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DECAY STRUCTURE OF REGULARITY-LOSS TYPE FOR SYMMETRIC HYPERBOLIC SYSTEMS WITH RELAXATION

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1. INTRODUCTION

This article is a survey of the paper [17]. In this article, we consider the Cauchy problem for the first-order linear symmetric hyperbolic system of equations with relaxation:

$$A^0u_t + \sum_{j=1}^{n} A^j u_{x_j} + Lu = 0 \tag{1.1}$$

with

$$u|_{t=0} = u_0. \tag{1.2}$$

Here $u = u(t,x) \in \mathbb{R}^m$ over $t > 0$, $x \in \mathbb{R}^n$ is an unknown function, $u_0 = u_0(x) \in \mathbb{R}^m$ over $x \in \mathbb{R}^n$ is a given function, and A^j $(j = 0, 1, \cdots, n)$ and L are $m \times m$ real constant matrices, where integers $m \geq 1$, $n \geq 1$ denote dimensions. Throughout this paper, it is assumed that all A^j $(j = 0, 1, \cdots, n)$ are symmetric, A^0 is positive definite and L is nonnegative definite with a nontrivial kernel. Notice that L is not necessarily symmetric. For this general linear degenerately dissipative system it is interesting to study its decay structure under additional conditions on the coefficient matrices and further investigate the corresponding time-decay property of solutions to the Cauchy problem.

When the degenerate relaxation matrix L is symmetric, Umeda-Kawashima-Shizuta [20] proved the large-time asymptotic stability of solutions for a class of equations of hyperbolic-parabolic type with applications to both electro-magneto-fluid dynamics and magnetohydrodynamics. The key idea in [20] and the later generalized work [15] that first introduced the so-called Kawashima-Shizuta condition is to design the compensating matrix to capture the dissipation of systems over the degenerate kernel space of L. The typical feature of the time-decay property of solutions established in those work is that the high frequency part decays exponentially while the low frequency part decays polynomially with the rate of the heat kernel.

For clearness and for later use let us precisely recall the results in [20, 15] mentioned above. Taking the Fourier transform of (1.1) with respect to x yields

$$A^0\hat{u}_t + i|\xi|A(\omega)\hat{u} + L\hat{u} = 0. \tag{1.3}$$

Here and hereafter, $\xi \in \mathbb{R}^n$ denotes the Fourier variable, $\omega = \xi/|\xi| \in S^{n-1}$ is the unit vector whenever $\xi \neq 0$, and we define $A(\omega) := \sum_{j=1}^{n} A^j\omega_j$ with $\omega = (\omega_1, \cdots, \omega_n) \in S^{n-1}$. The following two conditions for the coefficient matrices are needed:

Condition (A)$_0$: A^0 is real symmetric and positive definite, A^j $(j = 1, \cdots, n)$ are real symmetric, and L is real symmetric and nonnegative definite with the nontrivial kernel.

Namely, we assume that

$$(A^j)^T = A^j \quad \text{for} \quad j = 0, 1, \cdots, n, \qquad L^T = L,$$

$$A^0 > 0, \quad L \geq 0 \quad \text{on} \quad \mathbb{C}^m, \qquad \mathrm{Ker}(L) \neq \{0\}.$$

Here and in the sequel, the superscript T stands for the transpose of matrices, and given a matrix X, $X \geq 0$ means that $\mathrm{Re}\,\langle Xz, z\rangle \geq 0$ for any $z \in \mathbb{C}^m$, while $X > 0$ means that $\mathrm{Re}\,\langle Xz, z\rangle > 0$ for any $z \in \mathbb{C}^m$ with $z \neq 0$, where $\langle \cdot, \cdot \rangle$ denotes the standard complex inner product in $\mathbb{C}^m$. Also, for simplicity of notations, given a real matrix X, we use X_1 and X_2 to denote the symmetric and skew-symmetric parts of X, respectively, namely, $X_1 = (X + X^T)/2$ and $X_2 = (X - X^T)/2$.

Condition (K): There is a real compensating matrix $K(\omega) \in C^\infty(S^{n-1})$ with the following properties: $K(-\omega) = -K(\omega)$, $(K(\omega)A^0)^T = -K(\omega)A^0$ and

$$(K(\omega)A(\omega))_1 > 0 \quad \text{on} \quad \mathrm{Ker}(L) \tag{1.4}$$

for each $\omega \in S^{n-1}$.

Remark 1. *Under the condition* (A)$_0$, *the positivity* (1.4) *in the condition* (K) *holds if and only if*

$$\alpha(K(\omega)A(\omega))_1 + L > 0 \quad \text{on} \quad \mathbb{C}^m$$

for each $\omega \in S^{n-1}$, *where* α *is a suitably small positive constant.*

Under the conditions (A)$_0$ and (K) one has:

Theorem 1.1 (Decay property of the standard type ([20, 15])). *Assume that both the conditions* (A)$_0$ *and* (K) *hold. Then the Fourier image* $\hat{u}$ *of the solution* u *to the Cauchy problem* (1.1)-(1.2) *satisfies the pointwise estimate:*

$$|\hat{u}(t,\xi)| \leq Ce^{-c\rho(\xi)t}|\hat{u}_0(\xi)|, \tag{1.5}$$

where $\rho(\xi) := |\xi|^2/(1 + |\xi|^2)$. *Furthermore, let* $s \geq 0$ *be an integer and suppose that the initial data* u_0 *belong to* $H^s \cap L^1$. *Then the solution* u *satisfies the decay estimate:*

$$\|\partial_x^k u(t)\|_{L^2} \leq C(1+t)^{-n/4-k/2}\|u_0\|_{L^1} + Ce^{-ct}\|\partial_x^k u_0\|_{L^2} \tag{1.6}$$

for $k \leq s$. *Here* C *and* c *are positive constants.*

Unfortunately, when the degenerate relaxation matrix L is not symmetric, Theorem 1.1 can not be applied any longer. In fact, this is the case for some concrete systems, for example, the Timoshenko system [6, 7] and the Euler-Maxwell system [3, 19, 18], where the linearized relaxation matrix L indeed has a nonzero skew-symmetric part while it was still proved that solutions decay in time in some different way that we shall point out later on. Therefore, our purpose of this article is to formulate some new structural conditions in order to extend Theorem 1.1 to the general system (1.1) when L is not symmetric, which can include both the Timoshenko system and the Euler-Maxwell system.

More precisely, we introduce a constant matrix S which satisfies some properties in Condition (S) in Section 2. When the relaxation matrix L is not symmetric, we have a partial positivity on $\mathrm{Ker}(L_1)^\perp$ only. In this situation, we try finding a real compensating matrix S to make a positivity on $\mathrm{Ker}(L)^\perp$. Then, employing further the condition (K), we can construct a full positivity on $\mathbb{C}^m$. As the consequence, we can show the following weaker estimates:

$$|\hat{u}(t,\xi)| \le Ce^{-c\eta(\xi)t}|\hat{u}_0(\xi)|, \tag{1.7}$$

where $\eta(\xi) := |\xi|^2/(1+|\xi|^2)^2$, and

$$\|\partial_x^k u(t)\|_{L^2} \le C(1+t)^{-n/4-k/2}\|u_0\|_{L^1} + C(1+t)^{-\ell/2}\|\partial_x^{k+\ell}u_0\|_{L^2} \tag{1.8}$$

for $k+\ell \le s$. See Theorem 2.1 for the details. We note that these estimates (1.7) and (1.8) are weaker than (1.5) and (1.6), respectively. In particular, the decay estimate (1.7) is of the regularity-loss type.

Similar decay properties of the regularity-loss type have been recently observed for several interesting systems. We refer the reader to [6, 7, 12] (cf. [1, 14]) for the dissipative Timoshenko system, [3, 19, 18] for the Euler-Maxwell system, [5, 8] for a hyperbolic-elliptic system in radiation gas dynamics, [9, 10, 11, 13, 16] for a dissipative plate equation, and [2, 4] for the Vlasov-Maxwell-Boltzmann system.

Notations. For a nonnegative integer k, we denote by ∂_x^k the totality of all the k-th order derivatives with respect to $x = (x_1, \cdots, x_n)$.

Let $1 \le p \le \infty$. Then $L^p = L^p(\mathbb{R}^n)$ denotes the usual Lebesgue space over $\mathbb{R}^n$ with the norm $\|\cdot\|_{L^p}$. For a nonnegative integer s, $H^s = H^s(\mathbb{R}^n)$ denotes the s-th order Sobolev space over $\mathbb{R}^n$ in the L^2 sense, equipped with the norm $\|\cdot\|_{H^s}$. We note that $L^2 = H^0$.

Finally, in this paper, we use C or c to denote various positive constants without confusion.

2. Main results

In this section we shall introduce new structural conditions to investigate the decay structure and time-decay property for the system (1.1) when L is not necessarily symmetric, and then state under those conditions the main results which are the generalization of Theorem 1.1. Our structural conditions are formulated as follows.

Condition (A): A^0 is real symmetric and positive definite, A^j ($j = 1, \cdots, n$) are real symmetric, while L is not necessarily real symmetric but is nonnegative definite with the nontrivial kernel.

Namely, it is assumed that

$$(A^j)^T = A^j \quad \text{for} \quad j = 0, 1, \cdots, n,$$
$$A^0 > 0, \quad L \ge 0 \quad \text{on} \quad \mathbb{C}^m, \qquad \mathrm{Ker}(L) \ne \{0\}.$$

Condition (S): There is a real constant matrix S with the following properties: $(SA^0)^T = SA^0$ and

$$(SL)_1 + L_1 \ge 0 \quad \text{on} \quad \mathbb{C}^m, \qquad \mathrm{Ker}((SL)_1 + L_1) = \mathrm{Ker}(L). \tag{2.1}$$

Remark 2. *Under the conditions* (A) *and* (S), *the positivity* (1.4) *in the condition* (K) *holds if and only if*

$$\alpha(K(\omega)A(\omega))_1 + (SL)_1 + L_1 > 0 \quad on \quad \mathbb{C}^m \tag{2.2}$$

for each $\omega \in S^{n-1}$, *where* α *is a suitably small positive constant.*

When we use the condition (S), we additionally assume either the condition $(\mathrm{S})_1$ or $(\mathrm{S})_2$ below.

Condition $(\mathrm{S})_1$: For each $\omega \in S^{n-1}$, the matrix S in the condition (S) satisfies

$$i(SA(\omega))_2 \geq 0 \quad \text{on} \quad \mathrm{Ker}(L_1).$$

Condition $(\mathrm{S})_2$: For each $\omega \in S^{n-1}$, the matrix S in the condition (S) satisfies

$$i(SA(\omega))_2 \geq 0 \quad \text{on} \quad \mathbb{C}^m. \tag{2.3}$$

Under the above structural conditions, we can state our main results on the decay property for the system (1.1). The first one uses the condition $(\mathrm{S})_1$.

Theorem 2.1 (Decay property of the regularity-loss type). *Assume that the conditions* (A), (S), $(\mathrm{S})_1$ *and* (K) *hold. Then the Fourier image* $\hat{u}$ *of the solution* u *to the Cauchy problem* (1.1)-(1.2) *satisfies the pointwise estimate:*

$$|\hat{u}(t,\xi)| \leq Ce^{-c\eta(\xi)t}|\hat{u}_0(\xi)|, \tag{2.4}$$

where $\eta(\xi) := |\xi|^2/(1+|\xi|^2)^2$. *Moreover, let* $s \geq 0$ *be an integer and suppose that the initial data* u_0 *belong to* $H^s \cap L^1$. *Then the solution* u *satisfies the decay estimate:*

$$\|\partial_x^k u(t)\|_{L^2} \leq C(1+t)^{-n/4-k/2}\|u_0\|_{L^1} + C(1+t)^{-\ell/2}\|\partial_x^{k+\ell}u_0\|_{L^2} \tag{2.5}$$

for $k+\ell \leq s$. *Here* C *and* c *are positive constants.*

Remark 3. *The decay estimate* (2.5) *is of the regularity-loss type because we have the decay rate* $(1+t)^{-\ell/2}$ *only by assuming the additional* l*-th order regularity on the initial data.*

Our second main result uses the stronger condition $(\mathrm{S})_2$ instead of $(\mathrm{S})_1$ and gives the decay estimate of the standard type.

Theorem 2.2 (Decay property of the standard type). *If the condition* $(\mathrm{S})_1$ *in Theorem 2.1 is replaced by the stronger condition* $(\mathrm{S})_2$, *then the pointwise estimate* (2.4) *and the decay estimate* (2.5) *in Theorem 2.1 can be refined as* (1.5) *and* (1.6) *in Theorem 1.1, respectively.*

It should be pointed out that Theorem 2.2 is a direct extension of Theorem 1.1 and is applicable to the system (1.1) with a non-symmetric relaxation matrix L. More specifically, we have:

Claim 2.3. *Theorem 1.1 holds as a corollary of Theorem 2.2. In other words, when* L *is real symmetric, Theorem 2.2 is reduced to Theorem 1.1.*

In fact, when L is real symmetric, the condition (A) is reduced to $(\mathrm{A})_0$. Moreover, in this case, we have $L = L_1$ so that the conditions (S) and $(\mathrm{S})_2$ are satisfied trivially with $S = 0$. This shows that Theorem 2.2 implies Theorem 1.1.

In Theorems 2.1 and 2.2, the decay estimates (2.5) and (1.6) can be derived by using the pointwise estimates (2.4) and (1.5), respectively. Before closing this section, we prove this fact.

Proof of the decay estimates in Theorems 2.1 and 2.2. We first prove (2.5) in Theorem 2.1. By virtue of the Plancherel theorem and the pointwise estimate (2.4), we obtain

$$\|\partial_x^k u(t)\|_{L^2}^2 = \int_{\mathbb{R}^n} |\xi|^{2k} |\hat{u}(t,\xi)|^2 d\xi \le C \int_{\mathbb{R}^n} |\xi|^{2k} e^{-c\eta(\xi)t} |\hat{u}_0(\xi)|^2 d\xi. \tag{2.6}$$

We divide the integral on the right-hand side of (2.6) into two parts I_1 and I_2 according to the low frequency region $|\xi| \le 1$ and the high frequency region $|\xi| \ge 1$, respectively. Since $\eta(\xi) \ge c|\xi|^2$ for $|\xi| \le 1$, we see that

$$I_1 \le C \sup_{|\xi|\le 1} |\hat{u}_0(\xi)|^2 \int_{|\xi|\le 1} |\xi|^{2k} e^{-c|\xi|^2 t} d\xi \le C(1+t)^{-n/2-k} \|u_0\|_{L^1}^2.$$

On the other hand, we have $\eta(\xi) \ge c|\xi|^{-2}$ in the region $|\xi| \ge 1$. Consequently, we obtain

$$I_2 \le C \sup_{|\xi|\ge 1} \frac{e^{-ct/|\xi|^2}}{|\xi|^{2\ell}} \int_{|\xi|\ge 1} |\xi|^{2(k+\ell)} |\hat{u}_0(\xi)|^2 d\xi \le C(1+t)^{-\ell} \|\partial_x^{k+\ell} u_0\|_{L^2}^2.$$

Therefore, substituting these estimates into (2.6), we get the desired decay estimate (2.5).

To prove (1.6) in Theorem 2.2, we make use of the pointwise estimate (1.5). Since $\rho(\xi) \ge c|\xi|^2$ for $|\xi| \le 1$ and $\rho(\xi) \ge c$ for $|\xi| \ge 1$, a similar computation as in the proof of (2.5) yields the decay estimate (1.6). Thus we got the desired decay estimates and this completes the proof. □

3. Energy method in the Fourier space

The aim of this section is to prove the pointwise estimates stated in Theorems 2.1 and 2.2 by employing the energy method in the Fourier space.

Proof of the pointwise estimate in Theorem 2.1. We derive the energy estimate for the system (1.3) in the Fourier space. Taking the inner product of (1.3) with $\hat{u}$, we have

$$\langle A^0 \hat{u}_t, \hat{u}\rangle + i|\xi|\langle A(\omega)\hat{u}, \hat{u}\rangle + \langle L\hat{u}, \hat{u}\rangle = 0.$$

Taking the real part, we get the basic energy equality

$$\frac{1}{2}\frac{d}{dt} E_0 + \langle L_1 \hat{u}, \hat{u}\rangle = 0, \tag{3.1}$$

where $E_0 := \langle A^0 \hat{u}, \hat{u}\rangle$. Next we create the dissipation terms. For this purpose, we multiply (1.3) by the matrix S in the condition (S) and take the inner product with $\hat{u}$. This yields

$$\langle SA^0 \hat{u}_t, \hat{u}\rangle + i|\xi|\langle SA(\omega)\hat{u}, \hat{u}\rangle + \langle SL\hat{u}, \hat{u}\rangle = 0.$$

Taking the real part of this equality, we get

$$\frac{1}{2}\frac{d}{dt}E_1 + |\xi|\langle i(SA(\omega))_2\hat{u},\hat{u}\rangle + \langle (SL)_1\hat{u},\hat{u}\rangle = 0, \tag{3.2}$$

where $E_1 := \langle SA^0\hat{u},\hat{u}\rangle$. Moreover, letting $K(\omega)$ be the compensating matrix in the condition (K), we multiply (1.3) by $-i|\xi|K(\omega)$ and take the inner product with $\hat{u}$. Then we have

$$-i|\xi|\langle K(\omega)A^0\hat{u}_t,\hat{u}\rangle + |\xi|^2\langle K(\omega)A(\omega)\hat{u},\hat{u}\rangle - i|\xi|\langle K(\omega)L\hat{u},\hat{u}\rangle = 0.$$

Taking the real part of the above equality, we obtain

$$-\frac{1}{2}|\xi|\frac{d}{dt}E_2 + |\xi|^2\langle (K(\omega)A(\omega))_1\hat{u},\hat{u}\rangle - |\xi|\langle i(K(\omega)L)_2\hat{u},\hat{u}\rangle = 0, \tag{3.3}$$

where $E_2 := \langle iK(\omega)A^0\hat{u},\hat{u}\rangle$.

Now we combine the energy equalities (3.1), (3.2) and (3.3). First, letting α be the positive number in Remark 2, we multiply (3.2) and (3.3) by $1+|\xi|^2$ and $\alpha_2\alpha$, respectively, and add these two equalities, where α_2 is a positive constant to be determined. This yields

$$\begin{aligned}&\frac{1}{2}(1+|\xi|^2)\frac{d}{dt}\mathcal{E} + (1+|\xi|^2)\langle (SL)_1\hat{u},\hat{u}\rangle + \alpha_2|\xi|^2\langle \alpha(K(\omega)A(\omega))_1\hat{u},\hat{u}\rangle\\ &= -|\xi|(1+|\xi|^2)\langle i(SA(\omega))_2\hat{u},\hat{u}\rangle + \alpha_2|\xi|\langle i\alpha(K(\omega)L)_2\hat{u},\hat{u}\rangle,\end{aligned} \tag{3.4}$$

where $\mathcal{E} := E_1 - \frac{\alpha_2|\xi|}{1+|\xi|^2}\alpha E_2$. Furthermore, we multiply (3.1) and (3.4) by $(1+|\xi|^2)^2$ and α_1, respectively, and add the resulting two equalities, where α_1 is a positive constant to be determined. This yields

$$\begin{aligned}&\frac{1}{2}(1+|\xi|^2)^2\frac{d}{dt}\Big(E_0 + \frac{\alpha_1}{1+|\xi|^2}\mathcal{E}\Big)\\ &+ (1+|\xi|^2)^2\langle L_1\hat{u},\hat{u}\rangle + \alpha_1\big\{(1+|\xi|^2)\langle (SL)_1\hat{u},\hat{u}\rangle + \alpha_2|\xi|^2\langle \alpha(K(\omega)A(\omega))_1\hat{u},\hat{u}\rangle\big\}\\ &= \alpha_1\big\{-|\xi|(1+|\xi|^2)\langle i(SA(\omega))_2\hat{u},\hat{u}\rangle + \alpha_2|\xi|\langle i\alpha(K(\omega)L)_2\hat{u},\hat{u}\rangle\big\}.\end{aligned} \tag{3.5}$$

We write the equality (3.5) as

$$\frac{1}{2}\frac{d}{dt}E + D_1 + D_2 = G, \tag{3.6}$$

where we define E, D_1, D_2 and G as

$$\begin{aligned}E &:= E_0 + \frac{\alpha_1}{1+|\xi|^2}\mathcal{E} = E_0 + \frac{\alpha_1}{1+|\xi|^2}\Big(E_1 + \frac{\alpha_2|\xi|}{1+|\xi|^2}\alpha E_2\Big),\\ (1+|\xi|^2)^2D_1 &:= (1+|\xi|^2)^2\langle L_1\hat{u},\hat{u}\rangle\\ &\quad + \alpha_1\big\{(1+|\xi|^2)\langle (SL)_1\hat{u},\hat{u}\rangle + \alpha_2|\xi|^2\langle \alpha(K(\omega)A(\omega))_1\hat{u},\hat{u}\rangle\big\},\\ (1+|\xi|^2)^2D_2 &:= \alpha_1|\xi|(1+|\xi|^2)\langle i(SA(\omega))_2P_1\hat{u},P_1\hat{u}\rangle,\\ (1+|\xi|^2)^2G &:= \alpha_1\alpha_2|\xi|\langle i\alpha(K(\omega)L)_2\hat{u},\hat{u}\rangle\\ &\quad - \alpha_1|\xi|(1+|\xi|^2)\big\{\langle i(SA(\omega))_2\hat{u},\hat{u}\rangle - \langle i(SA(\omega))_2P_1\hat{u},P_1\hat{u}\rangle\big\}.\end{aligned}$$

We estimate each term in (3.6). Because of the positivity of A^0, for suitably small $\alpha_1 > 0$ and $\alpha_2 > 0$, we see that

$$c_0|\hat{u}|^2 \le E \le C_0|\hat{u}|^2, \tag{3.7}$$

where c_0 and C_0 are positive constants not depending on (α_1, α_2). On the other hand, we can rewrite D_1 as

$$\begin{aligned}(1+|\xi|^2)^2 D_1 &= \alpha_1\alpha_2|\xi|^2\langle(\alpha(K(\omega)A(\omega))_1 + (SL)_1 + L_1)\hat{u}, \hat{u}\rangle \\ &\quad + \alpha_1((1+|\xi|^2) - \alpha_2|\xi|^2)\langle((SL)_1 + L_1)\hat{u}, \hat{u}\rangle \\ &\quad + (1+|\xi|^2)((1+|\xi|^2) - \alpha_1)\langle L_1\hat{u}, \hat{u}\rangle.\end{aligned}$$

Here, using the positivity (2.2) which is based on the condition (K), we have

$$\langle(\alpha(K(\omega)A(\omega))_1 + (SL)_1 + L_1)\hat{u}, \hat{u}\rangle \ge c_1|\hat{u}|^2, \tag{3.8}$$

where c_1 is a positive constant. Therefore we can estimate D_1 as

$$(1+|\xi|^2)^2 D_1 \ge \alpha_1\alpha_2 c_1|\xi|^2|\hat{u}|^2 + \alpha_1 c_2(1+|\xi|^2)|(I-P)\hat{u}|^2 + c_3(1+|\xi|^2)^2|(I-P_1)\hat{u}|^2, \tag{3.9}$$

where c_1 is the constant in (3.8), c_2 and c_3 are positive constants not depending on (α_1, α_2), and P and P_1 denote the orthogonal projections onto $\mathrm{Ker}(L)$ and $\mathrm{Ker}(L_1)$, respectively. Here we have used (2.1) in the condition (S) and the fact that $L_1 \ge 0$ on $\mathbb{C}^m$ which is due to the condition (A). Also we see that $D_2 \ge 0$ by the condition $(\mathrm{S})_1$.

Finally, we estimate each term in G. Note that

$$\langle i(K(\omega)L)_2\hat{u}, \hat{u}\rangle = \mathrm{Re}\langle iK(\omega)L\hat{u}, \hat{u}\rangle = \mathrm{Re}\langle iK(\omega)L(I-P)\hat{u}, \hat{u}\rangle,$$

where we used $LP = 0$. Thus we have

$$|\xi||\langle i\alpha(K(\omega)L)_2\hat{u}, \hat{u}\rangle| \le C|\xi||(I-P)\hat{u}||\hat{u}| \le \epsilon|\xi|^2|\hat{u}|^2 + C_\epsilon|(I-P)\hat{u}|^2 \tag{3.10}$$

for any $\epsilon > 0$, where C_ϵ is a constant depending on ϵ. For the remaining term in G, by using the equality

$$\begin{aligned}&\langle i(SA(\omega))_2\hat{u}, \hat{u}\rangle - \langle i(SA(\omega))_2P_1\hat{u}, P_1\hat{u}\rangle \\ &= \langle i(SA(\omega))_2P_1\hat{u}, (I-P_1)\hat{u}\rangle + \langle i(SA(\omega))_2(I-P_1)\hat{u}, \hat{u}\rangle,\end{aligned}$$

we estimate as

$$\begin{aligned}&|\xi|(1+|\xi|^2)\big|\langle i(SA(\omega))_2\hat{u}, \hat{u}\rangle - \langle i(SA(\omega))_2P_1\hat{u}, P_1\hat{u}\rangle\big| \\ &\le C|\xi|(1+|\xi|^2)|(I-P_1)\hat{u}||\hat{u}| \\ &\le \delta|\xi|^2|\hat{u}|^2 + C_\delta(1+|\xi|^2)^2|(I-P_1)\hat{u}|^2\end{aligned} \tag{3.11}$$

for any $\delta > 0$, where C_δ is a constant depending on δ. Consequently, we obtain

$$\begin{aligned}(1+|\xi|^2)^2|G| &\le \alpha_1(\alpha_2\epsilon + \delta)|\xi|^2|\hat{u}|^2 \\ &\quad + \alpha_1\alpha_2C_\epsilon|(I-P)\hat{u}|^2 + \alpha_1C_\delta(1+|\xi|^2)^2|(I-P_1)\hat{u}|^2.\end{aligned}$$

We choose $\epsilon > 0$ and $\delta > 0$ such that $\epsilon = c_1/4$ and $\delta = \alpha_2 c_1/4$. For this choice of (ϵ, δ), we take $\alpha_2 > 0$ and $\alpha_1 > 0$ so small that $\alpha_2 C_\epsilon \le c_2/2$ and $\alpha_1 C_\delta \le c_3/2$. Then, by using (3.9), (3.10) and (3.11), we conclude that $|G| \le D_1/2$ and

$$D_1 \ge c\Big\{\frac{|\xi|^2}{(1+|\xi|^2)^2}|\hat{u}|^2 + \frac{1}{1+|\xi|^2}|(I-P)\hat{u}|^2 + |(I-P_1)\hat{u}|^2\Big\}, \tag{3.12}$$

where c is a positive constant. Consequently, (3.6) becomes

$$\frac{d}{dt}E + D_1 + 2D_2 \le 0. \tag{3.13}$$

Moreover, it follows from (3.7) and (3.12) that $D_1 \ge c\eta(\xi)E$, where $\eta(\xi) = |\xi|^2/(1+|\xi|^2)^2$, and c is a positive constant. Also we have $D_2 \ge 0$. Thus (3.13) leads the estimate

$$\frac{d}{dt}E + c\eta(\xi)E \le 0.$$

Solving this differential inequality, we get $E(t,\xi) \le e^{-c\eta(\xi)t}E(0,\xi)$, which together with (3.7) gives the desired pointwise estimate (2.4). This completes the proof of Theorem 2.1. □

When the condition $(\mathrm{S})_1$ is replaced by $(\mathrm{S})_2$, the above computations can be simplified and we obtain the better pointwise estimate (1.5).

Proof of the pointwise estimate in Theorem 2.2. Under the assumption (2.3) in the condition $(\mathrm{S})_2$, the first term on the right-hand side of (3.4) becomes a good term and we obtain

$$\begin{aligned}&\frac{1}{2}(1+|\xi|^2)\frac{d}{dt}\mathcal{E} + (1+|\xi|^2)\langle (SL)_1\hat{u},\hat{u}\rangle + \alpha_2|\xi|^2\langle \alpha(K(\omega)A(\omega))_1\hat{u},\hat{u}\rangle \\ &\quad + |\xi|(1+|\xi|^2)\langle i(SA(\omega))_2\hat{u},\hat{u}\rangle = \alpha_2|\xi|\langle i\alpha(K(\omega)L)_2\hat{u},\hat{u}\rangle.\end{aligned} \tag{3.14}$$

In this case, we multiply (3.1) and (3.14) by $1+|\xi|^2$ and α_1, respectively, and combine the resultant two equalities. This yields

$$\frac{1}{2}\frac{d}{dt}\tilde{E} + \tilde{D}_1 + \tilde{D}_2 = \tilde{G}, \tag{3.15}$$

where we define as

$$\begin{aligned}&\tilde{E} := E_0 + \alpha_1\mathcal{E} = E_0 + \alpha_1\Big(E_1 + \frac{\alpha_2|\xi|}{1+|\xi|^2}\,\alpha E_2\Big),\\ &(1+|\xi|^2)\tilde{D}_1 := (1+|\xi|^2)\langle L_1\hat{u},\hat{u}\rangle \\ &\qquad + \alpha_1\big\{(1+|\xi|^2)\langle (SL)_1\hat{u},\hat{u}\rangle + \alpha_2|\xi|^2\langle \alpha(K(\omega)A(\omega))_1\hat{u},\hat{u}\rangle\big\},\\ &\tilde{D}_2 := \alpha_1|\xi|\langle i(SA(\omega))_2\hat{u},\hat{u}\rangle, \qquad (1+|\xi|^2)\tilde{G} := \alpha_1\alpha_2|\xi|\langle i\alpha(K(\omega)L)_2\hat{u},\hat{u}\rangle.\end{aligned}$$

Here, for suitably small $\alpha_1 > 0$ and $\alpha_2 > 0$, we see that

$$c_0|\hat{u}|^2 \le \tilde{E} \le C_0|\hat{u}|^2, \tag{3.16}$$

where c_0 and C_0 are positive constants not depending on (α_1, α_2). On the other hand, we can rewrite $\tilde{D}_1$ as

$$\begin{aligned}(1+|\xi|^2)\tilde{D}_1 &= \alpha_1\alpha_2|\xi|^2\langle(\alpha(K(\omega)A(\omega))_1 + (SL)_1 + L_1)\hat{u}, \hat{u}\rangle \\ &\quad + \alpha_1((1+|\xi|^2) - \alpha_2|\xi|^2)\langle((SL)_1 + L_1)\hat{u}, \hat{u}\rangle + (1-\alpha_1)(1+|\xi|^2)\langle L_1\hat{u}, \hat{u}\rangle.\end{aligned}$$

Then, as in the derivation of (3.9), for suitably small $\alpha_1 > 0$ and $\alpha_2 > 0$, we can estimate $\tilde{D}_1$ as

$$(1+|\xi|^2)\tilde{D}_1 \geq \alpha_1\alpha_2 c_1|\xi|^2|\hat{u}|^2 + \alpha_1 c_2(1+|\xi|^2)|(I-P)\hat{u}|^2 + c_3(1+|\xi|^2)|(I-P_1)\hat{u}|^2,$$

where c_1, c_2 and c_3 are positive constants not depending on (α_1, α_2). Also, making use of (3.10), we can estimate the term $\tilde{G}$ as

$$(1+|\xi|^2)|\tilde{G}| \leq \alpha_1\alpha_2\epsilon|\xi|^2|\hat{u}|^2 + \alpha_1\alpha_2 C_\epsilon|(I-P)\hat{u}|^2 \tag{3.17}$$

for any $\epsilon > 0$, where C_ϵ is a constant depending on ϵ but not on (ϵ, δ).

We choose $\epsilon > 0$ in (3.17) so small that $\epsilon = c_1/2$. For this choice of ϵ, we take $\alpha_2 > 0$ so small that $\alpha_2 C_\epsilon \leq c_2/2$. Then we obtain $|\tilde{G}| \leq \tilde{D}_1/2$ and

$$\tilde{D}_1 \geq c\Big\{\frac{|\xi|^2}{1+|\xi|^2}|\hat{u}|^2 + |(I-P)\hat{u}|^2 + |(I-P_1)\hat{u}|^2\Big\}, \tag{3.18}$$

where c is a positive constant. Consequently, (3.15) becomes

$$\frac{d}{dt}\tilde{E} + \tilde{D}_1 + 2\tilde{D}_2 \leq 0.$$

Here we note that $D_2 \geq 0$ by (2.3) in the condition $(\mathrm{S})_2$. Also we have from (3.16) and (3.18) that $\tilde{D}_1 \geq \rho(\xi)\tilde{E}$, where $\rho(\xi) = |\xi|^2/(1+|\xi|^2)$, and c is a positive constant. Thus we obtain $\frac{d}{dt}\tilde{E} + c\rho(\xi)\tilde{E} \leq 0$, which is solved as $\tilde{E}(t,\xi) \leq e^{-c\rho(\xi)t}\tilde{E}(0,\xi)$. This together with (3.16) gives the desired pointwise estimate (1.5). Thus the proof of Theorem 2.2 is complete. □

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