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Whitney preserving maps onto dendrites

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Abstract

This note is a survey on Whitney preserving maps. In particular we introduce next results.

(1) Let X be a continuum such that X contains a dense arc component and let D be a dendrite with a closed set of branch points. If $f : X \to D$ is a Whitney preserving map, then f is a homeomorphism.

(2) For each dendrite D' with a dense set of branch points there exist a continuum X' containing a dense arc component and a Whitney preseiving map $f' : X' \to D'$ such that f' is not a homeomorphism.

1 Introduction

In this note, all spaces are separable metrizable spaces and maps are continuous. We denote the interval [0, 1] by I. A compact metric space is called a *compactum* and *continuum* means a connected compactum. If X is a continuum $C(X)$ denotes the space of all subcontinua of X with the topology generated by the Hausdorff metric.

In this note we study maps called *Whitney preserving maps*. If $f : X \to Y$ is a map between continua, then define a map $\hat{f} : C(X) \to C(Y)$ by $\hat{f}(A) = f(A)$ for each $A \in C(X)$. A map $f : X \to Y$ is called a Whitney preserving map if there exist Whitney maps (see p105 of [12]) $\mu : C(X) \to I$ and $\nu : C(Y) \to I$ such that for each $s \in [0, \mu(X)]$, $\hat{f}(\mu^{-1}(s)) = \nu^{-1}(t)$ for some $t \in [0, \nu(Y)]$. In this case, we say that f is μ, ν-*Whitney preserving*. Let $f : X \to Y$ be a μ, ν-Whitney preserving map. Then it is easy to see that if $s, s' \in [0, \mu(X)]$ and $t, t' \in [0, \nu(Y)]$ satisfy $s \leq s'$, $\hat{f}(\mu^{-1}(s)) = \nu^{-1}(t)$ and $\hat{f}(\mu^{-1}(s')) = \nu^{-1}(t')$, then $t \leq t'$.

The notion of a Whitney preserving map is introduced by Espinoza (see [1] and [2]). In this article we study these maps.

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[2]Key words and phrases: Whitney preserving map. terminal continuum, hyperspace.

2 Whitney preserving maps onto dendrites

At first we give an example of a Whitney preserving map.

Example 2.1 (Example 2 of [1]) let $f : [0, \pi] \to S^1$ be a map defined by $f(t) = \mathrm{e}^{4ti}$. Then f is Whitney preserving. But f is not a homeomorphism.

In [1] Espinoza proved the following result.

Theorem 2.2 (Theorem 16 of [1]) *Let X be a continuum such that X contains a dense arc component. If $f : X \to I$ is a Whitney preserving map, then f is a homeomorphism.*

A Peano continuum is called a *dendrite* if it contains no simple closed curve. Let D be a dendrite. A point $e \in D$ is called an *end point of D* if $D \setminus \{e\}$ is connected. A point $b \in D$ is called a *branch point of D* if there exists a neighbourhood U of b such that for each neighbourhood V of b with $V \subset U$, $|\mathrm{Bd}(V)| \geq 3$. We denote the set of all end points in D by $E(D)$. Also we denote the set of all branch points of D by $B(D)$.

Recently the author proved the next theorem ([9], see also [8]).

Theorem 2.3 *Let X be a continuum such that X contains a dense arc component and let D be a dendrite with the closed set of branch points. Then a map $f : X \to D$ is a Whitney preserving map if and only if f is a homeomorphism.*

Corollary 2.4 *Let X be a continuum such that X contains a dense arc component and let T be a tree. Then a map $f : X \to T$ is a Whitney preserving map if and only if f is a homeomorphism.*

Generally, Theorem 2.3 does not hold when D is a graph by Example 2.1.

Remark. For every 1-dimensional continuum M there exists a 1-dimensional continuum $\hat{M}$ (other than M) such that there is a Whitney preserving map $f : \hat{M} \to M$ by Theorem 2.9 of [2].

It is natural to ask that whether Theorem 2.3 holds when D is any dendrite. In fact, this does not hold.

If X and Y be compacta, then $C(X,Y)$ denotes the set of all continuous maps from X to Y endowed with sup metric. Also $S(X,Y)$ denotes the set of all surjective maps in $C(X,Y)$. If $v, w \in X$, then we denote the set $\{f \in C(X,Y) | f(v) = f(w)\}$ by $C_{(v,w)}(X,Y)$. Also we denote the set $\{f \in$

$S(X,Y)|f(v)=f(w)\}$ by $S_{(v,w)}(X,Y)$. It is easy to see that $C_{(v,w)}(X,Y)$ and $S_{(v,w)}(X,Y)$ are closed subsets of $C(X,Y)$. Let $N \subset X$. Then we denote the set $\{f \in C(X,Y)|f^{-1}(f(x)) = \{x\}$ for each $x \in N\}$ by $A_N(X,Y)$. If N is a one point set $\{a\}$, then we denote the set $A_N(X,Y)$ by $A_a(X,Y)$. Let $x \in X$ and $r > 0$. Then we denote the set $\{f \in C(X,Y)|\text{diam}\, f^{-1}(f(x)) < r\}$ by $A_{x,r}(X,Y)$.

Finally, we denote the identity map on a space S by id_S.

A surjective map e from I onto a graph G is called an *Eulerian path* if e satisfies; (i) $e(0) = e(1)$, (ii) $|\{y \in G|e^{-1}(y)$ is nondegenerate $\}| < \infty$ and (iii) each fiber of e is finite. In [3] Espinoza and Illanes proved the next result.

Theorem 2.5 ([3]) *For each graph G which admits an Eulerian path, there exist a continuum X_G containing a dense arc component and a Whitney preseiving map $f : X_G \to G$ such that f is not a homeomorphism.*

In [9] the author showed that this result holds when G is a superdendrite. A dendrite D is called a *superdendrite* if $E(D)$ is dense in D. It is known that a dendrite D is a superdendrite if and only if $B(D)$ is dense in D

Lemma 2.6 ([9]) *Let X be a compactum and let D be a superdendrite. If v, w and a are points in X such that $a \notin \{v,w\}$, then $C_{(v,w)}(X,D) \cap A_a(X,D)$ is a dense G_δ-subset in $C_{(v,w)}(X,D)$.*

Lemma 2.7 ([9]) *Let X be a nondegenerate continuum and let D be a superdendrite. If v, w and a are points in X such that $a \notin \{v,w\}$, then $S_{(v,w)}(X,D) \cap A_a(X,D)$ is a dense G_δ-subset in $S_{(v,w)}(X,D)$.*

By Lemma 2.7 and Baire Category Theorem, we get the next corollary.

Corollary 2.8 ([9]) *Let X be a nondegenerate continuum, N a countable subset of X and D a superdendrite. If v, w are points in X such that $N \cap \{v,w\} = \emptyset$, then $S_{(v,w)}(X,D) \cap A_N(X,D)$ is a dense G_δ-subset in $S_{(v,w)}(X,D)$.*

By using Corollary 2.8 and arguments in [3], we can prove the following result.

Theorem 2.9 ([9]) *For each superdendrite D, there exist a continuum X_D containing a dense arc component and a Whitney preseiving map $f : X_D \to D$ such that f is not a homeomorphism.*

Recently the author generalized this result.

Theorem 2.10 ([10]) *For each 1-dimensional locally connected continuum without free arcs P, there exist a continuum X_P containing a dense arc component and a Whitney preseiving map $f : X_P \to P$ such that f is not a homeomorphism.*

Theorem 2.11 ([10]) *For each $n \geq 2$ and an n-dimensional manifold M, there exist a continuum X_M containing a dense arc component and a Whitney preseiving map $f : X_M \to M$ such that f is not a homeomorphism.*

3 Other topics related to Whitney preserving maps

A subcontinuum T of a continuum X is *terminal*, if every subcontinuum of X which intersects both T and its complement must contain T.

Now we give a notation. If $f : X \to Y$ is a map, let $\mathcal{A}_f = \{f^{-1}(y)|y \in Y\}$ and $\mathcal{A}'_f = \{C|C$ is a component of a fiber of $f\}$.

Let $f : X \to Y$ be a Whitney preserving map. Then $\mathcal{A}_f$ need not be a continuous decomposition of X. For example let $f : [0, \pi] \to S^1$ be a map defined by $f(t) = \mathrm{e}^{4t\mathrm{i}}$. Then f is Whitney preserving (cf. Example 2 of [1]). But f is not an open map.

In [7] the author proved next results.

Proposition 3.1 ([7]) *Let $f : X \to Y$ be a μ, ν-Whitney preserving map. Then $\mathcal{A}'_f$ is a continuous decomposition of X and each element of $\mathcal{A}'_f$ is terminal in X.*

A map $f : X \to Y$ between continua is called an *atomic map* if $f^{-1}(f(A)) = A$ for each $A \in C(X)$ such that $f(A)$ is nondegenerate. It is known that a map f of a continuum X onto a continuum Y is atomic if and only if every fiber of f is a terminal continuum of X.

A map $f : X \to Y$ between compacta is called a *Krasinkiewicz map* if any continuum in X either contains a component of a fiber of f or is contained in a fiber of f (cf. [6]). These maps are related to Whitney preserving maps.

Proposition 3.2 ([7]) *Let $f : X \to Y$ be a map such that $\mathcal{A}'_f$ does not contain a one point set. Then the following conditions are equivalent.*

(1) *$\mathcal{A}'_f$ is a continuous decomposition of X and each element of $\mathcal{A}'_f$ is terminal in X.*

(2) *$\mathcal{A}'_f$ is a continuous decomposition of X and f is a Krasinkiewicz map.*

By using Proposition 3.2 the author proved next results.

Theorem 3.3 ([8]) *Let X be a continuum such that X contains a dense arc component. If $f : X \to f(X)$ is a Whitney preserving map such that f is not a constant map, then f is a light map.*

Theorem 3.4 ([7]) *Let X, Y be continua and let $f : X \to Y$ be a monotone map such that $f^{-1}(y)$ is a nondegenerate continuum in X. Then the following conditions are equivalent.*

(1) *f is an open map and each fiber of f is terminal in X.*
(2) *f is an open Krasinkiewicz map.*
(3) *f is a Whitney preserving map.*

As an application of Theorem 3.4 we obtain next results.

Theorem 3.5 ([8]) *There exists a 1-dimensional continuum $T \subset I^2$, a Whitney map $\mu : C(T) \to I$ and $s_0, s_1 \in I$ such that*

(1) $0 < s_0 < s_1 < \mu(T)$,
(2) $\dim \mu^{-1}(s) = 1$ *for each* $s \in [0, s_0)$,
(3) $\dim \mu^{-1}(s_0) = 2$, *and*
(4) $\dim \mu^{-1}(s) = \infty$ *for each* $s \in (s_0, s_1]$.

Theorem 3.6 ([8]) *There exists a 1-dimensional continuum $T \subset I^2$ such that*

(1) $\dim C(T) = \infty$, *and*
(2) *for each Whitney map $w : C(T) \to I$ there exists $a_0 \in (0, w(T))$ such that* $\dim w^{-1}(s) = 1$ *for each* $s \in [0, a_0]$.

At last we give some results related to Whitney preserving maps.

Proposition 3.7 ([8]) *Let $f : X \to Y$ be a monotone μ, ν-Whitney preserving map and let $s_0 = \max\{s \in I \mid \hat{f}(\mu^{-1}(s)) = \nu^{-1}(0)\}$. Then $\hat{f}|_{\mu^{-1}([s_0, \mu(X)])} : \mu^{-1}([s_0, \mu(X)]) \to C(Y)$ is a homeomorphism. Hence $\mu^{-1}(s)$ is homeomorphic to $\hat{f}(\mu^{-1}(s))$ for each $s \in [s_0, \mu(X)]$.*

A topological property P is said to be a *Whitney property* provided that if a continuum X has property P, so does $\mu^{-1}(t)$ for each Whitney map μ for $C(X)$ and for each $t \in [0, \mu(X)]$. As a corollary of Proposition 3.7 we get the next result.

Corollary 3.8 ([8]) *Let $f : X \to Y$ be a monotone Whitney preserving map. If X has a topological property P which is a Whitney property, then so does Y.*

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