary is locally $n$-connected by using reflections, and

(ii) a CAT(0) space whose boundary is non-locally connected by using a hyperbolic isometry and a reflection.

Local $n$-connectivity of boundaries of hyperbolic CAT(0) spaces

We define a reflection of a geodesic space as follows: An isometry $r$ of a geodesic space $X$ is called a reflection of $X$, if

1. $r^2$ is the identity of $X$,
2. $X \setminus F_r$ has exactly two convex connected components $X^+_r$ and $X^-_r$ and
3. $rX^+_r = X^-_r$,

where $F_r$ is the fixed-points set of $r$. We note that “reflections” in this paper need not satisfy the condition (4) $\text{Int} F_r = \emptyset$ in [15].

**Theorem 1.** Suppose that a group $G$ acts geometrically (i.e. properly and cocompactly by isometries) on a hyperbolic CAT(0) space $X$. If

1. there exist some reflections $r_1, \ldots, r_n \in G$ of $X$ such that $G = \langle r_1, \ldots, r_n \rangle$
2. the boundary $\partial X$ of $X$ is $n$-connected,

then the boundary $\partial X$ is locally $n$-connected.

**Corollary 2.** Suppose that a hyperbolic Coxeter group $W$ acts geometrically on a hyperbolic CAT(0) space $X$. If the boundary $\partial X$ of $X$ is $n$-connected then $\partial X$ is locally $n$-connected.

From [8], we also obtain a corollary.

**Corollary 3.** Let $(W, S)$ be a hyperbolic Coxeter system and let $L = L(W, S)$ be the nerve of the Coxeter system $(W, S)$. For any hyperbolic CAT(0) space $X$ on which the hyperbolic Coxeter group $W$ acts geometrically, the following statements are equivalent:

1. $L$ is connected and $L - \sigma$ is connected for any simplex $\sigma$ of $L$,
2. $\check{H}^0(\partial X) = 0$ where $\check{H}^*$ denote the reduced Čech cohomology,
(iii) the boundary $\partial X$ of $X$ is connected, and
(iv) the boundary $\partial X$ of $X$ is locally connected.

Here the following problems are open.

**Problem.** If $G$ is a hyperbolic CAT(0) group whose boundary is $n$-connected then is the boundary locally $n$-connected?

**Problem.** For a non-elementary hyperbolic Coxeter group $W$ on which acts geometrically on a CAT(0) space $X$, is it the case that the following statements are equivalent?

(i) $\check{H}^i(\partial X) = 0$ for any $0 \leq i \leq n$,
(ii) $L$ is $n$-connected and $L - \sigma$ is $n$-connected for any simplex $\sigma$ of $L$,
(iii) the boundary $\partial X$ of $X$ is $n$-connected, and
(iv) the boundary $\partial X$ of $X$ is locally $n$-connected.

**Non-local connectivity of boundaries of CAT(0) spaces**

Let $X$ be a proper CAT(0) space and let $g$ be an isometry of $X$. The translation length of $g$ is the number $|g| := \inf\{d(x, gx) \mid x \in X\}$, and the minimal set of $g$ is defined as $\text{Min}(g) = \{x \in X \mid d(x, gx) = |g|\}$. An isometry $g$ of $X$ is said to be hyperbolic, if $\text{Min}(g) \neq \emptyset$ and $|g| > 0$ (cf. [3, p.229]). For a hyperbolic isometry $g$ of a proper CAT(0) space $X$, $g^\infty$ is the limit point of the boundary $\partial X$ to which the sequence $\{g^ix_0\}_i$ converges, where $x_0$ is a point of $X$. Here we note that the limit point $g^\infty$ is not depend on the point $x_0$.

A CAT(0) space $X$ is said to be almost geodesically complete, if there exists a constant $M > 0$ such that for each pair of points $x, y \in X$, there is a geodesic ray $\zeta : [0, \infty) \to X$ such that $\zeta(0) = x$ and $\zeta$ passes within $M$ of $y$. In [9, Corollary 3], R. Geoghegan and P. Ontaneda have proved that every non-compact cocompact proper CAT(0) space is almost geodesically complete. Here a CAT(0) space $X$ is said to be cocompact, if some group acts cocompactly by isometries on $X$.

On non-local connectivity of CAT(0) spaces, we obtained the following.

**Theorem 4.** Let $X$ be a proper and almost geodesically complete CAT(0) space, let $g$ be a hyperbolic isometry of $X$ and let $r$ be a reflection of $X$. If
\begin{enumerate}
    \item $g^\infty \not\in \partial F_r$,
    \item $g(\partial F_r) \subset \partial F_r$ and
    \item $\operatorname{Min}(g) \cap F_r = \emptyset$,
\end{enumerate}
then the boundary $\partial X$ of $X$ is non-locally connected.

Here we note that the action of the group $G$ on the CAT(0) space $X$ in Theorem 4 need not be proper and cocompact.

The conditions in Theorem 4 are rather technical. We introduce some remarks.

First, every CAT(0) space on which some group acts geometrically (i.e. properly and cocompactly by isometries) is proper ([3, p.132]) and almost geodesically complete ([9], [20]).

Also, in [22], Ruane has proved that $\partial \operatorname{Min}(g)$ is the fixed-points set of $g$ in $\partial X$, i.e.,

$$\partial \operatorname{Min}(g) = \{ \alpha \in \partial X \mid g\alpha = \alpha \}.$$ 

Hence, for example, if $\partial F_r \subset \partial \operatorname{Min}(g)$ then $g(\partial F_r) = \partial F_r$ and the condition (2) in Theorem 4 holds.

As an example of CAT(0) spaces on which some reflections act, there is the Davis complex of a Coxeter system. A Coxeter system $(W, S)$ determines the Davis complex $\Sigma(W, S)$ which is a CAT(0) space ([6], [19]). Then the Coxeter group $W$ acts geometrically on $\Sigma(W, S)$ and each $s \in S$ is a reflection of $\Sigma(W, S)$.

\section*{References}


