Analysis of a pressure-stabilized characteristics finite element scheme for a linearized Navier-Stokes equations (Mathematical Analysis in Fluid and Gas Dynamics)

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Analysis of a pressure-stabilized characteristics finite element scheme for a linearized Navier-Stokes equations

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1 Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^d (d = 2, 3)$, $T$ be a positive constant, $u: \Omega \times (0, T) \rightarrow \mathbb{R}^d$ be a velocity and $\phi: \Omega \times (0, T) \rightarrow \mathbb{R}$ be a scalar function. We consider a trajectory of fluid particle, which is important for flow problems. Let $X = X(\cdot; x, t^n): (0, T) \rightarrow \mathbb{R}^d$ be a solution of the ordinary differential equation,

$$\frac{dX}{dt} = u(X, t) \tag{1}$$

with a condition $X(t^n) = x$, where $n \in \mathbb{N} \cup \{0\}$, $\Delta t$ is a time increment and $t^n \equiv n\Delta t$. Then, it holds that

$$D\phi = \frac{d}{dt} \phi(X(t), t), \tag{2}$$

where

$$D \equiv \frac{\partial}{\partial t} + u \cdot \nabla \tag{3}$$

is a material derivation. Therefore, we can consider a first order approximation of the material derivative at $t = t^n (n \geq 1)$ as follows;

$$D\phi \approx \phi(X(t^n), t^n) - \Delta \phi \left( X(t^{n-1}), t^{n-1} \right) \Delta t$$

$$= \frac{\phi^n - \psi^{n-1} \circ X_1(u^n, \Delta t)}{\Delta t}(x), \tag{4}$$

where we have used notations, $\phi^n \equiv \phi(\cdot, t^n)$, for $w: \Omega \rightarrow \mathbb{R}^d$,

$$X_1(w, \Delta t)(x) \equiv x - w(x)\Delta t, \tag{5}$$

and, for $\psi: \Omega \rightarrow \mathbb{R}$,

$$\psi \circ X_1(w, \Delta t)(x) \equiv \psi(X_1(w, \Delta t)(x)). \tag{6}$$

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The approximation (4) of $D\phi/Dt$ is a basic idea to devise numerical schemes based on the method of characteristics. The idea can be combined with finite element and difference methods (e.g., see [5, 10, 12, 13, 15]). In this paper, we introduce a finite element scheme based on the method of characteristics for a linearized Navier-Stokes equations, and give stability and convergence results of the scheme. Although the system of the equations is linear and simpler than one of the Navier-Stokes equations, the results to be shown are useful for an analysis of a scheme for the Navier-Stokes equations [9, 11].

2 Statement of the problem and a characteristics finite element scheme

We consider a linearized Navier-Stokes problem; find $(u, p) : \Omega \times (0, T) \rightarrow \mathbb{R}^d \times \mathbb{R}$ such that

\[
\begin{array}{l}
\frac{Du}{Dt} - \nabla(2\nu D(u)) + \nabla p = f, \quad \text{in } \Omega \times (0, T), \\
\nabla \cdot u = 0, \quad \text{in } \Omega \times (0, T), \\
u = 0, \quad \text{on } \Gamma \times (0, T), \\
u = u^0, \quad \text{in } \Omega, \quad \text{at } t = 0,
\end{array}
\]

(7)

where $\nu$ is the velocity, $p$ is the pressure, $f : \Omega \times (0, T) \rightarrow \mathbb{R}^d$ is a given external force, $u^0 : \Omega \rightarrow \mathbb{R}^d$ is a given initial velocity, $\nu(>0)$ is a viscosity, $D(u)$ is a strain-rate tensor defined by

\[
D_{ij}(u) \equiv \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad (i, j = 1, \ldots, d),
\]

(8)

$D/Dt$ is a material derivation defined by

\[
\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + w \cdot \nabla
\]

(9)

and $w : \Omega \times (0, T) \rightarrow \mathbb{R}^d$ is a given velocity. If $w$ is replaced by $u$, (7) is the Navier-Stokes problem.

We use the same notation $(\cdot, \cdot)$ to represent $L^2(\Omega)$ inner product for scalar-, vector- and matrix-valued functions, and define a bilinear forms $a$ on $H^1(\Omega)^d \times H^1(\Omega)^d$ and $b$ on $H^1(\Omega)^d \times L^2(\Omega)$ by

\[
a(u, v) \equiv 2\nu(D(u), D(v)), \quad \text{and} \quad b(v, q) \equiv -(\nabla \cdot v, q),
\]

(10)

respectively. The weak form of the problem (7) is written as follows; find $\{(u, p)(t)\}_{t \in (0, T)} \subset V \times Q$ such that

\[
\begin{array}{l}
\left( \frac{Du}{Dt}, v \right) + a(u, v) + b(v, p) + b(u, q) = (f, v), \quad \forall (v, q) \in V \times Q, \\
u(0) = u^0,
\end{array}
\]

(11)

where $V \equiv H^1_0(\Omega)^d$ and $Q \equiv L^2(\Omega)$.

We state a characteristics finite element scheme for (7). Let $\mathcal{T}_h = \{K\}$ be a triangulation,

\[
\Omega_h \equiv \text{int} \left( \bigcup_{K \in \mathcal{T}_h} K \right) \quad \text{and} \quad \Gamma_h \equiv \partial \Omega_h.
\]

(12)

We define function spaces $X_h, M_h, V_h$ and $Q_h$ by

\[
X_h \equiv \{ v_h \in C^0(\bar{\Omega}_h)^d ; \ v_h|_K \in P_1(K)^d, \ \forall K \in \mathcal{T}_h \},
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\]

(13)
\[ M_h \equiv \{ q_h \in C^0(\Omega_h); \ q_h|_K \in P_1(K), \ \forall K \in \mathcal{T}_h \}, \]
\[ V_h \equiv X_h \cap H^1_0(\Omega) \quad \text{and} \quad Q_h \equiv M_h \cap L^2_0(\Omega_h), \]
respectively, where \( P_1(K) \) is a function space of piecewise linear functions on an element \( K \in \mathcal{T}_h \), \( P \) means a nodal point on \( \Gamma_h \), and \( L^2_0(\Omega) \) is a subspace of \( L^2(\Omega) \) defined by
\[ L^2_0(\Omega) \equiv \{ q \in L^2(\Omega); \ (q, 1) = 0 \}. \]

Let \( N_T \equiv [T/\Delta t] \) be a total number of time steps, \( \delta \) be a positive constant, \( h_K \) be a diameter of \( K \) and \( (\cdot, \cdot)_K \) be an inner product in \( L^2(K)^d \). For \( u, \tilde{u} \) and \( w \in H^1(\Omega)^d \), we define a linear form \( \mathcal{M}_h(u, \tilde{u}; \Delta t, w) \) on \( V_h \) and a bilinear form \( \mathcal{C}_h \) on \( H^1(\Omega) \times H^1(\Omega) \) by
\[ \langle \mathcal{M}_h(u, \tilde{u}; \Delta t, w), v_h \rangle \equiv \left( \frac{u - \tilde{u} o X_{1}(w, t)}{\Delta t}, v_h \right) \quad \text{and} \quad \mathcal{C}_h(p, q) \equiv -\delta \sum_{K \in \mathcal{T}_h} h_K^2 (\nabla p, \nabla q)_K, \]
respectively. Let an approximate function \( u_h^0 \in V_h \) of \( u^0 \) be given. A pressure-stabilized characteristics finite element scheme for (7) is to find \( (u_h^n, p_h^n)_{n=1}^{N_T} \subset V_h \times Q_h \) such that, for \( n = 1, \cdots, N_T \),
\[ \langle \mathcal{M}_h(u_h^n, u_h^{n-1}; \Delta t, w^{n-1}), v_h \rangle + a(u_h^n, v_h) + b(v_h, p_h^n) + b(u_h^n, q_h) + \mathcal{C}_h(p_h^n, q_h) = (f^n, v_h), \quad \forall (v_h, q_h) \in V_h \times Q_h, \]

(18)
The scheme (18) can deal with high Reynolds number (small viscosity) problems by the method of characteristics. The material derivative term \( (Du/Dt, v) \) in (11) is approximated by \( \langle \mathcal{M}_h(u_h^n, u_h^{n-1}; \Delta t, w^{n-1}), v_h \rangle \). When we find \( (u_h^n, p_h^n) \) in the scheme (18), a composite function \( u_h^{n-1} o X_{1}(w_h^{n-1}, \Delta t) \) is a known function and a coefficient matrix of the system of the linear equations is symmetric. The advantage enables us to use symmetric linear iterative solvers, i.e., CG, CR, MINRES [2]. Since the coefficient matrix is independent of step number \( n \), it is enough to make the matrix at only the first time step. The scheme employs a cheap element \( P1/P1 \), it is useful for large scale computation, especially in 3D. Although \( P1/P1 \) element does not satisfy the inf-sup condition [8], the scheme works by a pressure-stabilization term \( \mathcal{C}_h \).

We impose assumption for a given velocity \( w \) and review a proposition in [15].

**Hypothesis 2.1.** A function \( w \) satisfies
\[ \begin{cases} w \in C^0([0, T]; W^{1,\infty}(\Omega)), \\ w = 0 \ \text{on} \ \Gamma, \\ \text{div} w = 0 \ \text{in} \ \Omega. \end{cases} \]

(19)

**Proposition 2.2** ([15]). Under Hypothesis 2.1 and an inequality:
\[ \Delta t < \frac{1}{\|w\|_{C^0(W^{1,\infty})}}, \]
(20)
it holds that, for any \( t \in [0, T] \),
\[ X_1(w(\cdot, t), \Delta t)(\Omega) = \Omega. \]
(21)
In the following sections, we assume that Hypothesis 2.1 and the inequality (20) hold, and that, for the sake of simplicity, \( \Omega \) is convex polygonal domain (\( \Omega = \Omega_h \)).
3 Stability and convergence

In this section, we consider stability and convergence of the scheme (18). We use $c$ to represent the generic positive constant independent of discretization parameters and solutions, which can take different values at different places, and $c(A)$ is a positive constant, which depends on $A$. Constants $c_0, c_1$ and $c_2$ have particular meanings in this paper,

\[ c_0 = c_0(\|w\|_{C^1(L^\infty)}), \quad c_1 = c_1(\|w\|_{C^0(W^{1,m})}) \quad \text{and} \quad c_2 = c_2(\|w\|_{C^0(W^{1,m})} \cap C^1(L^\infty)), \]

respectively.

3.1 Stability

This subsection is devoted to the stability of the scheme (18). We use norms and seminorms, $\|\cdot\|_k \equiv \|\cdot\|_{H^k(\Omega)} (k=0,1,2)$, $\|\cdot\|_V \equiv \|\cdot\|_1$, $\|\cdot\|_Q \equiv \|\cdot\|_0$, $\|(v,q)\|_{V \times Q} \equiv \{\|v\|_V^2 + \|q\|_Q^2\}^{1/2}$, $|q|_h \equiv \{\sum_{K \in \mathcal{T}_h} h_K^2 (\nabla q, \nabla q)_K\}^{1/2}$, $\|u\|_{L^\infty(L^2)} \equiv \max_{n} \|u^n\|_0$, $\|u\|_{L^\infty(H^1)} \equiv \max_{n} \|u^n\|_1$, $\|u\|_{L^\infty(X)} \equiv \|u\|_{L^\infty(X)} (X=L^2, H^1)$, $\|u\|_{L^2(X)} \equiv \|u\|_{L^2(X)} (X=L^2, H^1)$, $|p|_{L^2(M_h)} \equiv \{\sum_{n=m}^{N_7} \Delta t \|p^n\|_h^2\}^{1/2}$ and $|p|_{L^2(M_h)} \equiv |p|_{L^2(M_h)}$.

Setting a bilinear form $d_h$ on $(V_h \times Q_h) \times (V_h \times Q_h)$ by

\[ d_h((u,p);(v,q)) \equiv a(u,v) + b(v,p) + b(u,q) + \mathcal{C}_h(p,q), \]

we have another representation of the scheme (18);

\[ \langle \mathcal{M}_h(u_h^n, u_h^{n-1}; \Delta t, w^{n-1}), v_h \rangle + d_h((u_h^n, p_h^n); (v_h, q_h)) = (f^n, v_h), \quad \forall (v_h, q_h) \in V_h \times Q_h. \]

We prepare two lemmas for the stability. First one is an estimate of a composite function appearing $\mathcal{M}_h$;

Lemma 3.1. For any $v \in L^2(\Omega)^d$ and $t \in [0, T]$, it holds that

\[ \|v \circ X_1(w(\cdot,t), \Delta t)\|_0 \leq (1 + c_1 \Delta t) \|v\|_0. \]

We omit the proof, because it is similar to the proof of Lemma 1 in [15].

P1/P1 element is employed in the scheme (18), i.e., both finite dimensional function spaces $V_h$ and $Q_h$ consist of P1 element, and P1/P1 element does not satisfy the inf-sup condition [8];

\[ \inf_{q_h \in Q_h} \sup_{v_h \in V_h} \frac{b(v_h, q_h)}{\|v_h\|_V \|q_h\|_Q} \geq \beta^* > 0. \]

However, $a$, $b$ and $\mathcal{C}_h$ satisfies the following inf-sup condition.

Lemma 3.2 ([11]). There is a positive constant $\gamma$ such that

\[ \inf_{(u_h, p_h) \in V_h \times Q_h} \sup_{(v_h, q_h) \in V_h \times Q_h} \frac{\mathcal{A}_h((u_h, p_h); (v_h, q_h))}{\|v_h\|_V \|q_h\|_Q} \geq \gamma. \]
From the above lemmas, we obtain a stability result.

**Theorem 3.3** (stability). Let $\Delta t_0 (< 1)$ be a fixed number. Assume $\Delta t \in (0, \Delta t_0]$.

(i) It holds that

$$
\|u_h\|_{L^{\infty}(L^2)} + \sqrt{\delta}\|p_h\|_{L^1(M_h)} + \sqrt{\delta}\|p_h\|_{L^2(H^1)} \leq c_1(\|u_0\|_0 + \|f\|_{L^2(L^2)}).
$$

(ii) The following two inequalities hold:

$$
\sqrt{\delta}\|u_h\|_{L^1(H^1)} + \sqrt{\delta}\|p_h\|_{L^1(M_h)} + \|\overline{D}\Delta u_h\|_{L^1(L^2)} + \sqrt{\delta}\|D\Delta u_h\|_{L^1(L^2)} + \sqrt{\delta}\|D\Delta p_h\|_{L^1(M_h)} \\
\leq c_1(\|u_0\|_0 + \|p_0\|_0 + \|f\|_{L^2(L^2)}).
$$

Proof. First we prove (i). Substituting $(u_h^{n}, -p_h^{n}) \in V_h \times Q_h$ into $(v_h, q_h)$ in the scheme (18), we have

$$
\langle \mathcal{M}_h(u_h^{n}, u_h^{n-1}; \Delta t, u^{n-1}), u_h^{n} \rangle + a(u_h^{n}, u_h^{n}) + \mathcal{C}_h(p_h^{n}, -p_h^{n}) = (f^n, u_h^{n}).
$$

We evaluate the four terms;

$$
\langle \mathcal{M}_h(u_h^{n}, u_h^{n-1}; \Delta t, u^{n-1}), u_h^{n} \rangle = \frac{1}{\Delta t} \left\{ \frac{1}{2} (\|u_h^{n}\|_0^2 - \|u_h^{n-1}\|_0^2) + \frac{1}{2} \|u_h^{n} - u_h^{n-1}\|_0^2 \right\}
$$

$$
\geq \frac{1}{\Delta t} \left( \frac{1}{2} \|u_h^{n}\|_0^2 - \frac{1}{2} \|u_h^{n-1}\|_0^2 \right) - c_1 \|u_h^{n-1}\|_0^2 \quad (\because \text{Lemma 3.1})
$$

$$
= \overline{D}\Delta \left( \frac{1}{2} \|u_h^n\|_0^2 \right) - c_1 \|u_h^{n-1}\|_0^2,
$$

$$
a(u_h^n, u_h^n) = 2\nu \|D(u_h^n)\|_0^2 \geq c\nu \|u_h^n\|_1^2 \quad (\because \text{Korn's inequality} [6]),
$$

$$
\mathcal{C}_h(p_h^n, -p_h^n) = \delta \|p_h^n\|_h^2
$$

and

$$
(f^n, u_h^n) \leq \frac{1}{2} (\|f^n\|_0^2 + \|u_h^n\|_0^2) \quad (\because ab \leq (a^2 + b^2)/2),
$$

where $\overline{D}\Delta a^n \equiv (a^n - a^{n-1})/\Delta t$. The inequalities (33) implies

$$
\overline{D}\Delta \left( \frac{1}{2} \|u_h^n\|_0^2 \right) + c\nu \|u_h^n\|_1^2 + \delta \|p_h^n\|_h^2 \leq \frac{1}{2} (\|f^n\|_0^2 + \|u_h^n\|_0^2) \quad (n = 1, \cdots, N_T),
$$

and (29) holds from the discrete Gronwall lemma [20].

Next, we prove (30) of (ii). Substituting $0 \in V_h$ into $v_h$ in the scheme (18), we have

$$
b(u_h^n, q_h) + \mathcal{C}_h(p_h^n, q_h) = 0 \quad (\forall q_h \in Q_h, n = 1, \cdots, N_T),
$$

and, then,

$$
b(\overline{D}\Delta u_h^n, q_h) + \mathcal{C}_h(\overline{D}\Delta p_h^n, q_h) = 0 \quad (\forall q_h \in Q_h, n = 2, \cdots, N_T).
$$

Substituting $(\overline{D}\Delta u_h^n, 0) \in V_h \times Q_h$ into $(v_h, q_h)$ in the scheme (18), we have

$$
\langle \mathcal{M}_h(u_h^n, u_h^{n-1}; \Delta t, w^{n-1}), \overline{D}\Delta u_h^n \rangle + a(u_h^n, \overline{D}\Delta u_h^n) + b(\overline{D}\Delta u_h^n, p_h^n) = (f^n, \overline{D}\Delta u_h^n) \quad (n = 1, \cdots, N_T),
$$

(37)
and, by (36),
\[ \langle \mathcal{A}_h(u_h^n, u_h^{n-1}; : \Delta t, w^{n-1}), \overline{D}_N u_h^n \rangle + a(u_h^n, \overline{D}_N u_h^n) - \mathcal{C}_h(\overline{D}_N p_h^n, p_h^n) = (f^n, \overline{D}_N u_h^n) \quad (n = 2, \ldots, N_T). \]  

We evaluate the four terms in (38);
\[ \langle \mathcal{A}_h(u_h^n, u_h^{n-1}; : \Delta t, w^{n-1}), \overline{D}_N u_h^n \rangle = \left( \overline{D}_N u_h^n + \frac{1}{\Delta t}(u_h^{n-1} - u_h^{n-1} \circ X_1(w^{n-1}, \Delta t)), \overline{D}_N u_h^n \right) \]
\[ \geq \vert \overline{D}_N u_h^n \vert_0^2 - \frac{1}{\Delta t} \vert u_h^{n-1} - u_h^{n-1} \circ X_1(w^{n-1}, \Delta t) \vert_0 \vert \overline{D}_N u_h^n \vert_0 \]
\[ \geq \frac{3}{4} \vert \overline{D}_N u_h^n \vert_0^2 - c_0 \vert D(u_h^{n-1}) \vert_0^2 \]  

\[ a(u_h^n, \overline{D}_N u_h^n) = \overline{D}_N \left( \frac{1}{2} a(u_h^n, u_h^n) \right) + \frac{\Delta t}{2} a(\overline{D}_N u_h^n, \overline{D}_N u_h^n) \]
\[ = \overline{D}_N \left( v \vert D(u_h^n) \vert_0^2 + v^2 \Delta t \vert \overline{D}_N u_h^n \vert_0^2 \right), \]
\[ - \mathcal{C}_h(\overline{D}_N p_h^n, p_h^n) = \overline{D}_N \left( \frac{1}{2} \mathcal{C}_h(p_h^n, p_h^n) \right) - \frac{\Delta t}{2} \mathcal{C}_h(\overline{D}_N p_h^n, \overline{D}_N p_h^n) \]
\[ = \overline{D}_N \left( \frac{\delta}{2} \vert p_h^n \vert_h^2 \right) + \frac{\delta \Delta t}{2} \vert \overline{D}_N p_h^n \vert_h^2, \]

and
\[ (f^n, \overline{D}_N u_h^n) = \vert f^n \vert_0^2 + \frac{1}{4} \vert \overline{D}_N u_h^n \vert_0^2. \]

Combining (39) with (38), we have
\[ \overline{D}_N \left( v \vert D(u_h^n) \vert_0^2 + \frac{\delta}{2} \vert p_h^n \vert_h^2 \right) + \frac{1}{2} \vert \overline{D}_N u_h^n \vert_0^2 + v^2 \Delta t \vert \overline{D}_N u_h^n \vert_0^2 + \frac{\delta \Delta t}{2} \vert \overline{D}_N p_h^n \vert_0^2 \]
\[ \leq \vert f^n \vert_0^2 + c_0 \vert D(u_h^{n-1}) \vert_0^2 \quad (n = 2, \ldots, N_T), \]

which implies (30) by the discrete Gronwall lemma.

Finally we prove (31) of (ii). From Lemma 3.2, \( \|p_h^n\|_0 \) is evaluated as follows;
\[ \|p_h^n\|_0 \leq \|(u_h^n, p_h^n)\|_{V \times Q} \]
\[ \leq \frac{1}{\gamma} \sup_{(v_h, q_h) \in V_h \times Q_h} \frac{(f^n, v_h) - \langle \mathcal{A}_h(u_h^n, u_h^{n-1}; : \Delta t, w^{n-1}), v_h \rangle}{\|(v_h, q_h)\|_{V \times Q}} \]
\[ \leq \frac{1}{\gamma} \left( \|f^n\|_V + \|\mathcal{A}_h(u_h^n, u_h^{n-1}; : \Delta t, w^{n-1})\|_V \right) \]
\[ \leq \frac{1}{\gamma} \left( \|f^n\|_0 + \|\overline{D}_N u_h^n\|_0 + c_0 \vert u_h^{n-1} \vert_1 \right), \]
where we have used the following inequality in the last inequality of (41);

\[
\|A_{h}(u_{h}^{n-1};\Delta t,w^{n-1})\|_{V_{h}'} \leq \frac{1}{\Delta t} \|u_{h}^{n} - u_{h}^{n-1} \circ X_{1}(w^{n-1},\Delta t)\|_{0}
\]

\[
= \|\bar{D}_{\Delta t}u_{h}^{n}\| + \frac{1}{\Delta t} \|u_{h}^{n-1} - u_{h}^{n-1} \circ X_{1}(w^{n-1},\Delta t)\|_{0}
\]

\[
\leq \|\bar{D}_{\Delta t}u_{h}^{n}\|_{0} + \frac{1}{\Delta t} \|u_{h}^{n-1} - u_{h}^{n-1} \circ X_{1}(w^{n-1},\Delta t)\|_{0}
\]

\[
\leq \|\bar{D}_{N}u_{h}^{n}\|_{0} + \frac{1}{\Delta t} \|u_{h}^{n-1} - u_{h}^{n-1} \circ X_{1}(w^{n-1},\Delta t)\|_{0}
\]

\[
\leq \|\bar{D}_{N}u_{h}^{n}\|_{0} + c_{0}|u_{h}^{n-1}|_{1}.
\]

(42)

Combining (41) with (30), we have

\[
\|p_{h}\|_{L_{(2)}^{2}(L^{2})} \leq c_{0}(\|f\|_{L_{(2)}^{2}(L^{2})} + \|\bar{D}_{\Delta t}u_{h}^{n}\|_{L_{(2)}^{2}(L^{2})} + \|u_{h}\|_{L_{(1)}^{2}(H^{1})})
\]

\[
\leq c_{1}(\|u_{h}^{1}\|_{1} + |p_{h}^{1}|_{h} + \|f\|_{L_{(2)}^{2}(L^{2})})
\]

(43)

Remark 3.4. In the right hand side of (31), \(\|u_{h}^{1}\|_{1}\) can be estimated by (30), and, for \(|p_{h}^{1}|_{h}\), a detailed evaluation is required.

3.2 Convergence

This subsection is devoted to convergence of the scheme (18). At first we define a Stokes projection.

**Definition 3.5 (Stokes projection).** For \((u,p) \in (V \cap H^{2}(\Omega)^{d}) \times (Q \cap H^{1}(\Omega))\), \((w_{h},r_{h}) \in V_{h} \times Q_{h}\) is a Stokes projection of \((u,p)\) valued

\[
a(w_{h},v_{h}) + b(v_{h},r_{h}) + b(w_{h},q_{h}) + \mathcal{E}_{h}(r_{h},q_{h}) = \langle \mathcal{G}_{h}, (v_{h},q_{h}) \rangle,
\]

\[
\forall (v_{h},q_{h}) \in V_{h} \times Q_{h},
\]

(44)

where \(\mathcal{E}_{h}\) is a linear form on \(V_{h} \times Q_{h}\) defined by

\[
\langle \mathcal{E}_{h}(v_{h},q_{h}) \rangle \equiv (g,v_{h}) - \delta \sum_{K \in \mathcal{H}_{h}} h_{K}^{2}(g, \nabla q_{h})_{K} \quad \text{with} \quad g \equiv -2\nu \nabla D(u) + \nabla p.
\]

(45)

For the Stokes projection, the following error estimate holds.

**Proposition 3.6.** Suppose \((u,p) \in (V \cap H^{2}(\Omega)^{d}) \times (Q \cap H^{2}(\Omega))\), and \(u\) satisfies

\[
b(u,q) = 0 \quad (q \in Q).
\]

(46)

Let \((w_{h},r_{h})\) be a Stokes projection of \((u,p)\) by (44). Then, there exists a positive constant \(c_{s}\), independent of \(h\), such that

\[
\|u - w_{h}\|_{L_{(2)}^{2}(L^{2})} + \sqrt{\delta} \|p - r_{h}\|_{L_{(2)}^{2}(L^{2})} \leq c_{s} h(\|u\|_{2} + h\|p\|_{2}).
\]

(47)

We omit the proof, because papers [3] and [7] give the result.

**Theorem 3.7.** Let \(\Delta t_{0}(< 1)\) be a fixed positive number, \(\Delta \in (0,\Delta t_{0}]\), and \((u,p)\) and \((u_{h},p_{h})\) be solutions of (11) and (18), respectively, where \(u_{h}^{0}\) is a first component of the Stokes projection of \((u^{0},0)\). Suppose \(u \in C^{0}([0,T];V \cap H^{2}) \cap H^{2}(0,T;L^{2}) \cap H^{1}(0,T;H^{1})\) and \(p \in C^{0}([0,T];Q \cap H^{1}) \cap H^{1}(0,T;H^{2})\). Then, it holds that

\[
\|u - u_{h}\|_{L_{(2)}^{2}(L^{2})} + \sqrt{\delta} \|u - u_{h}\|_{L_{(2)}^{2}(H^{1})} + \sqrt{\delta} |p - p_{h}|_{L_{(2)}^{2}(M_{h})}
\]

\[
\leq c_{1}(\|u\|_{2} + h\|p\|_{2}).
\]
\[ \leq C_2 \left( \|u\|_{H^2(0,T;L^2)} + \|u\|_{H^1(0,T;H^1)} + \|u\|_{L^2(0,T;H^2)} \right) \\
+ h \left( \|u^0\|_2 + \sqrt{\delta} (\|v\|_{H^2(\Omega)} + \|p\|_{H^1(\Omega)}) + \|(u,p)\|_{H^1(0,T;H^2 \times H^2)} \right). \]  
(48)

**Proof.** Let \((\hat{u}_h, \hat{p}_h)(t)\) be the Stokes projection of \((u, p)(t) \in H^2(\Omega)^d \times H^2(\Omega),\)

\[ e_h^n \equiv u_h^n - \hat{u}_h^{n}, \quad e_h^n \equiv p_h^n - \hat{p}_h^n \quad \text{and} \quad \eta_h(t) \equiv (u - \hat{u}_h)(t), \]

For any \((v_h, q_h) \in V_h \times Q_h\), it holds that, from (11) and (18),

\[ \langle \mathcal{M}_h(u_h^{n}, u_h^{n-1}; \Delta t, w^{n-1}), v_h \rangle + a(e_h^{n}, v_h) + b(v_h, e_h^{n}) + b(e_h^{n}, q_h) + \mathscr{C}_h(e_h^{n}, q_h) \]

\[ = (f^n, v_h) - (-2\nu \nabla D(u^n) + \nabla p^n, v_h) + \delta \sum_K h_K^2 (-2\nu \nabla D(u^n) + \nabla p^n, \nabla q_h)_K \]

\[ = \left( \frac{Du^n}{D\nu_h^n}, v_h \right) + \delta \sum_K h_K^2 (-2\nu \nabla D(u^n) + \nabla p^n, \nabla q_h)_K. \]

(50)

and from an identity

\[ \mathcal{M}_h(u_h^{n}, u_h^{n-1}; \Delta t, w^{n-1}) = \mathcal{M}_h(e_h^{n}, e_h^{n-1}; \Delta t, w^{n-1}) - \mathcal{M}_h(\eta_h^{n}, \eta_h^{n-1}; \Delta t, w^{n-1}) + \mathcal{M}_h(u^n, u^{n-1}; \Delta t, w^{n-1}), \]

(51)

we have

\[ \langle \mathcal{M}_h(e_h^{n}, e_h^{n-1}; \Delta t, w^{n-1}), v_h \rangle + a(e_h^{n}, v_h) + b(v_h, e_h^{n}) + b(e_h^{n}, q_h) + \mathcal{M}_h(e_h^{n}, q_h) \]

\[ = \langle \mathcal{M}_h(\eta_h^{n}, \eta_h^{n-1}; \Delta t, w^{n-1}) - \mathcal{M}_h(u^n, u^{n-1}; \Delta t, w^{n-1}), v_h \rangle + \left( \frac{Du^n}{D\nu_h^n}, v_h \right) + \delta \sum_K h_K^2 (-2\nu \nabla D(u^n) + \nabla p^n, \nabla q_h)_K \]

\[ = \left( \frac{\eta_h^n - \eta_h^{n-1} \circ X_1(w^{n-1}, \Delta t)}{\Delta t} - \frac{u^n - u^{n-1} \circ X_1(w^{n-1}, \Delta t)}{\Delta t} + \frac{Du^n}{D\nu_h^n} , v_h \right) + \delta \sum_K h_K^2 (-2\nu \nabla D(u^n) + \nabla p^n, \nabla e_h^n)_K. \]

(52)

Substituting \((e_h^{n}, e_h^{n-1})\) into \((v_h, q_h)\) in (52), we have

\[ \langle \mathcal{M}_h(e_h^{n}, e_h^{n-1}; \Delta t, w^{n-1}), e_h^{n} \rangle + a(e_h^{n}, e_h^{n}) - \mathcal{M}_h(e_h^{n}, e_h^{n}) \]

\[ = \left( \frac{\eta_h^n - \eta_h^{n-1} \circ X_1(w^{n-1}, \Delta t)}{\Delta t} - \frac{u^n - u^{n-1} \circ X_1(w^{n-1}, \Delta t)}{\Delta t} + \frac{Du^n}{D\nu_h^n} , e_h^{n} \right) - \delta \sum_K h_K^2 (-2\nu \nabla D(u^n) + \nabla p^n, \nabla e_h^n)_K \]

\[ \equiv I_1^n + I_2^n. \]

(53)

The two terms \(I_1\) and \(I_2\) are evaluated as follows. We have

\[ \left\| \frac{\eta_h^n - \eta_h^{n-1} \circ X_1(w^{n-1}, \Delta t)}{\Delta t} \right\|_0 \]

\[ = \left\| \int_0^1 \left( \frac{\partial \eta_h}{\partial t} + (w^{n-1}(x) \cdot \nabla) \eta_h \right)(x - sw^{n-1}(x) \Delta t, t^n - s \Delta t) ds \right\|_0 \]

\[ = \left\| \int_\Omega \left\{ \int_0^1 \left( \frac{\partial \eta_h}{\partial t} + (w^{n-1}(x) \cdot \nabla) \eta_h \right)(x - sw^{n-1}(x) \Delta t, t^n - s \Delta t) ds \right\}^2 dx \right\}^{1/2} \]

\[ \leq \left\{ \int_0^1 ds \int_\Omega \left( \frac{\partial \eta_h}{\partial t} + (w^{n-1}(x) \cdot \nabla) \eta_h \right)(x - sw^{n-1}(x) \Delta t, t^n - s \Delta t)^2 dx \right\}^{1/2} \]

\[ \leq C_0 \left\{ \int_0^1 ds \int_\Omega \left( \frac{\partial \eta_h}{\partial t}(x - sw^{n-1}(x) \Delta t, t^n - s \Delta t)^2 dx \right) \right\}^{1/2} \]

(54)
\[ + \left( \int_0^1 ds \int_{\Omega} \nabla \eta_h(x - s w^{n-1}(x) \Delta t, t^n - s \Delta t)^2 \, dx \right)^{1/2} \]
\[ \leq c_1 \left[ \left( \int_0^1 \left\| \frac{\partial \eta_h}{\partial t}(\cdot, t^n - s \Delta t) \right\|_{0}^2 \, ds \right)^{1/2} + \left( \int_{n-1}^n \left\| \nabla \eta_h(\cdot, t) \right\|_{0}^2 \, dt \right)^{1/2} \right] \]
\[ \leq \frac{c_1}{\sqrt{\Delta t}} \left[ \left( \int_{t^{n-1}}^{t^n} \left\| \frac{\partial \eta_h}{\partial t}(\cdot, t) \right\|_{0}^2 \, dt \right)^{1/2} + \left( \int_{t^{n-1}}^{t^n} \left\| \nabla \eta_h(\cdot, t) \right\|_{0}^2 \, dt \right)^{1/2} \right] \]
\[ = \frac{c_1}{\sqrt{\Delta t}} \left( \Vert \frac{\partial \eta_h}{\partial t} \Vert_{L^2(t^{n-1}, t^n; L^2)} + \Vert \nabla \eta_h \Vert_{L^2(t^{n-1}, t^n; L^2)} \right) \]
\[ \leq \frac{c_1 h}{\sqrt{\Delta t}} \left( \Vert \frac{\partial u}{\partial t}, \frac{\partial p}{\partial t} \Vert_{L^2(t^{n-1}, t^n; H^2 \times H^2)} + \Vert (u, p) \Vert_{L^2(t^{n-1}, t^n; H^2 \times H^2)} \right) \]

(54)

and

\[ \left\| \frac{D u}{D t^n} - \frac{u^n - u^{n-1} \circ X_1(w^{n-1}, \Delta t)}{\Delta t} \right\|_0 \leq c_1 \sqrt{\Delta t} \left( \Vert u \Vert_{H^2(t^{n-1}, t^n; L^2)} + \Vert u \Vert_{L^2(t^{n-1}, t^n; H^1)} + \Vert u \Vert_{H^1(t^{n-1}, t^n; H^2)} \right). \]

(55)

For any \( \alpha_0 > 0 \), it holds that, from the inequalities (54) and (55),

\[ I_1^n \leq \frac{c_2}{\alpha_0} \left\{ \Delta t \left( \Vert u \Vert_{H^2(t^{n-1}, t^n; L^2)} + \Vert u \Vert_{L^2(t^{n-1}, t^n; H^1)} + \Vert u \Vert_{H^1(t^{n-1}, t^n; H^2)} \right) \right\} + \alpha_0 \epsilon_h^n \| \theta \|_0^2. \]

(56)

For \( I_2 \), we have

\[ I_2^n \leq \delta \sum_K h_K^2 \| \nabla e_h^{n-1} \|_{L^2(K)} \left( 2 \| \nabla D(u^n) \|_{L^2(K)} + \| \nabla p^n \|_{L^2(K)} \right) \]
\[ \leq 2 \delta \sum_K h_K^2 \| \nabla e_h^{n-1} \|_{L^2(K)} \| u^n \|_{H^2(K)} + \sum_K h_K^2 \| \nabla e_h^{n-1} \|_{L^2(K)} \| \nabla p^n \|_{L^2(K)} \]
\[ \leq \delta \sum_K h_K^2 \left( \alpha_1 \| u^n \|_{H^2(K)}^2 + \frac{c}{\alpha_1} \| u^n \|_{H^1(K)}^2 \right) + \sum_K h_K^2 \left( \alpha_2 \| \nabla e_h^{n-1} \|_{L^2(K)}^2 + \frac{c}{\alpha_2} \| \nabla p^n \|_{L^2(K)}^2 \right) \]
\[ \leq (\alpha_1 + \alpha_2) \delta \| \epsilon_h \|_0^2 + c \delta h^2 \left( \frac{\nu}{\alpha_1} \| u^n \|_0^2 + \frac{1}{\alpha_2} \| p^n \|_0^2 \right). \]

(57)

Combining (56) and (57) with (53), we have, for any positive numbers \( \delta, \alpha_0, \alpha_1 \) and \( \alpha_2 \) and \( n = 1, \cdots, N_T \),

\[ D_n \left( \frac{1}{2} \| e_h^n \|_0^2 + \| \nabla D(u^n) \|_{L^2}^2 + \delta \| \epsilon_h \|_0^2 \right) \]
\[ \leq \alpha_0 \| e_h^n \|_0^2 + c_1 \| e_h^{n-1} \|_0^2 + (\alpha_1 + \alpha_2) \delta \| \epsilon_h \|_0^2 + c \delta h^2 \left( \frac{\nu}{\alpha_1} \| u^n \|_0^2 + \frac{1}{\alpha_2} \| p^n \|_0^2 \right) \]
\[ + \frac{c_2}{\alpha_0} \left\{ \Delta t \left( \| u^n \|_{H^2(t^{n-1}, t^n; L^2)}^2 + \| u^n \|_{L^2(t^{n-1}, t^n; H^1)}^2 + \| u^n \|_{H^1(t^{n-1}, t^n; H^2)}^2 \right) + \frac{h^2}{\Delta t} \left( \| (u, p) \|_{L^2(t^{n-1}, t^n; H^2 \times H^2)}^2 \right) \right\}, \]

(58)

and, then, for \( \alpha_0 = 1/(4 \Delta t) \), \( \alpha_1 = 1/(4 \nu) \) and \( \alpha_2 = 1/4 \),

\[ D_n \left( \frac{1}{2} \| e_h^n \|_0^2 + \| \nabla D(u^n) \|_{L^2}^2 + \frac{\delta}{2} \| \epsilon_h \|_0^2 \right) \]

(59)
\[ \leq \frac{1}{4\Delta t_0} \Vert e_h^0 \Vert_0^2 + c_1 \Vert e_h^{n-1} \Vert_0^2 + c\delta h^2 \left( v^2 \Vert u^n \Vert_2^2 + \Vert p^n \Vert_1^2 \right) \\
+ c_2 \left\{ \Delta t \left( \Vert u \Vert_{H^2(0,T;L^2)}^2 + \Vert u \Vert_{H^1(0,T;H^1)}^2 + \Vert u \Vert_{H^2(0,T;H^2)}^2 \right) + \frac{h^2}{\Delta t} \Vert (u,p) \Vert_{H^1(0,T;H^2 \times H^2)}^2 \right\}. \] (59)

By the discrete Gronwall inequality, it holds that

\[ \Vert e_h \Vert_{L^\infty(L^2)} + \sqrt{\delta} \Vert D(e_h) \Vert_{l^2(L^2)} + \sqrt{\delta} |\epsilon_h|_{l^2(M_h)} \leq c_2 \left\{ \Vert e_h^0 \Vert_0 + \Delta t \left( \Vert u \Vert_{H^2(0,T;L^2)} + \Vert u \Vert_{H^1(0,T;H^1)} + \Vert u \Vert_{H^2(0,T;H^2)} \right) \right\}. \] (60)

which implies (48).

4 Conclusions

We have introduced a characteristics finite element scheme for a linearized Navier-Stokes equations. The scheme employs P1/P1 element, and the coefficient matrix appearing in the scheme is symmetric. These advantages reduces computational time and cost by half. Therefore, the scheme is useful especially for three dimensional computation. We have shown stability and convergence results with the optimal \( L^2 \)-error estimate for the velocity. Although the system of the equations is linear, the analysis is useful even in the nonlinear case. We note that the convergence of the pressure can be proved under some assumptions.

References


