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# Mapping class groups of non-compact surfaces

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## 1. Introduction

The mapping class groups of surfaces of finite type have been studied by many authors (cf. [7, 9]). In this article we define a kind of mapping class groups for non-compact surfaces and discuss some properties of them. This article is a preliminary report on the joint work with Taras Banakh [3].

Suppose $S$ is a surface (i.e., a connected separable metrizable topological 2-manifold possibly with boundary). Let $\mathcal{H}_c(S)$ denote the group of homeomorphisms of $S$ with compact support and $\mathcal{H}_c(S)_1$ denote the normal subgroup of $\mathcal{H}_c(S)$ consisting of $h \in \mathcal{H}_c(S)$ which is isotopic to the identity $\mathrm{id}_S$ by an isotopy with compact support. Then, we can define the mapping class group of $S$ as the quotient group

$$\mathcal{M}_c(S) = \mathcal{H}_c(S)/\mathcal{H}_c(S)_1.$$

When $S$ is compact, the mapping class group $\mathcal{M}_c(S)$ reduces to the ordinary one based upon the full homeomorphism groups and free boundaries (cf. [9]). When $S$ is a surface of finite type, the definition of $\mathcal{M}_c(S)$ differs from the usual one. In [7] all boundaries are fixed and the punctures correspond to free boundaries, while in our definition, all boundaries are free and the punctures correspond to fixed boundaries by the compact support condition.

The group $\mathcal{M}_c(S)$ is a countable group, but not necessarily finitely generated in general. If we consider the full homeomorphism group of $S$, then the associated mapping class group has the uncountable cardinal in general. In this sense, $\mathcal{M}_c(S)$ is the "smallest" mapping class group defined for a general surface $S$ and we expect that the study of this group help a further study of larger mapping class groups of $S$.

The definition of the group $\mathcal{M}_c(S)$ is also justified by the topological properties of the homeomorphism group endowed with the Whitney topology. For any topological manifold $M$, the full homeomorphism group $\mathcal{H}(M)^w$ endowed with the Whitney topology is a topological group and the subgroup $\mathcal{H}_c(M)^w$ is locally contractible. It is also seen that $\mathcal{H}_c(M)_1^w$ is the identity path-component of both the full group $\mathcal{H}(M)^w$ and the subgroup $\mathcal{H}_c(M)^w$ ([1]). Moreover, for any surface $S$, if $S$ is compact, the group $\mathcal{H}_c(S)^w$ is a topological $l_2$-manifold and $\mathcal{H}_c(S)_1^w$ is homeomorphic to $l_2$ except for several cases, and if $S$

is non-compact, then the group $\mathcal{H}_c(S)^w$ is a topological $l_2 \times \mathbb{R}^\infty$-manifold (in fact, it is homeomorphic to $l_2 \times \mathbb{R}^\infty$ or $l_2 \times \mathbb{R}^\infty \times \mathbb{N}$) and $\mathcal{H}_c(S)_1^w$ is always homeomorphic to $l_2 \times \mathbb{R}^\infty$ ([1, 2]). Here, $l_2$ is the separable Hilbert space, $\mathbb{R}^\infty$ is the standard direct limit of the Euclidean spaces $\mathbb{R}^n$ ($n \geq 1$) and $\mathbb{N}$ is the discrete space of natural numbers. In any case, we have the standard expression

$$\mathcal{M}_c(S) = \pi_0(\mathcal{H}_c(S)^w, \mathrm{id}_S).$$

Our main goal is formally expressed as follows:

**Problem 1.1.** Describe the (geometric) group structure of the group $\mathcal{M}_c(S)$.

As the 1st step to this problem, we consider the following questions.

**Question 1.1.**

(1) When is $\mathcal{M}_c(S)$ trivial ?
(2) When is $\mathcal{M}_c(S)$ finitely generated ?

Since the group $\mathcal{M}_c(S)$ is not necessarily finitely generated, we need some notions in the geometric group theory which are applicable to general countable groups (extending some basic notions for finitely generated groups).

Suppose $G$ is an abstract countable group. The $\mathbb{Z}$*-rank* of $G$ is defined by

$$r_{\mathbb{Z}}(G) = \sup\{n \geq 0 \mid \mathbb{Z}^n \hookrightarrow G\}.$$

Note that $r_{\mathbb{Z}}(G) = 0$ means that $G$ is a torsion group. One of the most important invariants in the geometric group theory is the notion of asymptotic dimension (cf. [6, 8]).

**Definition 1.1.** The *asymptotic dimension* $\mathrm{asdim}(G)$ of a countable group $G$ is defined by the following conditions:

(1) $\mathrm{asdim}(G) \leq n \iff$ for each finite subset $F \subset G$ there is a cover $\mathcal{U}$ of the group $G$ such that $\bigcup_{U \in \mathcal{U}} U^{-1}U$ is finite and for each $x \in G$ the set $xF$ meets at most $n+1$ sets $U \in \mathcal{U}$.
(2) $\mathrm{asdim}(G) = n \iff \mathrm{asdim}(G) \leq n,\ \mathrm{asdim}(G) \not\leq n-1$
(3) $\mathrm{asdim}(G) = \infty \iff \mathrm{asdim}(G) \not\leq n$ for any $n \geq 0$

It is known that $\mathrm{asdim}(\mathbb{Z}^n) = n$ ($n \geq 0$), $r_{\mathbb{Z}}(G) \leq \mathrm{asdim}(G)$ and if $G$ is abelian, then $r_{\mathbb{Z}}(G) = \mathrm{asdim}(G)$.

Using these notions, we can answer Question 2.1 as follows [2, 3].

**Theorem 1.1.** For a non-compact surface $S$, the following conditions are equivalent:

(1) $\mathcal{M}_c(S)$ is trivial;
(2) $r_{\mathbb{Z}}(\mathcal{M}_c(S)) = 0$;

(3) $\operatorname{asdim}(\mathcal{M}_c(S)) = 0$;

(4) $S$ is homeomorphic to $N \setminus K$, where $N$ is the disk, the annulus, or the Möbius band, and $K$ is a non-empty compact subset of a boundary circle of $N$.

**Theorem 1.2.** For a surface $S$, the following conditions are equivalent:

(1) $\mathcal{M}_c(S)$ is finitely generated;

(2) $\mathcal{M}_c(S)$ is finitely presented;

(3) $r_{\mathbb{Z}}(\mathcal{M}_c(S)) < \infty$;

(4) $\operatorname{asdim}(\mathcal{M}_c(S)) < \infty$;

(5) $S$ is of semi-finite type.

Moreover, if $S$ is not of semi-finite type, then $\mathcal{M}_c(S)$ includes a free abelian group of infinite rank.

We have to explain the notion of semi-finite type. A surface $S$ is said to be of *finite type* if $S$ is homeomorphic to $N \setminus F$ for some compact surface $N$ and a finite subset $F \subset N \setminus \partial N$. Here $\partial N$ denotes the boundary of the 2-manifold $N$. We need a small modification of this notion.

**Definition 1.2.** We say that a surface $S$ is of *semi-finite type* if $S$ is homeomorphic to $N \setminus (K \cup F)$ for some compact surface $N$, a compact subset $K \subset \partial N$ and a finite subset $F \subset N \setminus \partial N$.

This condition is justified by the next propoeition [3].

**Proposition 1.1.** For a surface $S$, the following conditions are equivalent:

(1) $\pi_1(S)$ is finitely presented;

(2) $H_1(S; \mathbb{Z})$ is finitely generated;

(3) $S$ is of semi-finite type.

M. Bestvina, K. Bromberg, K. Fujiwara [4] have obtained the following important result.

**Theorem 1.3.** $\operatorname{asdim} \mathcal{M}_c(S) < \infty$ for any surface $S$ of semi-finite type.

## 2. Relation among the mapping class groups of a non-compact surface and its compact subsurfaces

In this section we study a relation between the mapping class group $\mathcal{M}_c(S)$ of a surface $S$ and those of compact subsurfaces of $S$.

Suppose $S$ is a surface. We assume that a 2-submanifold $N$ of $S$ is a closed subset of $S$ and $\mathrm{Fr}_S N$ is transversal to the boundary $\partial S$ so that $\mathrm{Fr}_S N$ is a proper 1-submanifold of $S$. Thus $\check{N} := S \setminus \mathrm{Int}_S N$ is also a 2-submanifold of $S$. Here, $\mathrm{Int}_S N$ and $\mathrm{Fr}_S N$ denote the topological interior and frontier of $N$ in $S$ respectively. The symbol $\mathcal{C}(N)$ denotes the collection of connected components of $N$. Let $\mathcal{N}(S)$ denote the collection of all compact connected 2-submanifolds $N$ of $S$ such that each $L \in \mathcal{C}(\check{N})$ is non-compact.

For a subset $A$ of $S$, let $\mathcal{H}_c(S, A) = \{h \in \mathcal{H}_c(S) : h|_A = \mathrm{id}_A\}$ and $\mathcal{H}_c(S, A)_1$ denote the normal subgroup of $\mathcal{H}_c(S, A)$ consisting of $h \in \mathcal{H}_c(S, A)$ which is isotopic to $\mathrm{id}_S$ by an isotopy rel $A$ with compact support. The mapping class group of $S$ relative to $A$ is defined by

$$\mathcal{M}_c(S, A) = \mathcal{H}_c(S, A)/\mathcal{H}_c(S, A)_1.$$

For $N \in \mathcal{N}(S)$, the restriction map $\mathcal{H}_c(S, \check{N}) \to \mathcal{H}_c(N, \mathrm{Fr}_S N)$ induces an isomorphism

$$\mathcal{M}_c(S, \check{N}) \cong \mathcal{M}_c(N, \mathrm{Fr}_S N).$$

If $N_1, N_2 \in \mathcal{N}(S)$ and $N_1 \subset N_2$, then the inclusion maps $\mathcal{H}_c(S, \check{N}_1) \subset \mathcal{H}_c(S, \check{N}_2) \subset \mathcal{H}_c(S)$ induce homomorphisms

$$(*) \qquad \begin{tikzcd} \mathcal{M}(S, \check{N}_1) \arrow[rr, "\varphi_{N_1,N_2}"] \arrow[dr, "\varphi_{N_1}"'] & & \mathcal{M}(S, \check{N}_2) \arrow[dl, "\varphi_{N_2}"] \\ & \mathcal{M}_c(S). & \end{tikzcd} \qquad (N_1 \subset N_2 \ \text{ in } \ \mathcal{N}(S)).$$

The class $\mathcal{N}(S)$ is directed by the inclusion and the diagram $(*)$ forms a direct system of groups in the upper side and a morphism from this direct system to the group $\mathcal{M}_c(S)$.

**Proposition 2.1.** Suppose $S$ is a non-compact surface.

(1) The diagram $(*)$ is a direct limit in the category of groups.

(2) The homomorphism $\varphi_N : \mathcal{M}_c(S, \check{N}) \to \mathcal{M}_c(S)$ is injective for any $N \in \mathcal{N}(S)$ (cf. [10]).

We can also consider the pure mapping class group $\mathcal{PM}_c(S)$ of a surface $S$. Suppose $S$ is a surface and $A$ is a subset of $S$. The group $\mathcal{H}_c(S, A)$ includes the normal subgroup

$$\mathcal{H}_c^\partial(S, A) = \{h \in \mathcal{H}_c(S, A) : h(C) = C \text{ for each circle component } C \text{ of } \partial S\}.$$

Since $\mathcal{H}_c(S, A)_1 \subset \mathcal{H}_c^\partial(S, A)$, we obtain the *pure mapping class group*

$$\mathcal{PM}_c(S, A) = \mathcal{H}_c^\partial(S, A)/\mathcal{H}_c(S, A)_1 \ \lhd \ \mathcal{M}_c(S, A).$$

Note that each $h \in \mathcal{H}_c(S, A)$ preserves any line component of $\partial S$ since $h$ has compact support. This subgroup fits into the short exact sequence

$$1 \to \mathcal{PM}_c(S) \subset \mathcal{M}_c(S) \xrightarrow{\lambda} \Sigma_f(\mathcal{C}_c(\partial S)) \to 1,$$

where $\Sigma_f(\mathcal{C}_c(\partial S))$ is the group of finite permutations of the set $\mathcal{C}_c(\partial S)$ of circle components of $\partial S$ and for $[h] \in \mathcal{M}_c(S)$ the induced permutation $\lambda([h])$ is defined by $\lambda([h])(C) = h(C)$ for $C \in \mathcal{C}_c(\partial S)$.

The diagram $(*)$ in Proposition 2.1 restricts to the diagram of subgroups:

$$(**) \qquad \begin{tikzcd} \mathcal{PM}(S,\check{N}_1) \arrow[rr, "\varphi_{N_1,N_2}"] \arrow[dr, "\varphi_{N_1}"'] & & \mathcal{PM}(S,\check{N}_2) \arrow[dl, "\varphi_{N_2}"] \\ & \mathcal{PM}_c(S) & \end{tikzcd} \qquad (N_1 \subset N_2 \text{ in } \mathcal{N}(S)).$$

**Proposition 2.2.** Suppose $S$ is a non-compact surface.

(1) The diagram $(**)$ is a direct limit in the category of groups.

(2) The homomorphism $\varphi_N : \mathcal{PM}_c(S,\check{N}) \to \mathcal{PM}_c(S)$ is injective for any $N \in \mathcal{N}(S)$

Note that if $M$ is an $\mathcal{M}_c(S)$-module, then for each $N \in \mathcal{N}(S)$ the homomorphism $\varphi_N : \mathcal{M}_c(S,\check{N}) \to \mathcal{M}_c(S)$ induces an $\mathcal{M}(S,\check{N})$-module structure on $M$. The same observation applies to the pure mapping class groups. Since the group homology commutes with direct limits (cf. [5, Ch. V, Section 5, Exercises 3]), we have the following conclusions.

**Proposition 2.3.**

(1) $H_*(\mathcal{M}_c(S), M) = \operatorname*{dirlim}_{N \in \mathcal{N}(S)} H_*(\mathcal{M}_c(S,\check{N}), M)$ for any $\mathcal{M}_c(S)$-module $M$.

(2) $H_*(\mathcal{PM}_c(S), M) = \operatorname*{dirlim}_{N \in \mathcal{N}(S)} H_*(\mathcal{PM}_c(S,\check{N}), M)$ for any $\mathcal{PM}_c(S)$-module $M$.

Proposition 2.3 enables us to deduce from the stability results on the homology of the mapping class groups of compact surfaces the corresponding conclusions on the mapping class groups of non-compact surfaces.

## 3. Surfaces of semi-finite type

In this final section we provide with a criterion which detects surfaces of semi-finite type and some related lemmas which lead to Theorem 1.1 (4) and Theorem 1.2 (5).

Consider the following conditions on a surface $S$:

$(\#_1)$ $S$ includes no handle,

$(\#_2)$ $S$ does not include two boundary circles of $S$,

$(\#_3)$ $S$ is not separated by a circle $C$ in $\operatorname{Int} S$ into two non-compact connected 2-submanifolds.

**Proposition 3.1.** A surface $S$ is of semi-finite type iff there exists $N \in \mathcal{N}(S)$ such that each $L \in \mathcal{C}(\check{N})$ satisfies the conditions $(\#_1)$, $(\#_2)$, $(\#_3)$.

The symbols $\mathbb{D}$ and $\mathbb{A}$ denote the disk and the annulus respectively.

**Lemma 3.1.** Suppose $S$ is a non-compact orientable surface and satisfies the conditions $(\#_1)$, $(\#_2)$, $(\#_3)$.

(1) If $S$ has no boundary circle, then $S \approx \mathbb{D} \setminus K$ for a non-empty compact subset $K$ of $\partial\mathbb{D}$.

(2) If $S$ has a boundary circle, then $S \approx \mathbb{A} \setminus K$ for a non-empty compact subset $K$ of one boundary circle of $\mathbb{A}$.

Suppose $S$ is a non-compact surface and $N \in \mathcal{N}(S)$.

**Lemma 3.2.** *$N$ is a retract of $S$.*

**Lemma 3.3.** Suppose $L \in \mathcal{C}(\check{N})$. In each of the cases (1) $\sim$ (3) below, the Dehn twist $h$ along the circle $C$ satisfies the condition: $[h^n] \in \varphi_N(\mathcal{M}_c(S, \check{N}))$ iff $n = 0$.

(1) $C$ is a meridian of a handle (or a Klein bottle with a hole) $H$ in $L$.

(2) Suppose $L$ contains two boundary circles $C_1, C_2$ of $S$ (or a Möbius band and a boundary circle $C_1$ of $S$). Then we can connect them by an arc and thicken their union so to obtain a disk with two holes (or a Möbius band with a hole) $H$ with a circle frontier $C = \mathrm{Fr}_S H$.

(3) $L$ is separated by a circle $C$ in Int $L$ into two non-compact connected 2-submanifolds $L_1$ and $L_2$.

**Lemma 3.4.** Suppose the homomorphism $\varphi_N : \mathcal{M}_c(S, \check{N}) \to \mathcal{M}_c(S)$ is surjective. Then, each $L \in \mathcal{C}(\check{N})$ has the following properties:

(1) $L$ satisfies the conditions $(\#_1)$, $(\#_2)$, $(\#_3)$.

(2) $L$ contains neither "two disjoint Möbius bands" nor "a Möbius band and a bounary circle of $S$".

(3) if either $S$ is orientable or $N$ is non-orientable, then

  (i) $L$ meets exactly one boundary circle of $N$ and

  (ii) $N$ meets at most one boundary circle of $M$ for any connected 2-submanifold $M$ of $L$.

In particular, $S$ has only finitely many boundary circles and $S \setminus K$ is orientable for some compact subset $K$ of $S$.

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