

On the steady supersonic flow past a curved cone

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1 Problem and the Main Result

We are concerned with the problem of the steady potential supersonic flow past a Lipschitz curved cone. The flow and the cone have axis-symmetry. The surface of the cone is formed by the rotation of the graph $\{y = b(x), x > 0\}$, where the function $b(x)$ is a Lipschitz function satisfying the following (see Fig. 1):

(A1) $b(x) < 0$ for $x > 0$ and

$$b(x) = b_0x, \quad x \in [0, t_0]$$

for some constants $b_0 < 0$ and $t_0 > 0$; moreover

$$b'_+(x) = \lim_{t \rightarrow x+0} \frac{b(t) - b(x)}{t - x} \in BV([0, +\infty)),$$

and $b'_+(x)$ equals to some negative constant for $x > t_*$ for some $t_* > t_0$.

With the coordinates x and y , the equations of the flow can be written as

$$(\rho u)_x + (\rho v)_y = -\frac{\rho v}{y}, \tag{1.1}$$

$$v_x - u_y = 0, \tag{1.2}$$

with the Bernoulli equation:

$$\frac{u^2 + v^2}{2} + \frac{c^2}{\gamma - 1} = \frac{u_\infty^2}{2} + \frac{c_\infty^2}{\gamma - 1}.$$

Here u and v are components of the flow velocity in the direction of the axis of the cone (or in the x -direction) and in the y -direction respectively; $(u_\infty, 0)$ is the velocity of the incoming flow; ρ is the density of the flow and c the sonic speed with $c = \sqrt{\gamma A \rho^{\gamma-1}}$ for some constant $A > 0$; $c_\infty = \sqrt{\gamma A \rho_\infty^{\gamma-1}}$ and ρ_∞ is the density of the incoming flow. Moreover we assume that

(A2) The velocity of the incoming flow is supersonic, i.e., $u_\infty > c_\infty$.

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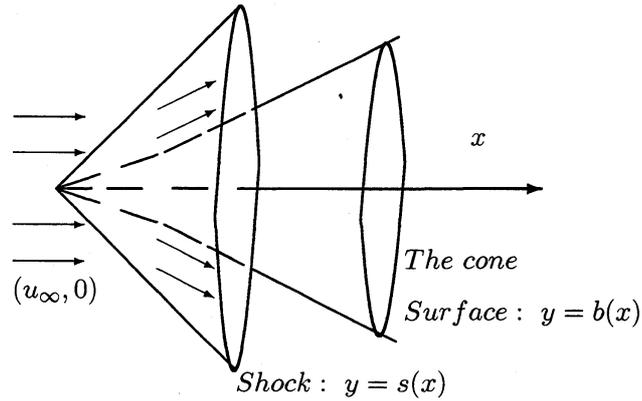


Figure 1: Supersonic conical flow

The original problem is reduced to the problem of finding the global weak solution to (1.1-1.2) in Ω with

$$(u, v)|_{x \leq 0} = (u_\infty, 0), \quad (1.3)$$

$$(u, v) \cdot \vec{n}|_{\partial\Omega} = 0, \quad (1.4)$$

where $\Omega = \{(x, y) | x > 0, y < b(x)\}$ and $\vec{n} = \vec{n}(x, y)$ is the normal to $\partial\Omega$ at differential points of b .

This problem has been studied by many authors, for instance, Courant, Friedrichs, Shuxing Chen, Zhouping, Xin, Huicheng Yin and W. Lien, Tai-Ping Liu etc. We generalize Lien and Liu's work as follows (see also [24]).

Theorem 1.1. *Suppose that the conditions **A(1)** and **A(2)** are satisfied, and that $1 < \gamma < 3$ and $-b_* < b_0 < 0$. If $T.V.b'(\cdot)$ is sufficiently small and u_∞ is sufficiently large, then problem(1.1)-(1.4) has a global solution $U(x, y)$ with bounded total variation. The solution contains a 1-shock front, which is a small perturbation of $y = s_0x$, and the solution in between the shock front and surface of the cone is a small perturbation of the self-similar solution of the problem. Here s_0 denote the slope of the corresponding shock front when the location of the surface of the cone is given by $y = b_0x$, and*

$$b_* = \left(\frac{1}{2} \left(-1 + \sqrt{\frac{\gamma+7}{\gamma-1}} \right) \right)^{1/2}.$$

I will talk about this result and briefly recall some related results and explain the proof of the above theorem.

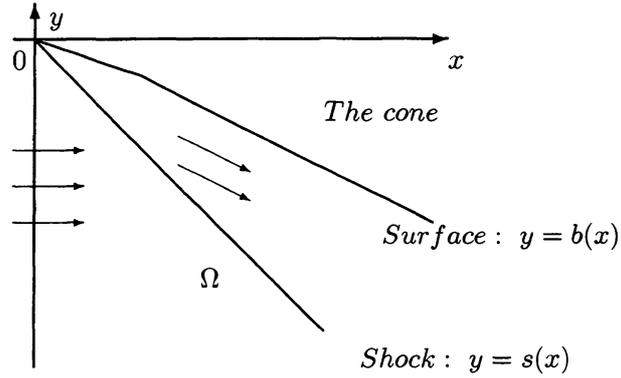


Figure 2: Supersonic flow

2 Some Related Results

In this section we recall some results on the following aspects:

- 1 Steady supersonic flow past a wedge;
- 2 Steady Supersonic Flow past a cone (previous results).

2.1 Steady supersonic flow past a wedge

We assume the lower part of the wedge is given by the graph of the function $y = b(x)$, see Fig.3. Here without confusion, we use the same notations as that in the case of the conical flow. In the case of potential flow, the governing equations are

$$(\rho u)_x + (\rho v)_y = 0, \quad (2.1)$$

$$v_x - u_y = 0, \quad (2.2)$$

Bernoulli equations:

$$\frac{u^2 + v^2}{2} + \frac{c^2}{\gamma - 1} = \frac{u_\infty^2}{2} + \frac{c_\infty^2}{\gamma - 1},$$

The first equation means the conservation of mass, while the second one means that the flow is irrotational. Here $c = \sqrt{\gamma A \rho^{\gamma-1}}$ for some $A > 0$; $c_\infty = \sqrt{\gamma A \rho_\infty^{\gamma-1}}$; (u, v) is the velocity at (x, y) satisfying the following conditions:

$$(u, v)|_{x \leq 0} = (u_\infty, 0),$$

$$(u, v) \cdot \vec{n}|_{\partial\Omega} = 0,$$

where $\Omega = \{(x, y) | x > 0, y < b(x)\}$ and $\vec{n} = \vec{n}(x, y)$ is the normal to $\partial\Omega$ at differential points of b . here we assume again that the incoming flow is supersonic, that is, $u_\infty > c_\infty$.

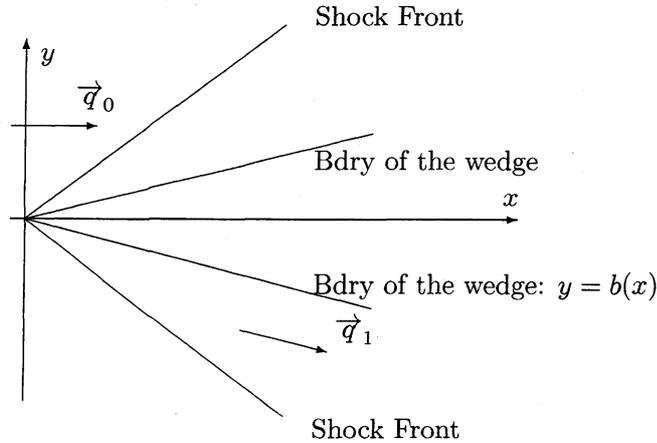


Figure 3: $q_0 = (u_\infty, 0)$: supersonic. q_1 : supersonic or subsonic.

For the straight wedge with the attack angle less than a critical value, Courant and Friedrichs gave in their famous book [13] the solution containing one straight shock issuing from the vertex by making use of the shock polar. The state behind shock is the constant state q_1 , see Fig.3. The shock polar is given by the Rankine-Hugoniot equations as shown in shown in the Fig. 4, the line Oq_{sup} has the same slope as the boundary of the wedge. Therefore, for the wedge having an attack angle less than the critical value, there are two solutions, one of which is called supersonic-supersonic shock with $q_1 = q_{sup}$, while the another is called supersonic-subsonic shock with $q_1 = q_{sub}$. Many authors consider the perturbation of these two solutions.

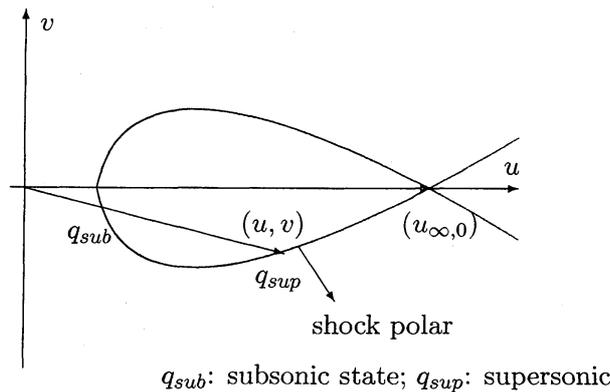


Figure 4: Shock polar

(1) Case: The perturbation problem of supersonic-supersonic shock solution.

The local piecewise smooth solutions containing a curved supersonic shock issuing from the vertex were obtained by Chaohao Gu for potential flow in [17], and by T. T. Li for the full Euler equations in [18], and by D. G. Schaffer in [22]. In the above works, the flow is assumed to be supersonic. Therefore, the equations are of hyperbolic type, and the characteristic method are used to construct the solutions. In addition, with the sharp estimates on the decay of solutions, Chen constructed the global piecewise solutions containing reflecting shocks for the concave and the convex wedges with piecewise smooth wedge in [6],[7] and [8], Yin constructed the global solutions containing one shock for the smooth wedge in [25]. In a different approach, Shuxing Chen constructed the piecewise smooth solution with a smooth shock surface for 3-d case in [5].

For the case of the wedge with Lipschitz boundary, besides the leading shock issuing from the vertex of the wedge, there may be many shocks in between the leading shock front and the boundary of the wedge. Also the leading shock may disappear. Therefore it is hard to use the characteristic method, we use instead the Glimm difference method to construct the global weak solutions. For the potential flow, Zhang established the global weak solutions for the wedge with opening angle less than a critical value and small total variation of the tangent along the boundary in [27] and [28]. The asymptotic behaviour was also studied in [29]. For the compressible flows governed by the full Euler equations, in [1] we proved the following:

Theorem 2.1. *Suppose that $|\arctan b'(0)|$ is less than a critical value and $TV(b')$ is sufficiently small, and that the incoming flow is supersonic. Then (2.1-2.3) has a global solution (u, v, p) . Moreover, $\lim_{x \rightarrow +\infty} (u, v, p)(x, y) = (u_+, v_+, p_+)$ uniformly in y for some constant supersonic state (u_+, v_+, p_+) , and the state (u_+, v_+, p_+) can be determined by the shock polar through U_∞ and the relation: $v_+/u_+ = \lim_{x \rightarrow +\infty} b'(x)$.*

In the above works, we use the modified Glimm scheme to construct the approximate solutions. To deal with the interactions of the waves at the boundary, we introduce a new functional containing angles only to control the increasing strengthes of the reflecting waves, and use the fact that some weighted linear functional over the strengthes is decreasing at the boundary. In addition, the key point in the proof of this theorem is to show the following

$$|K_1||K_2| < 1$$

which implies the decay of the strengthes after the reflections at both boundary and the leading shock front. Here K_1 and K_2 are the coefficients of the reflecting waves at the boundary and the leading shock front. The L^1 -stability for solutions under the small BV perturbations of the incoming flow was given by Guiqiang Chen and Tianhong Li in [2].

(2) Case: The perturbation problems of the supersonic-subsonic shock. In this case, the shock is also called the transonic shock. For the wedge which is the small perturbation of the straight one, the existence and stability of the transonic shock have been studied by S.

Chen, Z. Xin and Yin, Beixiang Fang and Guiqiang Chen, see for instance [15], [4] and [26] and the references therein. In such a case, the equations become of elliptic type in the domain between the shock front and the wedge. Therefore some conditions are needed to impose on the velocity and the pressure. The weighted Holder space or weighted Sobolev space were introduced to describe the regularities of the solutions near the vertex and the infinity.

It is mentioned that Shiffman and Bers dealt with Subsonic flow past a obstacle, and C. Morawetz [21], Chen, Dafermos, Dehua Wang and Slemrod used compensated compactness approach to studied the transonic flow past a obstacle, see [3]. Recently, Elling and Liu [14] studied the unsteady case.

One of open problems is to find the global weak solution for the steady supersonic flow past a wedge with an opening angle larger than the critical value. In such case, the experiments show that the shock is detached from the wedge. Formal analysis shows that the equations may be of mixed type. That is to say, it is a transonic shock. New ideas are needed to deal with this problem.

2.2 Some results on Steady Supersonic Flow past a cone

For the circular cone, in [13] Courant-Friedrich gave the solutions by making use of the shock polar and the apple curves, see Fig. 5. Due to their results, there are two solutions for the cone with an opening angle less than a critical value, one of which is of supersonic-supersonic shock solution. In such case, the equations are of hyperbolic type over the flow field. Many authors consider the small perturbation of this solution.

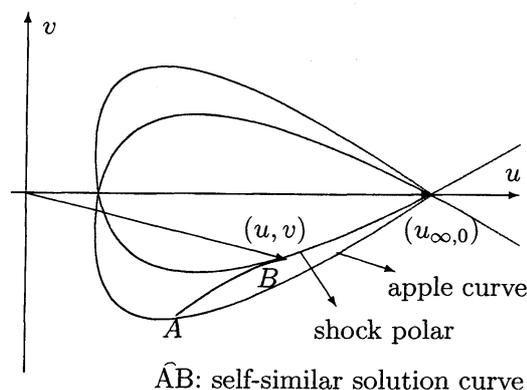


Figure 5: Apple Curve

The local piecewise smooth solution near the vertex is given in [9]. Such solution contains a smooth conical shock front issuing from the vertex. One of global results was given by W. Lien

and Tai-Ping Liu in [20]. They assume that the cone has small opening angle and has small total variation of b' and that the initial strength of the relatively strong shock is sufficiently weak and that the Mach number of incoming flow is sufficiently large. Then for $1 < \gamma < 3$ the boundary value problem (1.1-1.4) has a global solution with bounded total variation. To get the global weak solution, they used a modified Glimm type difference scheme to construct the approximate solutions. Different from the standard Glimm scheme, the Riemann solutions and the self-similar solutions are used as the building block. The smallness of the opening angle and the strength of the leading shock play an important role in proving the monotonicity of Glimm type functional.

For smooth function b , the global piecewise smooth solution is constructed by Shuxing Chen, Zhoupin Xin and Huicheng Yin in [11]. In the paper the attack angle is assumed to be small and the Mach number of incoming flow is assumed to be sufficiently large. $1 < \gamma < 3$ is also assumed. Then they considered the case that $b - b_0$ is small in suitable weighted sobolev norm and the derivatives of b (up to some order) are decay at $+\infty$. The solution they constructed has a smooth conical shock front issuing from the vertex of the cone. In their proof the local piecewise smooth solution constructed in [9] was used. Then to extend this solution, the weighted Sobolev space are introduced to describe the regularity of the solutions and the Hardy type inequality are established for the solution. Yin and his students also made the improvement of this result, see [26] and the reference therein.

In Courant and Friedrichs' result, the supersonic-subsonic shock solutions were also given for the circular cone. For the cone that is a smooth perturbation of the straight circular one, Yin and students and Beixiang Fang and Guiqiang Chen constructed the piecewise smooth solutions, see [26] and [4] and the references therein. Such solutions also have smooth conical shock front issuing from the vertex of the cone, but the equations are of elliptic type in and between the shock front and the surface of the cone. In addition, some conditions on the velocities are imposed at $x = +\infty$ to find the solution. For the three dimensional case, self-similar solutions were given by S. Chen in [10].

An open problem is to determine the solution for the steady supersonic flow past a cone of general shape.

3 Remarks on the Proof of the Main Result

Our solution is still a small perturbation of the solution for circular cone. Therefore, we first studied the asymptotic expansion of the solution for circular cone. In this case $b \equiv b_0 x$. Let $y = s_0 x$ be the location of shock front. Let $\sigma = y/x$. Then equations for Circular conical flow

can be reduced to:

$$(-\sigma^2(1 - \frac{u^2}{c^2}) - \frac{uv}{c^2}\sigma)u_\sigma + (\frac{uv}{c^2} + (1 - \frac{v^2}{c^2})\sigma)v_\sigma + v = 0, \quad (3.1)$$

$$u_\sigma + \sigma v_\sigma = 0, \quad s_0 < \sigma < b_0, \quad (3.2)$$

$$\rho(us_0 - v) = \rho_\infty u_\infty s_0, \quad \sigma = s_0, \quad (3.3)$$

$$u + vs_0 = u_\infty, \quad \sigma = s_0, \quad (3.4)$$

$$ub_0 = v, \quad \sigma = b_0; \quad (3.5)$$

with

$$(u(\sigma), v(\sigma)) = (u_\infty, 0), \quad \sigma < s_0. \quad (3.6)$$

It can be reduced to

$$u_\sigma = \frac{vc^2}{(1 + \sigma^2)c^2 - (v - \sigma u)^2}, \quad (3.7)$$

$$v_\sigma = \frac{-vc^2}{\sigma((1 + \sigma^2)c^2 - (v - \sigma u)^2)}, \quad (3.8)$$

$$\rho_\sigma = \frac{\rho vc^2(v - \sigma u)}{\sigma((1 + \sigma^2)c^2 - (v - \sigma u)^2)}. \quad (3.9)$$

We have the following.

Lemma 3.1. For $1 < \gamma < 3$ and $b_0 \in (-b_*, 0)$ and $\rho_\infty > 0$, there exist constants $K' > 0$, $K'' > 0$ and $K''' > 0$, independent of u_∞ , such that for $u_\infty > K'''$, the problem (3.1-3.5) has a unique solution $(u(\sigma), v(\sigma), s_0)$ constituted by a supersonic conical shock front issuing from the vertex. Moreover,

$$\lim_{u_\infty \rightarrow +\infty} \frac{u(\sigma)}{u_\infty} = \frac{1}{1 + b_0^2} = \cos^2 \theta_0, \quad (3.10)$$

$$\lim_{u_\infty \rightarrow +\infty} \frac{v(\sigma)}{u_\infty} = \frac{b_0}{1 + b_0^2} = \sin \theta_0 \cos \theta_0, \quad (3.11)$$

$$s_0 = b_0 + O(1)u_\infty^{-\frac{2}{\gamma-1}}, \quad (3.12)$$

$$\frac{\rho}{\rho_\infty} = \left\{ \frac{\gamma-1}{2} \frac{u_\infty^2 s_0^2}{c_\infty^2 (1 + s_0^2)} \right\}^{\frac{1}{\gamma-1}} (1 + O(1)u_\infty^{-2}), \quad (3.13)$$

and

$$\lim_{u_\infty \rightarrow +\infty} \left(\frac{u(\sigma)}{c(\sigma)} \right)^2 = \frac{2}{(\gamma-1)b_0^2(1+b_0^2)} > 1, \quad (3.14)$$

$$\cos(\theta_0 \pm \theta_{ma}^0) > 0, \quad (3.15)$$

where $\theta_0 = \arctan b_0$ and $\theta_{ma}^0 = \lim_{u_\infty \rightarrow +\infty} \theta_{ma}$; $O(1)$ stands for a bounded quantity as $u_\infty \rightarrow +\infty$.

Such expansions can be found for instance in the book of Van Dyke [23]. The proof is given in [11] and [24] and it plays a crucial role in our analysis. Now we regard x as time variable, then the homogeneous system is strictly hyperbolic. The origin problem can be regarded as a initial boundary value problem of a hyperbolic system with a singular source term. Then we approximate the boundary by piecewise line segments as in [28] and use the modified Glimm scheme by Lien and Liu [20]. To get the bounds on the total variations of the approximate solutions, we introduced a weighted Glimm type functional. Besides the terms given by Lien and Liu, we introduced the linear terms to control the moving centers and the strengthes of the waves produced by the flow around the corner. Then, by making use of the above expansion for straight circular cone, we establish the estimates on coefficients for the wave-reflections for large Mach numbers, which show that the total amount of the waves in between the leading shock and the cone will vanish. This also implies the decreasing of the Glimm type functionals. Standard arguments [16] give the main result then.

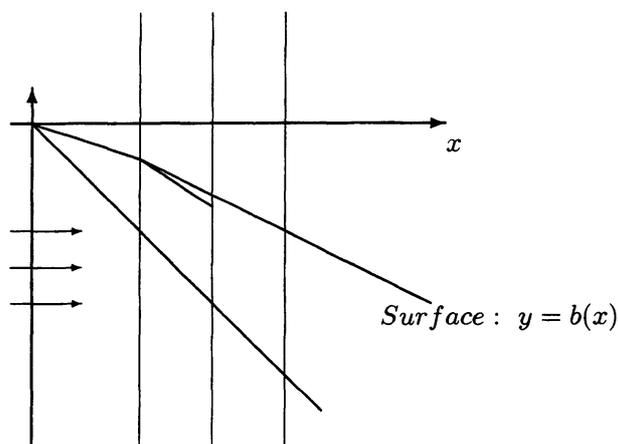


Figure 6: Glimm Scheme

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