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Analysis of a pressure-stabilized characteristics finite element scheme for a linearized Navier-Stokes equations

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1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^d (d=2,3)$, T be a positive constant, $u: \Omega\times(0,T)\to\mathbb{R}^d$ be a velocity and $\phi: \Omega\times(0,T)\to\mathbb{R}$ be a scalar function. We consider a trajectory of fluid particle, which is important for flow problems. Let $X=X(\cdot;x,t^n):(0,T)\to\mathbb{R}^d$ be a solution of the ordinary differential equation,

$$\frac{dX}{dt}=u(X,t) \tag{1}$$

with a condition $X(t^n)=x$, where $n\in\mathbb{N}\cup\{0\}$, Δt is a time increment and $t^n\equiv n\Delta t$. Then, it holds that

$$\frac{D\phi}{Dt}(x,t)=\frac{d}{dt}\phi\big(X(t),t\big), \tag{2}$$

where

$$\frac{D}{Dt}\equiv\frac{\partial\phi}{\partial t}+u\cdot\nabla \tag{3}$$

is a material derivation. Therefore, we can consider a first order approximation of the material derivative at $t=t^n (n\geq 1)$ as follows;

$$\begin{aligned}\frac{D\phi}{Dt}(x,t^n)&\approx\frac{\phi\big(X(t^n),t^n\big)-\phi\big(X(t^{n-1}),t^{n-1}\big)}{\Delta t}\\ &\approx\frac{\phi(x,t^n)-\phi\big(X_1(u^n,\Delta t)(x),t^{n-1}\big)}{\Delta t}\\ &=\frac{\phi^n-\phi^{n-1}\circ X_1(u^n,\Delta t)}{\Delta t}(x),\end{aligned} \tag{4}$$

where we have used notations, $\phi^n\equiv\phi(\cdot,t^n)$, for $w:\Omega\to\mathbb{R}^d$,

$$X_1(w,\Delta t)(x)\equiv x-w(x)\Delta t, \tag{5}$$

and, for $\psi:\Omega\to\mathbb{R}$,

$$\psi\circ X_1(w,\Delta t)(x)\equiv\psi\big(X_1(w,\Delta t)(x)\big). \tag{6}$$

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The approximation (4) of $D\phi/Dt$ is a basic idea to devise numerical schemes based on the method of characteristics. The idea can be combined with finite element and difference methods (e.g., see [5, 10, 12, 13, 15]). In this paper, we introduce a finite element scheme based on the method of characteristics for a linearized Navier-Stokes equations, and give stability and convergence results of the scheme. Although the system of the equations is linear and simpler than one of the Navier-Stokes equations, the results to be shown are useful for an analysis of a scheme for the Navier-Stokes equations [9, 11].

2 Statement of the problem and a characteristics finite element scheme

We consider a linearized Navier-Stokes problem; find $(u,p):\ \Omega\times(0,T)\to\mathbb{R}^d\times\mathbb{R}$ such that

$$\begin{cases} \dfrac{Du}{Dt_w}-\nabla(2\nu D(u))+\nabla p=f, & \text{in } \Omega\times(0,T),\\ \nabla\cdot u=0, & \text{in } \Omega\times(0,T),\\ u=0, & \text{on } \Gamma\times(0,T),\\ u=u^0, & \text{in } \Omega,\ \text{at } t=0, \end{cases} \tag{7}$$

where u is the velocity, p is the pressure, $f:\ \Omega\times(0,T)\to\mathbb{R}^d$ is a given external force, $u^0:\ \Omega\to\mathbb{R}^d$ is a given initial velocity, $\nu(>0)$ is a viscosity, $D(u)$ is a strain-rate tensor defined by

$$D_{ij}(u)\equiv\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\quad(i,j=1,\cdots,d), \tag{8}$$

D/Dt_w is a material derivation defined by

$$\frac{D}{Dt_w}\equiv\frac{\partial}{\partial t}+w\cdot\nabla \tag{9}$$

and $w:\ \Omega\times(0,T)\to\mathbb{R}^d$ is a given velocity. If w is replaced by u, (7) is the Navier-Stokes problem.

We use the same notation $(\cdot,\cdot)$ to represent $L^2(\Omega)$ inner product for scalar-, vector- and matrix-valued functions, and define a bilinear forms a on $H^1(\Omega)^d\times H^1(\Omega)^d$ and b on $H^1(\Omega)^d\times L^2(\Omega)$ by

$$a(u,v)\equiv 2\nu\big(D(u),D(v)\big),\quad\text{and}\quad b(v,q)\equiv-(\nabla\cdot v,q), \tag{10}$$

respectively. The weak form of the problem (7) is written as follows; find $\{(u,p)(t)\}_{t\in(0,T)}\subset V\times Q$ such that

$$\begin{cases} \left(\dfrac{Du}{Dt_w},v\right)+a(u,v)+b(v,p)+b(u,q)=(f,v), \quad \forall(v,q)\in V\times Q,\\ u(0)=u^0, \end{cases} \tag{11}$$

where $V\equiv H_0^1(\Omega)^d$ and $Q\equiv L_0^2(\Omega)$.

We state a characteristics finite element scheme for (7). Let $\mathscr{T}_h=\{K\}$ be a triangulation,

$$\Omega_h\equiv\operatorname{int}\Big(\bigcup_{K\in\mathscr{T}_h}K\Big)\quad\text{and}\quad\Gamma_h\equiv\partial\Omega_h. \tag{12}$$

We define function spaces X_h, M_h, V_h and Q_h by

$$X_h\equiv\big\{v_h\in C^0(\bar{\Omega}_h)^d;\ v_h|_K\in P_1(K)^d,\ \forall K\in\mathscr{T}_h\big\}, \tag{13}$$

$$M_h \equiv \{q_h \in C^0(\bar{\Omega}_h);\ q_h|_K \in P_1(K),\ \forall K \in \mathscr{T}_h\}, \tag{14}$$

$$V_h \equiv X_h \cap H^1_0(\Omega) \quad \text{and} \quad Q_h \equiv M_h \cap L^2_0(\Omega_h), \tag{15}$$

respectively, where $P_1(K)$ is a function space of piecewise linear functions on an element $K \in \mathscr{T}_h$, P means a nodal point on Γ_h, and $L^2_0(\Omega)$ is a subspace of $L^2(\Omega)$ defined by

$$L^2_0(\Omega) \equiv \{q \in L^2(\Omega);\ (q,1) = 0\}. \tag{16}$$

Let $N_T \equiv [T/\Delta t]$ be a total number of time steps, δ be a positive constant, h_K be a diameter of K and $(\cdot,\cdot)_K$ be an inner product in $L^2(K)^d$. For u, $\tilde{u}$ and $w \in H^1(\Omega)^d$, we define a linear form $\mathscr{M}_h(u,\tilde{u};\Delta t,w)$ on V_h and a bilinear form $\mathscr{C}_h$ on $H^1(\Omega) \times H^1(\Omega)$ by

$$\langle \mathscr{M}_h(u,\tilde{u};\Delta t,w),\ v_h\rangle \equiv \Big(\frac{u - \tilde{u}\circ X_1(w,\Delta t)}{\Delta t},\ v_h\Big) \quad \text{and} \quad \mathscr{C}_h(p,q) \equiv -\delta \sum_{K\in\mathscr{T}_h} h_K^2(\nabla p, \nabla q)_K, \tag{17}$$

respectively. Let an approximate function $u_h^0 \in V_h$ of u^0 be given. A pressure-stabilized characteristics finite element scheme for (7) is to find $\{(u_h^n,\ p_h^n)\}_{n=1}^{N_T} \subset V_h \times Q_h$ such that, for $n = 1,\cdots,N_T$,

$$\begin{aligned}\langle \mathscr{M}_h(u_h^n,u_h^{n-1};\Delta t,w^{n-1}),v_h\rangle + a(u_h^n,v_h) + b(v_h,p_h^n) + b(u_h^n,q_h) + \mathscr{C}_h(p_h^n,q_h)&\\ = (f^n,v_h),\quad \forall (v_h,q_h) \in V_h \times Q_h,&\end{aligned} \tag{18}$$

The scheme (18) can deal with high Reynolds number (small viscosity) problems by the method of characteristics. The material derivative term $(Du/Dt_w, v)$ in (11) is approximated by $\langle \mathscr{M}_h(u_h^n,u_h^{n-1};\Delta t,w^{n-1}),v_h\rangle$. When we find $(u_h^n,\ p_h^n)$ in the scheme (18), a composite function $u_h^{n-1}\circ X_1(w_h^{n-1},\Delta t)$ is a known function and a coefficient matrix of the system of the linear equations is symmetric. The advantage enables us to use symmetric linear iterative solvers, i.e., CG, CR, MINRES [2]. Since the coefficient matrix is independent of step number n, it is enough to make the matrix at only the first time step. The scheme employs a cheap element P1/P1, it is useful for large scale computation, especially in 3D. Although P1/P1 element does not satisfy the inf-sup condition [8], the scheme works by a pressure-stabilization term $\mathscr{C}_h$.

We impose assumption for a given velocity w and review a proposition in [15].

Hypothesis 2.1. *A function w satisfies*

$$\begin{cases} w \in C^0([0,T];W^{1,\infty}(\Omega)),\\ w = 0 \ \ on\ \Gamma,\\ \mathrm{div} w = 0 \ \ in\ \Omega.\end{cases} \tag{19}$$

Proposition 2.2 ([15]). *Under Hypothesis 2.1 and an inequality;*

$$\Delta t < \frac{1}{\|w\|_{C^0(W^{1,\infty})}}, \tag{20}$$

it holds that, for any $t \in [0,T]$,

$$X_1(w(\cdot,t),\Delta t)(\Omega) = \Omega. \tag{21}$$

In the following sections, we assume that Hypothesis 2.1 and the inequality (20) hold, and that, for the sake of simplicity, Ω is convex polygonal domain ($\Omega = \Omega_h$).

3 Stability and convergence

In this section, we consider stability and convergence of the scheme (18). We use c to represent the generic positive constant independent of discretization parameters and solutions, which can take different values at different places, and $c(A)$ is a positive constant, which depends on A. Constants c_0, c_1 and c_2 have particular meanings in this paper,

$$c_0 = c_0(\|w\|_{C^0(L^\infty)}), \quad c_1 = c_1(\|w\|_{C^0(W^{1,\infty})}) \quad \text{and} \quad c_2 = c_2(\|w\|_{C^0(W^{1,\infty})\cap C^1(L^\infty)}), \tag{22}$$

respectively.

3.1 Stability

This subsection is devoted to the stability of the scheme (18). We use norms and seminorms, $\|\cdot\|_k \equiv \|\cdot\|_{H^k(\Omega)}$ $(k=0,1,2)$, $\|\cdot\|_V \equiv \|\cdot\|_1$, $\|\cdot\|_Q \equiv \|\cdot\|_0$,

$$\|(v,q)\|_{V\times Q} \equiv \left\{\|v\|_V^2 + \|q\|_Q^2\right\}^{1/2}, \quad |q|_h \equiv \left\{\sum_{K\in\mathscr{T}_h} h_K^2(\nabla q, \nabla q)_K\right\}^{1/2},$$

$$\|u\|_{l^\infty_{(m)}(L^2)} \equiv \max_{n=m,\cdots,N_T} \|u^n\|_0, \quad \|u\|_{l^\infty_{(m)}(H^1)} \equiv \max_{n=m,\cdots,N_T} \|u^n\|_1, \quad \|u\|_{l^\infty(X)} \equiv \|u\|_{l^\infty_{(0)}(X)} \ (X = L^2, H^1),$$

$$\|u\|_{l^2_{(m)}(L^2)} \equiv \left\{\Delta t \sum_{n=m}^{N_T} \|u^n\|_0^2\right\}^{1/2}, \quad \|u\|_{l^2_{(m)}(H^1)} \equiv \left\{\Delta t \sum_{n=m}^{N_T} \|u^n\|_1^2\right\}^{1/2}, \quad \|u\|_{l^2(X)} \equiv \|u\|_{l^2_{(1)}(X)} \ (X = L^2, H^1),$$

$$|p|_{l^2_{(m)}(M_h)} \equiv \left\{\Delta t \sum_{n=m}^{N_T} |p^n|_h^2\right\}^{1/2} \quad \text{and} \quad |p|_{l^2(M_h)} \equiv |p|_{l^2_{(1)}(M_h)}. \tag{23}$$

Setting a bilinear form $\mathscr{A}_h$ on $(V_h \times Q_h) \times (V_h \times Q_h)$ by

$$\mathscr{A}_h((u,p);(v,q)) \equiv a(u,v) + b(v,p) + b(u,q) + \mathscr{C}_h(p,q), \tag{24}$$

we have another representation of the scheme (18);

$$\langle \mathscr{M}_h(u_h^n, u_h^{n-1}; \Delta t, w^{n-1}), v_h\rangle + \mathscr{A}_h((u_h^n, p_h^n);(v_h, q_h)) = (f^n, v_h), \quad \forall (v_h, q_h) \in V_h \times Q_h. \tag{25}$$

We prepare two lemmas for the stability. First one is an estimate of a composite function appearing $\mathscr{M}_h$;

Lemma 3.1. *For any $v \in L^2(\Omega)^d$ and $t \in [0,T]$, it holds that*

$$\|v \circ X_1(w(\cdot,t), \Delta t)\|_0 \le (1 + c_1 \Delta t)\|v\|_0. \tag{26}$$

We omit the proof, because it is similar to the proof of Lemma 1 in [15].

P1/P1 element is employed in the scheme (18), i.e., both finite dimensional function spaces V_h and Q_h consist of P1 element, and P1/P1 element does not satisfy the inf-sup condition [8];

$$\inf_{q_h\in Q_h} \sup_{v_h\in V_h} \frac{b(v_h, q_h)}{\|v_h\|_V \|q_h\|_Q} \ge \beta^* > 0. \tag{27}$$

However, a, b and $\mathscr{C}_h$ satisfies the following inf-sup condition.

Lemma 3.2 ([7]). *There is a positive constant γ such that*

$$\inf_{(u_h,p_h)\in V_h\times Q_h} \sup_{(v_h,q_h)\in V_h\times Q_h} \frac{\mathscr{A}_h((u_h,p_h);(v_h,q_h))}{\|(u_h,p_h)\|_{V\times Q}\|(v_h,q_h)\|_{V\times Q}} \ge \gamma. \tag{28}$$

From the above lemmas, we obtain a stability result.

Theorem 3.3 (stability). *Let $\Delta t_0(<1)$ be a fixed number. Assume $\Delta t \in (0,\Delta t_0]$.*
(i) It holds that

$$\|u_h\|_{l^\infty(L^2)}+\sqrt{\nu}\|u_h\|_{l^2(H^1)}+\sqrt{\delta}|p_h|_{l^2(M_h)} \le c_1(\|u_h^0\|_0+\|f\|_{l^2(L^2)}). \tag{29}$$

(ii) The following two inequalities hold;

$$\sqrt{\nu}\|u_h\|_{l^\infty_{(1)}(H^1)}+\sqrt{\delta}|p_h|_{l^\infty_{(1)}(M_h)}+\|\overline{D}_{\Delta t}u_h\|_{l^2_{(2)}(L^2)}+\sqrt{\nu\Delta t}\|\overline{D}_{\Delta t}u_h\|_{l^2_{(2)}(H^1)}+\sqrt{\delta\Delta t}|\overline{D}_{\Delta t}p_h|_{l^2_{(2)}(M_h)}$$
$$\le c_1(\|u_h^1\|_1+|p_h^1|_h+\|f\|_{l^2_{(2)}(L^2)}), \tag{30}$$

$$\|p_h\|_{l^2_{(2)}(L^2)} \le c_1(\|u_h^1\|_1+|p_h^1|_h+\|f\|_{l^2_{(2)}(L^2)}). \tag{31}$$

Proof. First we prove (i). Substituting $(u_h^n,-p_h^n)\in V_h\times Q_h$ into (v_h,q_h) in the scheme (18), we have

$$\langle \mathscr{M}_h(u_h^n,u_h^{n-1};\Delta t,w^{n-1}),u_h^n\rangle + a(u_h^n,u_h^n)+\mathscr{C}_h(p_h^n,-p_h^n)=(f^n,u_h^n). \tag{32}$$

We evaluate the four terms;

$$\langle \mathscr{M}_h(u_h^n,u_h^{n-1};\Delta t,w^{n-1}),u_h^n\rangle = \frac{1}{\Delta t}\left\{\frac{1}{2}(\|u_h^n\|_0^2-\|u_h^{n-1}\circ X_1(w^{n-1},\Delta t)\|_0^2)+\frac{1}{2}\|u_h^n-u_h^{n-1}\circ X_1(w^{n-1},\Delta t)\|_0^2\right\}$$
$$(\because (a-b)a=(a^2-b^2)/2+(a-b)^2/2)$$
$$\ge \frac{1}{\Delta t}\Big(\frac{1}{2}\|u_h^n\|_0^2-\frac{1}{2}\|u_h^{n-1}\|_0^2\Big)-c_1\|u_h^{n-1}\|_0^2 \qquad (\because \text{Lemma 3.1})$$
$$=\overline{D}_{\Delta t}\Big(\frac{1}{2}\|u_h^n\|_0^2\Big)-c_1\|u_h^{n-1}\|_0^2, \tag{33a}$$
$$a(u_h^n,u_h^n)=2\nu\|D(u_h^n)\|_0^2\ge c\nu\|u_h^n\|_1^2 \qquad (\because \text{Korn's inequality [6]}), \tag{33b}$$
$$\mathscr{C}_h(p_h^n,-p_h^n)=\delta|p_h^n|_h^2, \tag{33c}$$

and

$$(f^n,u_h^n)\le\frac{1}{2}(\|f^n\|_0^2+\|u_h^n\|_0^2) \qquad (\because ab\le(a^2+b^2)/2)), \tag{33d}$$

where $\overline{D}_{\Delta t}a^n\equiv(a^n-a^{n-1})/\Delta t$. The inequalities (33) implies

$$\overline{D}_{\Delta t}\Big(\frac{1}{2}\|u_h^n\|_0^2\Big)+c\nu\|u_h^n\|_1^2+\delta|p_h^n|_h^2\le\frac{1}{2}(\|f^n\|_0^2+\|u_h^n\|_0^2) \quad (n=1,\cdots,N_T), \tag{34}$$

and (29) holds from the discrete Gronwall lemma [20].

Next, we prove (30) of (ii). Substituting $0\in V_h$ into v_h in the scheme (18), we have

$$b(u_h^n,q_h)+\mathscr{C}_h(p_h^n,q_h)=0 \quad (\forall q_h\in Q_h,\ n=1,\cdots,N_T), \tag{35}$$

and, then,

$$b(\overline{D}_{\Delta t}u_h^n,q_h)+\mathscr{C}_h(\overline{D}_{\Delta t}p_h^n,q_h)=0 \quad (\forall q_h\in Q_h,\ n=2,\cdots,N_T). \tag{36}$$

Substituting $(\overline{D}_{\Delta t}u_h^n,0)\in V_h\times Q_h$ into (v_h,q_h) in the scheme (18), we have

$$\langle \mathscr{M}_h(u_h^n,u_h^{n-1};\Delta t,w^{n-1}),\overline{D}_{\Delta t}u_h^n\rangle + a(u_h^n,\overline{D}_{\Delta t}u_h^n)+b(\overline{D}_{\Delta t}u_h^n,p_h^n)=(f^n,\overline{D}_{\Delta t}u_h^n) \quad (n=1,\cdots,N_T), \tag{37}$$

and, by (36),

$$\langle \mathscr{M}_h(u_h^n, u_h^{n-1}; \Delta t, w^{n-1}), \overline{D}_{\Delta t} u_h^n \rangle + a(u_h^n, \overline{D}_{\Delta t} u_h^n) - \mathscr{C}_h(\overline{D}_{\Delta t} p_h^n, p_h^n) = (f^n, \overline{D}_{\Delta t} u_h^n) \quad (n = 2, \cdots, N_T). \tag{38}$$

We evaluate the four terms in (38);

$$\begin{aligned}
\langle \mathscr{M}_h(u_h^n, u_h^{n-1}; \Delta t, w^{n-1}), \overline{D}_{\Delta t} u_h^n \rangle &= \Big(\overline{D}_{\Delta t} u_h^n + \frac{1}{\Delta t}(u_h^{n-1} - u_h^{n-1} \circ X_1(w^{n-1}, \Delta t)), \overline{D}_{\Delta t} u_h^n\Big) \\
&= \|\overline{D}_{\Delta t} u_h^n\|_0^2 + \frac{1}{\Delta t}(u_h^{n-1} - u_h^{n-1} \circ X_1(w^{n-1}, \Delta t)), \overline{D}_{\Delta t} u_h^n) \\
&\geq \|\overline{D}_{\Delta t} u_h^n\|_0^2 - \frac{1}{\Delta t}\|u_h^{n-1} - u_h^{n-1} \circ X_1(w^{n-1}, \Delta t)\|_0 \|\overline{D}_{\Delta t} u_h^n\|_0 \\
&\geq \|\overline{D}_{\Delta t} u_h^n\|_0^2 - c_0 |u_h^{n-1}|_1 \|\overline{D}_{\Delta t} u_h^n\|_0 \\
&\geq \|\overline{D}_{\Delta t} u_h^n\|_0^2 - \Big(c_0 |u_h^{n-1}|_1^2 + \frac{1}{4}\|\overline{D}_{\Delta t} u_h^n\|_0^2\Big) \\
&\geq \frac{3}{4}\|\overline{D}_{\Delta t} u_h^n\|_0^2 - c_0 \|D(u_h^{n-1})\|_0^2 \quad (\because \text{Korn の不等式}),
\end{aligned} \tag{39a}$$

$$\begin{aligned}
a(u_h^n, \overline{D}_{\Delta t} u_h^n) &= \overline{D}_{\Delta t}\Big(\frac{1}{2} a(u_h^n, u_h^n)\Big) + \frac{\Delta t}{2} a(\overline{D}_{\Delta t} u_h^n, \overline{D}_{\Delta t} u_h^n) \\
&= \overline{D}_{\Delta t}\Big(\nu \|D(u_h^n)\|_0^2\Big) + \nu \Delta t \|D(\overline{D}_{\Delta t} u_h^n)\|_0^2,
\end{aligned} \tag{39b}$$

$$\begin{aligned}
-\mathscr{C}_h(\overline{D}_{\Delta t} p_h^n, p_h^n) &= \overline{D}_{\Delta t}\Big(-\frac{1}{2}\mathscr{C}_h(p_h^n, p_h^n)\Big) - \frac{\Delta t}{2}\mathscr{C}_h(\overline{D}_{\Delta t} p_h^n, \overline{D}_{\Delta t} p_h^n) \\
&= \overline{D}_{\Delta t}\Big(\frac{\delta}{2}|p_h^n|_h^2\Big) + \frac{\delta \Delta t}{2}|\overline{D}_{\Delta t} p_h^n|_h^2,
\end{aligned} \tag{39c}$$

and

$$(f^n, \overline{D}_{\Delta t} u_h^n) = \|f^n\|_0^2 + \frac{1}{4}\|\overline{D}_{\Delta t} u_h^n\|_0^2. \tag{39d}$$

Combining (39) with (38), we have

$$\begin{aligned}
\overline{D}_{\Delta t}\Big(\nu \|D(u_h^n)\|_0^2 + \frac{\delta}{2}|p_h^n|_h^2\Big) + \frac{1}{2}\|\overline{D}_{\Delta t} u_h^n\|_0^2 + \nu \Delta t \|D(\overline{D}_{\Delta t} u_h^n)\|_0^2 + \frac{\delta \Delta t}{2}|\overline{D}_{\Delta t} p_h^n|_h^2 \\
\leq \|f^n\|_0^2 + c_0 \|D(u_h^{n-1})\|_0^2 \quad (n = 2, \cdots, N_T),
\end{aligned} \tag{40}$$

which implies (30) by the discrete Gronwall lemma.

Finally we prove (31) of (ii). From Lemma 3.2, $\|p_h^n\|_0$ is evaluated as follows;

$$\begin{aligned}
\|p_h^n\|_0 &\leq \|(u_h^n, p_h^n)\|_{V \times Q} \\
&\leq \frac{1}{\gamma} \sup_{(v_h, q_h) \in V_h \times Q_h} \frac{\mathscr{A}_h((u_h^n, p_h^n); (v_h, q_h))}{\|(v_h, q_h)\|_{V \times Q}} \\
&\leq \frac{1}{\gamma} \sup_{(v_h, q_h) \in V_h \times Q_h} \frac{(f^n, v_h) - \langle \mathscr{M}_h(u_h^n, u_h^{n-1}; \Delta t, w^{n-1}), v_h \rangle}{\|(v_h, q_h)\|_{V \times Q}} \\
&\leq \frac{1}{\gamma}\Big(\|f^n\|_{V_h'} + \|\mathscr{M}_h(u_h^n, u_h^{n-1}; \Delta t, w^{n-1})\|_{V_h'}\Big) \\
&\leq \frac{1}{\gamma}\Big(\|f^n\|_0 + \|\overline{D}_{\Delta t} u_h^n\|_0 + c_0 |u_h^{n-1}|_1\Big),
\end{aligned} \tag{41}$$

where we have used the following inequality in the last inequality of (41);

$$\begin{aligned}
\|\mathscr{M}_h(u_h^n, u_h^{n-1}; \Delta t, w^{n-1})\|_{V_h'} &\le \frac{1}{\Delta t}\|u_h^n - u_h^{n-1}\circ X_1(w^{n-1},\Delta t)\|_0 \\
&= \|\overline{D}_{\Delta t}u_h^n + \frac{1}{\Delta t}(u_h^{n-1} - u_h^{n-1}\circ X_1(w^{n-1},\Delta t))\|_0 \\
&\le \|\overline{D}_{\Delta t}u_h^n\|_0 + \frac{1}{\Delta t}\|u_h^{n-1} - u_h^{n-1}\circ X_1(w^{n-1},\Delta t))\|_0 \\
&\le \|\overline{D}_{\Delta t}u_h^n\|_0 + c_0|u_h^{n-1}|_1.
\end{aligned} \tag{42}$$

Combining (41) with (30), we have

$$\begin{aligned}
\|p_h\|_{l^2_{(2)}(L^2)} &\le c_0(\|f\|_{l^2_{(2)}(L^2)} + \|\overline{D}_{\Delta t}u_h^n\|_{l^2_{(2)}(L^2)} + \|u_h\|_{l^2_{(1)}(H^1)}) \\
&\le c_1(\|u_h^1\|_1 + |p_h^1|_h + \|f\|_{l^2_{(2)}(L^2)}).
\end{aligned} \tag{43}$$

□

Remark 3.4. *In the right hand side of* (31), $\|u_h^1\|_1$ *can be estimated by* (30), *and, for* $|p_h^1|_h$, *a detailed evaluation is required.*

3.2 Convergence

This subsection is devoted to convergence of the scheme (18). At first we define a Stokes projection.

Definition 3.5 (Stokes projection). *For* $(u,p) \in (V\cap H^2(\Omega)^d)\times(Q\cap H^1(\Omega))$, $(w_h, r_h) \in V_h \times Q_h$ *is a Stokes projection of* (u,p) *provided*

$$a(w_h,v_h) + b(v_h,r_h) + b(w_h,q_h) + \mathscr{C}_h(r_h,q_h) = \langle \mathscr{F}_h, (v_h,q_h)\rangle, \quad \forall(v_h,q_h) \in V_h\times Q_h, \tag{44}$$

where $\mathscr{F}_h$ *is a linear form on* $V_h \times Q_h$ *defined by*

$$\langle \mathscr{F}_h, (v_h,q_h)\rangle \equiv (g,v_h) - \delta \sum_{K\in\mathscr{T}_h} h_K^2(g, \nabla q_h)_K \quad \text{with} \quad g \equiv -2\nu\nabla D(u) + \nabla p. \tag{45}$$

For the Stokes projection, the following error estimate holds.

Proposition 3.6. *Suppose* $(u,p) \in (V\cap H^2(\Omega)^d)\times(Q\cap H^2(\Omega))$, *and u satisfies*

$$b(u,q) = 0 \quad (q \in Q). \tag{46}$$

Let (w_h, r_h) *be a Stokes projection of* (u,p) *by* (44). *Then, there exists a positive constant* c_s, *independent of h, such that*

$$\|u - w_h\|_1 + \|p - r_h\|_0 \le c_s h(\|u\|_2 + h\|p\|_2). \tag{47}$$

We omit the proof, because papers [3] and [7] give the result.

Theorem 3.7. *Let* $\Delta t_0 (< 1)$ *be a fixed positive number,* $\Delta t \in (0, \Delta t_0]$, *and* (u,p) *and* (u_h, p_h) *be solutions of* (11) *and* (18), *respectively, where* u_h^0 *is a first component of the Stokes projection of* $(u^0, 0)$. *Suppose* $u \in C^0([0,T]; V\cap H^2)\cap H^2(0,T;L^2)\cap H^1(0,T;H^1)$ *and* $p \in C^0([0,T]; Q\cap H^1)\cap H^1(0,T;H^2)$. *Then, it holds that*

$$\|u - u_h\|_{l^\infty(L^2)} + \sqrt{\nu}\|u - u_h\|_{l^2(H^1)} + \sqrt{\delta}|p - p_h|_{l^2(M_h)}$$

$$\leq c_2\Big\{\Delta t\big(\|u\|_{H^2(0,T;L^2)}+\|u\|_{H^1(0,T;H^1)}+\|u\|_{L^2(0,T;H^2)}\big)$$
$$+h\Big(\|u^0\|_2+\sqrt{\delta}\big(\nu\|u\|_{l^2(H^2)}+\|p\|_{l^2(H^1)}\big)+\|(u,p)\|_{H^1(0,T;\,H^2\times H^2)}\Big)\Big\}. \quad (48)$$

Proof. Let $(\hat{u}_h,\hat{p}_h)(t)$ be the Stokes projection of $(u,p)(t)\in H^2(\Omega)^d\times H^2(\Omega)$,

$$e_h^n\equiv u_h^n-\hat{u}_h^n,\quad \varepsilon_h^n\equiv p_h^n-\hat{p}_h^n\quad\text{and}\quad \eta_h(t)\equiv(u-\hat{u}_h)(t). \quad (49)$$

For any $(v_h,q_h)\in V_h\times Q_h$, it holds that, from (11) and (18),

$$\begin{aligned}&\langle\mathscr{M}_h(u_h^n,u_h^{n-1};\Delta t,w^{n-1}),v_h\rangle+a(e_h^n,v_h)+b(v_h,\varepsilon_h^n)+b(e_h^n,q_h)+\mathscr{C}_h(\varepsilon_h^n,q_h)\\&=(f^n,v_h)-(-2\nu\nabla D(u^n)+\nabla p^n,v_h)+\delta\sum_K h_K^2(-2\nu\nabla D(u^n)+\nabla p^n,\nabla q_h)_K\\&=\Big(\frac{Du^n}{Dt_w},v_h\Big)+\delta\sum_K h_K^2(-2\nu\nabla D(u^n)+\nabla p^n,\nabla q_h)_K,\end{aligned} \quad (50)$$

and from an identity

$$\mathscr{M}_h(u_h^n,u_h^{n-1};\Delta t,w^{n-1})=\mathscr{M}_h(e_h^n,e_h^{n-1};\Delta t,w^{n-1})-\mathscr{M}_h(\eta_h^n,\eta_h^{n-1};\Delta t,w^{n-1})+\mathscr{M}_h(u^n,u^{n-1};\Delta t,w^{n-1}), \quad (51)$$

we have

$$\begin{aligned}&\langle\mathscr{M}_h(e_h^n,e_h^{n-1};\Delta t,w^{n-1}),v_h\rangle+a(e_h^n,v_h)+b(v_h,\varepsilon_h^n)+b(e_h^n,q_h)+\mathscr{C}_h(\varepsilon_h^n,q_h)\\&=\langle\mathscr{M}_h(\eta_h^n,\eta_h^{n-1};\Delta t,w^{n-1})-\mathscr{M}_h(u^n,u^{n-1};\Delta t,w^{n-1}),v_h\rangle+\Big(\frac{Du^n}{Dt_w},v_h\Big)+\delta\sum_K h_K^2(-2\nu\nabla D(u^n)+\nabla p^n,\nabla q_h)_K\\&=\Big(\frac{\eta_h^n-\eta_h^{n-1}\circ X_1(w^{n-1},\Delta t)}{\Delta t}-\frac{u^n-u^{n-1}\circ X_1(w^{n-1},\Delta t)}{\Delta t}+\frac{Du^n}{Dt_w},v_h\Big)+\delta\sum_K h_K^2(-2\nu\nabla D(u^n)+\nabla p^n,\nabla q_h)_K.\end{aligned} \quad (52)$$

Substituting $(e_h^n,-\varepsilon_h^n)$ into (v_h,q_h) in (52), we have

$$\begin{aligned}&\langle\mathscr{M}_h(e_h^n,e_h^{n-1};\Delta t,w^{n-1}),e_h^n\rangle+a(e_h^n,e_h^n)-\mathscr{C}_h(\varepsilon_h^n,\varepsilon_h^n)\\&=\Big(\frac{\eta_h^n-\eta_h^{n-1}\circ X_1(w^{n-1},\Delta t)}{\Delta t}-\frac{u^n-u^{n-1}\circ X_1(w^{n-1},\Delta t)}{\Delta t}+\frac{Du^n}{Dt_w},e_h^n\Big)-\delta\sum_K h_K^2(-2\nu\nabla D(u^n)+\nabla p^n,\nabla\varepsilon_h^n)_K\\&\equiv I_1^n+I_2^n.\end{aligned} \quad (53)$$

The two terms I_1 and I_2 are evaluated as follows. We have

$$\begin{aligned}\Big\|\frac{\eta_h^n-\eta_h^{n-1}\circ X_1(w^{n-1},\Delta t)}{\Delta t}\Big\|_0&=\Big\|\int_0^1\Big(\frac{\partial\eta_h}{\partial t}+(w^{n-1}(x)\cdot\nabla)\eta_h\Big)(x-sw^{n-1}(x)\Delta t,t^n-s\Delta t)ds\Big\|_0\\&=\Big[\int_\Omega\Big\{\int_0^1\Big(\frac{\partial\eta_h}{\partial t}+(w^{n-1}(x)\cdot\nabla)\eta_h\Big)(x-sw^{n-1}(x)\Delta t,t^n-s\Delta t)ds\Big\}^2dx\Big]^{1/2}\\&\leq\Big\{\int_0^1 ds\int_\Omega\Big(\frac{\partial\eta_h}{\partial t}+(w^{n-1}(x)\cdot\nabla)\eta_h\Big)(x-sw^{n-1}(x)\Delta t,t^n-s\Delta t)^2dx\Big\}^{1/2}\\&\leq c_0\Big[\Big\{\int_0^1 ds\int_\Omega\frac{\partial\eta_h}{\partial t}(x-sw^{n-1}(x)\Delta t,t^n-s\Delta t)^2dx\Big\}^{1/2}\end{aligned}$$

$$
\left. + \left\{ \int_0^1 ds \int_\Omega \nabla\eta_h \big(x - s w^{n-1}(x)\Delta t, t^n - s\Delta t\big)^2 dx \right\}^{1/2} \right]
$$

$$
\le c_1 \left[\left\{ \int_0^1 \left\| \frac{\partial \eta_h}{\partial t}(\cdot, t^n - s\Delta t) \right\|_0^2 ds \right\}^{1/2} + \left\{ \int_0^1 \|\nabla\eta_h(\cdot, t^n - s\Delta t)\|_0^2 ds \right\}^{1/2} \right]
$$

$$
\le \frac{c_1}{\sqrt{\Delta t}} \left[\left\{ \int_{t^{n-1}}^{t^n} \left\| \frac{\partial \eta_h}{\partial t}(\cdot, t) \right\|_0^2 dt \right\}^{1/2} + \left\{ \int_{t^{n-1}}^{t^n} \|\nabla\eta_h(\cdot, t)\|_0^2 \, dt \right\}^{1/2} \right]
$$

$$
= \frac{c_1}{\sqrt{\Delta t}} \left\{ \left\| \frac{\partial \eta_h}{\partial t} \right\|_{L^2(t^{n-1}, t^n; L^2)} + \|\nabla\eta_h\|_{L^2(t^{n-1}, t^n; L^2)} \right\}
$$

$$
\le \frac{c_1 h}{\sqrt{\Delta t}} \left\{ \left\| \left(\frac{\partial u}{\partial t}, \frac{\partial p}{\partial t} \right) \right\|_{L^2(t^{n-1},\, t^n;\, H^2 \times H^2)} + \|(u,p)\|_{L^2(t^{n-1},\, t^n;\, H^2 \times H^2)} \right\}
$$

$$
= \frac{c_1 h}{\sqrt{\Delta t}} \|(u,p)\|_{H^1(t^{n-1},\, t^n;\, H^2 \times H^2)}, \tag{54}
$$

and

$$
\left\| \left(\frac{Du}{Dt_w} \right)^n - \frac{u^n - u^{n-1} \circ X_1(w^{n-1}, \Delta t)}{\Delta t} \right\|_0 \le c_2 \sqrt{\Delta t} \left\{ \|u\|_{H^2(t^{n-1}, t^n; L^2)} + \|u\|_{H^1(t^{n-1}, t^n; H^1)} + \|u\|_{L^2(t^{n-1}, t^n; H^2)} \right\}. \tag{55}
$$

For any $\alpha_0 > 0$, it holds that, from the inequalities (54) and (55),

$$
I_1^n \le \frac{c_2}{\alpha_0} \left\{ \Delta t \left(\|u\|^2_{H^2(t^{n-1}, t^n; L^2)} + \|u\|^2_{H^1(t^{n-1}, t^n; H^1)} + \|u\|^2_{L^2(t^{n-1}, t^n; H^2)} \right) + \frac{h^2}{\Delta t} \|(u,p)\|^2_{H^1(t^{n-1},\, t^n;\, H^2 \times H^2)} \right\} + \alpha_0 \|e_h^n\|_0^2. \tag{56}
$$

For I_2, we have

$$
I_2^n \le \delta \sum_K h_K^2 \|\nabla \varepsilon_h^n\|_{L^2(K)} \left(2\nu \|\nabla D(u^n)\|_{L^2(K)} + \|\nabla p^n\|_{L^2(K)} \right)
$$

$$
\le 2\nu\delta \sum_K h_K^2 \|\nabla \varepsilon_h^n\|_{L^2(K)} \|u^n\|_{H^2(K)} + \delta \sum_K h_K^2 \|\nabla \varepsilon_h^n\|_{L^2(K)} \|\nabla p^n\|_{L^2(K)}
$$

$$
\le \nu\delta \sum_K h_K^2 \left(\alpha_1 \|\nabla \varepsilon_h^n\|^2_{L^2(K)} + \frac{c}{\alpha_1} \|u^n\|^2_{H^2(K)} \right) + \delta \sum_K h_K^2 \left(\alpha_2 \|\nabla \varepsilon_h^n\|^2_{L^2(K)} + \frac{c}{\alpha_2} \|\nabla p^n\|^2_{L^2(K)} \right)
$$

$$
\le (\alpha_1 \nu + \alpha_2) \delta |\varepsilon_h^n|_h^2 + c\delta h^2 \left(\frac{\nu}{\alpha_1} \|u^n\|_2^2 + \frac{1}{\alpha_2} \|p^n\|_1^2 \right). \tag{57}
$$

Combining (56) and (57) with (53), we have, for any positive numbers $\delta, \alpha_0, \alpha_1$ and α_2 and $n = 1, \cdots, N_T$,

$$
\overline{D}_{\Delta t} \left(\frac{1}{2} \|e_h^n\|_0^2 \right) + \nu \|D(e_h^n)\|_0^2 + \delta |\varepsilon_h^n|_h^2
$$

$$
\le \alpha_0 \|e_h^n\|_0^2 + c_1 \|e_h^{n-1}\|_0^2 + (\alpha_1 \nu + \alpha_2) \delta |\varepsilon_h^n|_h^2 + c\delta h^2 \left(\frac{\nu}{\alpha_1} \|u^n\|_2^2 + \frac{1}{\alpha_2} \|p^n\|_1^2 \right)
$$

$$
+ \frac{c_2}{\alpha_0} \left\{ \Delta t \left(\|u\|^2_{H^2(t^{n-1}, t^n; L^2)} + \|u\|^2_{H^1(t^{n-1}, t^n; H^1)} + \|u\|^2_{L^2(t^{n-1}, t^n; H^2)} \right) + \frac{h^2}{\Delta t} \|(u,p)\|^2_{H^1(t^{n-1},\, t^n;\, H^2 \times H^2)} \right\}, \tag{58}
$$

and, then, for $\alpha_0 = 1/(4\Delta t_0)$, $\alpha_1 = 1/(4\nu)$ and $\alpha_2 = 1/4$,

$$
\overline{D}_{\Delta t} \left(\frac{1}{2} \|e_h^n\|_0^2 \right) + \nu \|D(e_h^n)\|_0^2 + \frac{\delta}{2} |\varepsilon_h^n|_h^2
$$

$$\leq \frac{1}{4\Delta t_0}\|e_h^n\|_0^2 + c_1\|e_h^{n-1}\|_0^2 + c\delta h^2\left(\nu^2\|u^n\|_2^2 + \|p^n\|_1^2\right)$$

$$+ c_2\{\Delta t(\|u\|^2_{H^2(t^{n-1},t^n;L^2)} + \|u\|^2_{H^1(t^{n-1},t^n;H^1)} + \|u\|^2_{L^2(t^{n-1},t^n;H^2)}) + \frac{h^2}{\Delta t}\|(u,p)\|^2_{H^1(t^{n-1},\, t^n;\, H^2\times H^2)}\}. \quad (59)$$

By the discrete Gronwall inequality, it holds that

$$\|e_h\|_{l^\infty(L^2)} + \sqrt{\nu}\|D(e_h)\|_{l^2(L^2)} + \sqrt{\delta}|\varepsilon_h|_{l^2(M_h)}$$
$$\leq c_2\Big\{\|e_h^0\|_0 + \Delta t(\|u\|_{H^2(0,T;L^2)} + \|u\|_{H^1(0,T;H^1)} + \|u\|_{L^2(0,T;H^2)})$$
$$+ h(\sqrt{\delta}(\nu\|u\|_{l^2(H^2)} + \|p\|_{l^2(H^1)}) + \|(u,p)\|_{H^1(0,T;\, H^2\times H^2)})\}$$
$$\leq c_2\Big\{\Delta t(\|u\|_{H^2(0,T;L^2)} + \|u\|_{H^1(0,T;H^1)} + \|u\|_{L^2(0,T;H^2)})$$
$$+ h(\|u^0\|_2 + \sqrt{\delta}(\nu\|u\|_{l^2(H^2)} + \|p\|_{l^2(H^1)}) + \|(u,p)\|_{H^1(0,T;\, H^2\times H^2)})\Big\}, \quad (60)$$

which implies (48). □

4 Conclusions

We have introduced a characteristics finite element scheme for a linearized Navier-Stokes equations. The scheme employs P1/P1 element, and the coefficient matrix appearing in the scheme is symmetric. These advantages reduces computational time and cost by half. Therefore, the scheme is useful especially for three dimensional computation. We have shown stability and convergence results with the optimal L^2-error estimate for the velocity. Although the system of the equations is linear, the analysis is useful even in the nonlinear case. We note that the convergence of the pressure can be proved under some assumptions.

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