Semipositivity of relative canonical bundles via Kahler-Ricci flows (Potential theory and fiber spaces)

Author(s)
Boucksom, S.; Tsuji, Hajime

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Semipositivity of relative canonical bundles via Kähler-Ricci flows

S. Boucksom and H. Tsuji

Abstract
In this paper, we shall discuss the fact that the fiberwise Kähler-Ricci flow preserves the semipositivity on a smooth projective family. The full accounts will be given in [B-T].

1 Introduction
In [Ka1], Y. Kawamata proved a semipositivity of the direct image of a relative pluricanonical systems. The second author extended the result to the case of logpluricanonical systems in terms of the generalized Kähler-Einstein metric by using the method in [T4] ([T7]).

In February in 2010, the second author attended the talk given by R. Berman in Luminy about [B]. Inspired by this talk the authors began to work on the stability of the semipositivity of the fiberwise Kähler-Ricci flows on a smooth projective family. This enables us to provide the homotopy version of the semipositivity of relative canonical bundles (cf. Theorem 7). This provides us a new tool to explore the projective (or possibly) Kähler families. For example, as a consequence we may give an alternative proof of the quasiprojectivity of the moduli space of polarized varieties with semiample canonical sheaves.

This is a research announcement and the full accounts will be given in [B-T].

1.1 Kähler-Einstein metrics
Let $X$ be a compact Kähler manifold. It is important to construct a canonical Kähler metric on $X$.

Let $(X, \omega)$ be a compact Kähler manifold. $(X, \omega)$ is said to be Kähler-Einstein, if there exists a constant $c$ such that

$$\text{Ric}(\omega) = c \cdot \omega$$

holds, where the Ricci tensor: $\text{Ric}(\omega)$ is defined by

$$\text{Ric}(\omega) = -\sqrt{-1} \partial \bar{\partial} \log \det \omega.$$ 

This means that $X$ admits a Kähler-Einstein metrics, then $c_1(X)$ is positive or negative or 0.

Theorem 1 ([A, Y1]) Let $X$ be a compact Kähler manifold.
(1) If \( c_1(X) < 0 \), then there exists a Kähler-Einstein metric \( \omega \) such that
\[-\text{Ric}(\omega) = \omega.\]

(2) If \( c_1(X) = 0 \), for every Kähler class \( c \), there exists a Ricci flat Kähler metric \( \omega \) such that \([\omega] = c\) and
\[\text{Ric}(\omega) = 0.\]

\[\square\]

1.2 Twisted Kähler-Einstein metrics

Let \( X \) be a smooth projective variety defined over \( \mathbb{C} \) and let \((L, h_L)\): a (singular) hermitian \( \mathbb{Q} \)-line bundle on \( X \) with \( \sqrt{-1}\Theta_{h_L} \geq 0 \).

\( \omega \) is said to be a twisted Kähler-Einstein metrics associated with \((L, h_L)\), if
\[-\text{Ric}(\omega) + \sqrt{-1}\Theta_{h_L} = \omega\]
holds in the sense of current.

\textbf{Theorem 2 (T7)} If \( h_L \) is \( C^\infty \) on a nonempty Zariski open subset and \( \mathcal{I}(h_L) \simeq \mathcal{O}_X \). Then there exists a closed positive current \( \omega \) on \( X \) such that

(1) There exists a nonempty Zariski open subset \( U \) of \( X \) such that \( \omega|U \) is \( C^\infty \),

(2) \(-\text{Ric}(\omega) + \sqrt{-1}\Theta_{h_L} = \omega\) holds on \( U \),

(3) \((\omega^n)^{-1} \cdot h_L\) is an AZD of \( K_X + L \). \( \square \)

1.3 Bergman metrics

Let \( X \) be a smooth projective variety and let \((L, h_L)\) be a singular hermitian line bundle on \( X \). We set
\[K(X, K_X + L, h_L) := \sum_i |\sigma_i|^2,\]
where \( \{\sigma_i\} \) is an orthonormal basis of \( H^0(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{I}(h_L)) \) with respect to the inner product:
\[(\sigma, \tau) := \int_X \sigma \cdot \overline{\tau} \cdot h_L.\]

We call \( K(X, K_X + L, h_L) \) the Bergman kernel of \( X \) with respect to \((L, h_L)\). If \(|H^0(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{I}(h_L))|\) is very ample, then the pull back of the Fubini-Study metric
\[\omega := \sqrt{-1}\partial\bar{\partial}\log K(X, K_X + L, h_L)\]
is a Kähler form on \( X \). We call it the Bergman metric on \( X \) with respect to \((L, h_L)\).
1.4 Dynamical construction of K-E-metrics

Let $X$ be a smooth projective $n$-fold with ample $K_X$ and $(A, h_A)$ be a sufficiently ample line bundle with $C^\infty$-metric $h_A$. We set $K_1 = K(X, K_X + A, h_A), h_1 = K_1^{-1}$. And inductively we define

$$K_m = K(X, mK_X + A, h_{m-1}), h_m = K_m^{-1}$$

for $m \geq 2$. Then we have the following rather unexpected result.

**Theorem 3** ([T]) $dV_E = \lim_{m \to \infty} \sqrt[m]{(m!)^{-n}K_m}$ is the K-E volume form on $X$, i.e., $\omega_E = -Ric dV_E$ is K-E-form. □

1.5 Kähler-Ricci flow

Let $X$ be a compact Kähler manifold and let $\omega_0$: $C^\infty$-Kähler form on $X$.

We consider the initial value problem:

$$\frac{\partial}{\partial t} \omega(t) = -Ric(\omega(t)) - \omega(t)$$

on $X \times [0, T)$,

$$\omega(0) = \omega_0,$$

where $Ric(\omega(t)) = -\sqrt{-1}\partial\overline{\partial} \log \det \omega(t)$ and $T$ is the maximal existence time for the $C^\infty$-solution. This type of Kähler-Ricci flow was first considered by the second author in [T1]. Then by taking the exterior derivative of the both sides of (1),

$$[\omega(t)] = (1 - e^{-t})2\pi c_1(K_X) + e^{-t}[\omega_0] \in H^{1,1}(X, \mathbb{R})$$

Let $\mathcal{K}(X)$ denote the Kähler cone of $X$. Then the following holds:

**Proposition 1** ([T1])

$$T = \sup\{t|\omega(t) \in \mathcal{K}(X)\}$$

holds. □

The next question is what happens on $\omega(t)$ after exiting the Kähler cone. Let $PE(X)$ denote the pseudoeffective cone $\subseteq H^{1,1}(X, \mathbb{R})$.

**Definition 1** Let $T$ be a closed positive $(1, 1)$ current on $X$. $T$ is said to be of minimal singularities, if for every closed positive $(1, 1)$-current $T'$ with $[T'] = [T]$, there exists a $L^1$-function $\varphi$ such that

$$T' = T + \sqrt{-1}\partial\overline{\partial}\varphi$$

and is bounded from above. □

The following proposition is an easy consequence of [Le, p.26, Theorem 5].

**Proposition 2** Let $\eta \in PE(X)$ be a pseudoeffective class. Then there exists a closed positive $(1, 1)$-current $T_{min}$ with minimal singularities which represents $\eta$. □
A closed semipositive current $T$ with $[T] \in PE(X)$ is said to be of almost minimal singularities if we write $T$ as $T = T_{min} + \sqrt{-1} \partial \overline{\partial} \varphi$ for some $\varphi \in L^{1}(X)$, $e^{-\varphi} \in L^{p}(X)$ holds for every $p \geq 1$.

For a pseudoeffective $\mathbb{R}$-line bundle $F$ on a smooth projective manifold $M$, we say that the decomposition:

$$F = P + N(P, N \in \text{Div}(M) \otimes \mathbb{R})$$

is said to be a Zariski composition, if there exists a closed semipositive $(1, 1)$ current $T$ on $M$ such that

1. $T$ is a closed semipositive current of almost minimal singularities in $2\pi c_{1}(F)$,
2. $T_{sing} = 2\pi N$ in the sense of currents, where $T = T_{abc} + T_{sing}$ is the Lebesgue decomposition.

Let $X$ be a smooth projective variety with pseudoeffective $K_{X}$. Then we have the following lemma by [B-C-H-M].

**Lemma 1** There exists a sequence: $T = T_{0} < T_{1} < \cdots < T_{j} < \cdots$ such that for each $j$, there exists a modification $\pi_{j}: X_{j} \rightarrow X$ such that $\pi_{j}^{*}(e^{-t}L + (1 - e^{-t})K_{X})$ admits a Zariski decomposition:

$$\pi_{j}^{*}(e^{-t}L + (1 - e^{-t})K_{X}) = P_{t} + N_{t}$$

such that $N_{t}$ is independent of $t \in [T_{j}, T_{j+1})$. \(\square\)

Then we have the following theorem.

**Theorem 4** Let $X$ be a smooth projective variety with pseudoeffective canonical class. Let $(L, h_{L})$ be a $C^{\infty}$-hermitian line bundle such that $\omega_{0} := \sqrt{-1} \Theta_{h_{L}}$ is a Kähler form on $X$. Then the initial value problem:

$$\frac{\partial}{\partial t} \omega(t) = -\text{Ric}(\omega(t)) - \omega(t) \text{ on } X \times [0, \infty), \quad (2)$$

$\omega(0) = \omega_{0}$ has the unique long time solution $\omega(t)$ such that

1. For $t \in [T_{j}, T_{j+1})$, $\omega(t)$ is $C^{\infty}$ on a nonempty Zariski open subset $U(T_{j})$ depending on $T_{j} \in [0, \infty)$ defined as in Lemma 1.
2. For $t \in [T_{j}, T_{j+1})$, $\omega(t)$ satisfies the equation (2) on $U(T_{j})$.
3. $\omega(t)$ is a closed semipositive current with almost minimal singularity in $(1 - e^{-t})2\pi c_{1}(K_{X}) + e^{-t}c_{1}(L)$. \(\square\)

## 2 Proof of Theorem 4

Let $X$ be a smooth projective variety with pseudoeffective canonical class and let $(L, h_{L})$ be a $C^{\infty}$-hermitian line bundle on $X$ such that $\omega_{0} = \sqrt{-1} \Theta_{h_{L}}$ is a Kähler form.
2.1 Discretization of Kähler-Ricci flows

Let $a$ be a positive integer. We consider the following successive equations:

$$a(\omega_{m,a} - \omega_{m-1,a}) = -\text{Ric}_{\omega_{m,a}} - \omega_{m,a}$$

(3)

for $m \geq 1$ under the initial condition $\omega_{0,a} = \omega_0$. We see that the cohomology class $[\omega_{m,a}]$ satisfies the equations:

$$a([\omega_{m,a}] - [\omega_{m-1,a}]) = 2\pi c_1(K_X) - [\omega_{m,a}]$$

(4)

Hence we see that

$$[\omega_{m,a}] = \left(1 - \left(1 + \frac{1}{a}\right)^{-m}\right)2\pi c_1(K_X) + \left(1 + \frac{1}{a}\right)^{-m}[\omega_0]$$

(5)

We define the singular hermitian metric

$$h_{m,a} := n!(\omega_{m,a}^n)^{-\frac{1}{a+1}} \cdot h_{m-1}^{\frac{a}{m-a+1}}$$

(6)

on

$$(1 - t_{m,a})L + t_{m,a}K_X,$$  (7)

where

$$t_{m,a} = 1 - \left(1 + \frac{1}{a}\right)^{-m}.$$  (8)

$$\omega(m, a) := t_{m,a}(-\text{Ric}\Omega) + (1 - t_{m,a})\omega_0$$

(9)

Then the $\{u_{m,a}\}_{m=0}^\infty$ satisfies the successive differential equations:

$$a(u_{m,a} - u_{m-1,a}) = \log \frac{(\omega(m,a) + \sqrt{-1}\partial\bar{\partial}u_{m,a})^n}{\Omega} - u_{m,a}.$$  (10)

Now we introduce the following notation:

$$\delta_{a}u_{m,a} := a(u_{m,a} - u_{m-1,a}),$$

(11)

i.e., $\delta_{a}u_{m,a}$ denotes the (backward) difference at $u_{m,a}$.

Then (10) is denoted as:

$$\delta_{a}u_{m,a} = \log \frac{(\omega(m,a) + \sqrt{-1}\partial\bar{\partial}u_{m,a})^n}{\Omega} - u_{m,a}.$$  (12)

Later we shall see that this equation corresponds to the parabolic Monge-Ampère equation:

$$\frac{\partial u}{\partial t} = \log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}u)^n}{\Omega} - u,$$  (13)

where

$$\omega_t := (1 - e^{-t})(-\text{Ric}\Omega) + e^{-t}\omega_0$$

(14)
with the initial condition: $u = 0$ on $X \times \{0\}$.

And there are correspondences:

$$\frac{m}{a} \leftrightarrow t, \ u_{m, a} \leftrightarrow u(t), \ \omega(m, a) \leftrightarrow \omega_t$$

and

$$\delta_a u_{m, a} \leftrightarrow \frac{\partial u}{\partial t}.$$  

We set

$$T := \sup\{t \in \mathbb{R} | 2\pi(1 - e^{-t})c_1(K_X) + e^{-t}[\omega_0] \in \mathcal{K}\}.$$  

(15)

Since the Kähler-Ricci flow corresponds to the minimal model with scalings in [B-C-H-M] in an obvious manner, we have the following lemma.

**Lemma 2 ([B-C-H-M])** The followings holds:

1. $e^{-T} \in \mathbb{Q}$,
2. $(1 - e^{-T})K_X + e^{-T}L$ is semiample.  

By Lemma 2, there exists a $C^\infty$-function $\phi$ such that

$$\omega_{T, \phi} := (1 - e^{-T})(\text{Ric}\Omega + \sqrt{-1}\partial\overline{\partial}\phi) + e^{-T}\omega_0$$  

(16)

is a $C^\infty$-semipositive form on $X$ and is strictly positive on a nonempty Zariski open subset of $X$. We set

$$\omega(m, a)_{\phi} := \left(1 - \left(1 + \frac{1}{a}\right)^{-m}\right)(\text{Ric}\Omega + \sqrt{-1}\partial\overline{\partial}\phi) + \left(1 + \frac{1}{a}\right)^{-m}\omega_0$$  

(17)

$$= \omega(m, a) + \left(1 - \left(1 + \frac{1}{a}\right)^{-m}\right)\sqrt{-1}\partial\overline{\partial}\phi$$

We set

$$m(a) := \sup \left\{ m \mid \left(1 - \left(1 + \frac{1}{a}\right)^{-m}\right)c_1(K_X) + \left(1 + \frac{1}{a}\right)^{-m}[\omega_0] \in \mathcal{K} \right\}.$$  

(18)

Then since

$$\omega(m, a)_{\phi} := \frac{1 - \left(1 + \frac{1}{a}\right)^{-m}}{1 - e^{-T}}\omega_{T, \phi} + \frac{\left(1 + \frac{1}{a}\right)^{-m} - e^{-T}}{1 - e^{-T}}\omega_0.$$  

(19)

for every $m < m(a)$, $\omega(m, a)_{\phi}$ is a $C^\infty$-Kähler form on $X$ and for $m = m(a)$, $\omega(m, a)_{\phi} = \omega_{T, \phi}$ holds.

**Theorem 5** (3) has a smooth solution $\omega_{m, a}$ as long as $[\omega(m, a)] \in \mathcal{K}$. And (10) has $C^\infty$-solution as $[\omega(m, a)] \in \mathcal{K}$.  

**Lemma 3** Suppose that $T$ is finite, then we see that

$$\omega(T) := \lim_{t \uparrow T} \omega(t)$$

exists in $C^\infty$-topology on $X \setminus E$ and is a well defined as a limit of closed positive current on $X$.  

$\square$
2.2 Beyond the Kähler cone

After exiting the Kähler cone, the singular solution of the Kähler-Ricci flow can be constructed as follows.

**Theorem 6** There exists a sequence of closed semipositive currents \( \{\omega_{m,a}\}_{m=0}^{\infty} \) such that

1. For every \( m \geq 0 \), \( \omega_{m,a} \) is a closed semipositive current on \( X \),
2. There exists a nonempty Zariski open subset \( U_m \) of \( X \) such that \( h_{m,a}|U_m \) is \( C^\infty \),
3. \( h_{m,a} \) is an AZD of the \( \mathbb{Q} \)-line bundle \((1-t_{m,a})L+t_{m,a}K_X\),
4. \( \omega_{m,a} = \sqrt{-1} \Theta_{h_{m,a}} \) is a well defined closed semipositive current on \( X \),
5. \( \{\omega_{m,a}\}_{m=0}^{\infty} \) satisfies the equations (3) on \( U_m \). \( \square \)

The following lemma is a slight refinement of Lemma 1.

**Lemma 4** There exists a sequence of positive number \( T = T_0 < T_1 < \cdots < T_j < \cdots \) such that for every \( t \in [T_j, T_{j+1}) \)

1. There exists a modification \( \pi_j : X_j \to X \) such that \( \pi_j^*(e^{-t}L+(1-e^{-t})K_X) \)
   admits a Zariski decomposition:
   \[
   \pi_j^*(e^{-t}L+(1-e^{-t})K_X) = P_t + N_t (P_t, N_t \in \text{Div}(X_j) \otimes \mathbb{R}),
   \]
   where \( P_t \) is nef and \( N_t \) is effective and
   \[
   H^0(X_j, \mathcal{O}_{X_j}([mP_j])) \simeq H^0(X_j, \mathcal{O}_{X_j}(m\pi_j^*(e^{-t}L+(1-e^{-t})K_X)))
   \]
   holds for every \( m \) such that \( me^{-t} \in \mathbb{Z} \).
2. \( N_t \) is independent of \( t \in [T_j, T_{j+1}) \),
3. If \( e^{-t} \in \mathbb{Q} \), then \( P_t \) is semiample. \( \square \)

We set \( N_j := N_t (t \in [T_j, T_{j+1})) \). Let \( \tau_j \) be the multivalued holomorphic section of \( N_j \) with divisor \( N_j \). Then there exists a \( C^\infty \)-hermitian metric \( \| \| \) such that \( \omega_{T_j} + \sqrt{-1} \partial \bar{\partial} \log \| \tau_j \| \) is a closed semipositive current. We set

\[
\phi_j := \log \| \tau_j \|^2.
\] (20)

Suppose that we have already defined \( u_{0,a}(\phi_j) \) such that for every \( \epsilon > 0 \), there exists a constant \( C(\epsilon) \)

\[
u_{0,a}(\phi_j) \geq \epsilon \phi_j + C(\epsilon)
\] (21)

holds. We set

\[
\omega_j(m, a) := \left(1 - e^{-T_j} \left(1 + \frac{1}{a}\right)^{-m}\right)(-\text{Ric } \Omega) + e^{-T_j} \left(1 + \frac{1}{a}\right)^{-m} \omega_0.
\] (22)

We consider the Ricci iteration:

\[
\delta_a u_{m,a}(\phi_j) = \log \frac{\omega(m,a)\phi_j + \sqrt{-1} \partial \bar{\partial} u_{m,a}(\phi_j)}{\Omega \cdot e^{-\phi_j}} - u_{m,a}(\phi_j).
\] (23)

The rest of the proof is similar to the case \( t \in [0, T) \).
3 Semipositivity of a Kähler-Ricci flow

In this section we shall sketch the proof of the fact that the relative Kähler-Ricci flows preserve the semipositivity in the horizontal direction on projective families.

3.1 Main results

Let $f : X \to S$ be a smooth projective family and let $\omega$ be a relative Kähler form on $X$. We set $n := \dim X - \dim S$ and $k := \dim S$. We define the relative Ricci form $\text{Ric}_{X/S,\omega}$ of $\omega$ by

$$\text{Ric}_{X/S,\omega} = -\sqrt{-1} \partial \bar{\partial} \log \left( \omega^n \wedge f^* |ds_1 \wedge \cdots \wedge ds_k|^2 \right), \quad (24)$$

where $(s_1, \cdots, s_k)$ is a local coordinate on $S$. Then it is easy to see that $\text{Ric}_{X/S,\omega}$ is independent of the choice of the local coordinate $(s_1, \cdots, s_k)$. The Kähler-Ricci flow preserves the semipositivity in the following sense.

**Theorem 7** Let $f : X \to S$ be a smooth projective family of varieties with pseudoeffective canonical bundles. Let $L$ be an ample line bundle on $X$ and let $h_L$ be a $C^\infty$-hermitian metric on $L$ with strictly positive curvature. Suppose that there exists a $C^\infty$-relative volume form $\Omega$ on $f : X \to S$ such that $\text{Ric} \Omega + \sqrt{-1} \Theta_{h_L}$ is also a Kähler form on $X$. We set $\omega_0 := \sqrt{-1} \Theta_{h_L}$. We consider the normalized Kähler-Ricci flow:

$$\frac{\partial}{\partial t} \omega(t) = -\text{Ric}_{X/S,\omega(t)} - \omega(t)$$

on $X$ with the initial condition $\omega(0) = \omega_0$, where $\text{Ric}_{X/S,\omega(t)}$ denotes the relative Ricci form of $\omega(t)$ on $X$.

Then $\omega(t)$ is a closed semipositive current on $X$ for every $t \in [0, \infty)$. \[\square\]

In Theorem 7, the semipositivity of $\omega(t)$ corresponds to the pseudoeffectivity of $(1 - e^{-t})K_{X/S} + e^{-t}L$. And as $t$ goes to infinity, we observe that the relative canonical bundle $K_{X/S}$ is pseudoeffective.

Similarly we have the following theorem.

**Theorem 8** Let $f : X \to S$ be a smooth projective family of varieties with pseudoeffective canonical bundles. Let $L$ be an ample line bundle on $X$ and let $h_L$ be a $C^\infty$-hermitian metric on $L$ with strictly positive curvature. Let $K$ be a closed semipositive current on $X$ such that $K$ is $C^\infty$ on a nonempty Zariski open subset of $X$ and $[K] \in 2\pi c_1(K_{X/S})$. We set $\omega_0 := \sqrt{-1} \Theta_{h_L}$. We consider the Kähler-Ricci flow:

$$\frac{\partial}{\partial t} \omega(t) = -\text{Ric}_{X/S,\omega(t)} - K$$

on $X$ with the initial condition $\omega(0) = \omega_0$, where $\text{Ric}_{X/S,\omega(t)}$ denotes the relative Ricci form of $\omega(t)$ on $X$.

Then $\omega(t)$ is a closed semipositive current on $X$ for every $t \in [0, \infty)$. Moreover as $t$ goes to infinity, $\omega(t)$ converges to a current solution of $-\text{Ric}_{X/S,\omega(t)} = K$. \[\square\]
3.2 Some conjecture for the Kähler case

We expect that the similar statement holds even in the case that \( f : X \to S \) is a smooth Kähler fibration.

**Conjecture 1** Let \( X \) be a compact Kähler manifold with pseudoeffective canonical bundle. And let \( \omega_0 \) be a \( C^\infty \)-Kähler form on \( X \). Suppose that there exists a \( C^\infty \)-volume form \( \Omega \) such that

\[
\text{Ric} \Omega + \omega_0
\]

is also a Kähler form on \( X \). Then there exists a family of closed semipositive current \( \omega(t) \) on \( X \) such that

\[
(1) \quad \omega(0) = \omega_0,
\]

\[
(2) \quad \text{for every } T > 0, \text{ there exists a nonempty Zariski open subset } U(T) \text{ depending on } T \text{ such that } \omega(t) \text{ is Kähler form on } U(T) \times [0, T),
\]

\[
(3) \quad [\omega(t)] = 2\pi (e^{-t}[\omega_0] + (1 - e^{-t})c_1(K_X)) \quad \text{holds for every } t \in [0, \infty),
\]

\[
(4) \quad \text{on } U(t) \times [0, T) \omega(t) \text{ satisfies the differential equation:}
\]

\[
\frac{\partial \omega(t)}{\partial t} = -\text{Ric}_{\omega(t)} - \omega(t).
\]

\[\square\]

**Conjecture 2** Let \( f : X \to S \) be a smooth Kähler family with pseudoeffective canonical bundles. Let \( \omega_0 \) be a \( C^\infty \)-Kähler form on \( X \). Suppose that there exists a \( C^\infty \)-relative volume form \( \Omega \) on \( f : X \to S \) such that \( \text{Ric} \Omega + \omega_0 \) is also a Kähler form on \( X \). We consider the normalized Kähler-Ricci flow:

\[
\frac{\partial}{\partial t} \omega(t) = -\text{Ric}_{X/S, \omega(t)} - \omega(t)
\]

on \( X \) with the initial condition \( \omega(0) = \omega_0 \), where \( \text{Ric}_{X/S, \omega(t)} \) denotes the relative Ricci form of \( \omega(t) \) on \( X \).

Then \( \omega(t) \) is a closed semipositive current on \( X \) for every \( t \in [0, \infty) \). \( \square \)

This conjecture will lead us to the invariance of plurigenera in the Kähler case.

4 Proof of Theorem 7

The essential technical difficulty here is the fact that we cannot apply the direct calculation of the variation, since the Kähler-Ricci flow in Theorem 4 has singularities. We overcome this difficulty by using the dynamical construction of the solution of the Ricci iterations as in [LC].
4.1 The relative Ricci iterations to the relative Kähler-Ricci flow

Let $f : X \rightarrow S$ be a smooth projective family of varieties with pseudoeffective canonical bundles. Let $L$ be an ample line bundle on $X$ and let $h_L$ be a $C^\infty$-hermitian metric on $L$ with strictly positive curvature. Suppose that there exists a $C^\infty$-relative volume form $\Omega$ on $f : X \rightarrow S$ such that $\text{Ric} \Omega + \sqrt{-1}\Theta_{h_L}$ is also a Kähler form on $X$. We set $\omega_0 := \sqrt{-1}\Theta_{h_L}$. We consider the normalized Kähler-Ricci flow:

$$\frac{\partial \omega(t)}{\partial t} = -\text{Ric}_{\omega(t)} - \omega(t)$$

(25)

on $X$ with the initial condition $\omega(0) = \omega_0$, where $\text{Ric}_{\omega(t)}$ denotes the relative Ricci form on $X$.

For every $s \in S$, we consider Lemma 1. Then by the invariance of the twisted plurigenra, we see that for every $C > 0$ the sequence

$$T = T_0 < T_1 < \cdots < T_j < \cdots < C$$

(26)

in Lemma 1 are constant on a nonempty Zariski open subset $S(C)$ of $S$.

Suppose that we have already proven the (logarithmic) plurisubharmonic variation property of the solution $\omega(t)$ of (25) for every $t < C$ on $f^{-1}(S(C))$. Then the removable singularity theorem for plurisubharmonic function implies the logarithmic plurisubharmonic variation property of the solution $\omega(t)$ over the whole $X$.

Hence we may and do assume that the sequence $T_0 < \cdots < T_j < \cdots$ are constant over the whole $S$ without loss of generality. Moreover since the assertion of Theorem 7 is local in $S$, we may and do assume that $S$ is the unit open polydisk $\Delta^k$ in $\mathbb{C}^k$.

The plurisubharmonic variation property of the Ricci iteration is proven by the parallel argument as follows.

We set

$$m(a) := \sup \left\{ m \left| \left(1 + \frac{1}{a}\right)^{-m} > e^{-T_0} \right. \right\}.$$  

(27)

First we shall consider the relative Ricci iteration:

$$\delta_a \omega_{m,a} = -\text{Ric}_{\omega_{m,a,s}}, \omega_{m,a,0} = \omega_0$$

(28)

on $X$ for $0 \leq m < m(a)$. This is equivalent to the fiberwise Ricci iteration:

$$\delta_a \omega_{m,a,z} = -\text{Ric}_{\omega_{m,a,s}}, \omega_{m,a,0} = \omega_0|_{X_s}$$

(29)

on $X_s$ for $0 \leq m < m(a)$. Then by the proof of Theorem 4, letting $a$ tends to infinity, we may construct the solution of the relative Kähler-Ricci flow:

$$\frac{\partial \omega(t)}{\partial t} = -\text{Ric}_{\omega(t)} - \omega(t)$$

(30)

on $X \times [0, T_0)$.

Then as in the previous section, we may continue this process beyond the critical time $T_0$ and we obtain the long time existence of the current solution of the relative Kähler-Ricci flow on $X$. 
4.2 Auxiliary Ricci iterations

We prove Theorem 7 by decomposing the Ricci iterations by a dynamical system of Bergman kernels and apply the plurisubharmonic variation properties of Bergman kernels due to Berndtsson. The main difficulty is to deal with Q-line bundles. We deal with Q-line bundles in terms of the auxiliary Ricci iterations.

**Lemma 5** For every $0 \leqq m \leqq m(a)$, $\omega_{m,a}$ is semipositive on $X$. □

We prove Lemma 5 by induction on $m$.

For $m = 0$ $\omega_{0,a} = \omega_{0}$ is a Kähler form on $X$ by the assumption. Hence Lemma 5 holds for $m = 0$. Suppose that $\omega_{m,a}$ is semipositive on $X$. We shall prove that $\omega_{m+1,a}$ is also semipositive on $X$.

To prove this assertion, we consider the auxiliary Ricci iteration which connects $\omega_{m,a}$ and $\omega_{m+1,a}$.

First we define the Q-line bundle $L_m$ by

$$L_m := \left(1 - \left(1 + \frac{1}{a}\right)^{-m}\right)K_{X/S} + \left(1 + \frac{1}{a}\right)^{-m}L. \quad (31)$$

Let $q = q(m+1)$ be a positive integer such that $qL_{m+1}$ is a genuine line bundle on $X$. Since

$$L_{m+1} = \left(1 - \left(1 + \frac{1}{a}\right)^{-(m+1)}\right)K_{X/S} + \left(1 + \frac{1}{a}\right)^{-(m+1)}L$$

is of the form $\beta(K_{X/S} + \alpha L)$ for some positive rational numbers $\alpha$ and $\beta$. By B-C-H-M, we have that the relative logcanonical ring:

$$R(X, K_{X/S} + \alpha L) = \oplus_{\nu=0}^{\infty} f_{\ast}\mathcal{O}_X(\lfloor\nu(K_{X/S} + \alpha L)\rfloor)$$

is a finitely generated algebra over $\mathcal{O}_S$. By the invariance of twisted plurigeera, we see that each $f_{\ast}\mathcal{O}_X(\lfloor\nu(K_{X/S} + \alpha L)\rfloor)$ is a vector bundle over $S$ which is biholomorphic to the unit open polydisk $\Delta^k$. We take a sufficiently large positive integer $\nu_0$ and take a set of generators $\{\sigma_i\}$ of $f_{\ast}\mathcal{O}_X(\nu_0!(K_{X/S} + \alpha L))$ (In this case $K_{X/S} + \alpha L$ is relatively ample. But later we also consider the case $K_{X/S} + \alpha L$ is big, but not relatively ample). Then we set

$$h_{m,a,0} := \left(\sum_{i} |\sigma_i|^2\right)^{-\frac{\beta}{\nu_0}} \quad (32)$$

and

$$\omega_{m,a,0} := \sqrt{-1}\Theta_{h_{m,a,0}}. \quad (33)$$

Then $h_{m,a,0}$ is a hermitian metric of $L_{m+1} = \beta(K_{X/S} + \alpha L)$ with semipositive curvature on $X$. Now we shall consider the following Ricci iteration:

$$-\text{Ric}_{\omega_{m,a,\ell}} + (q - a - 1)\omega_{m,a,\ell-1} + a\omega_{m,a} = q\omega_{m,a,\ell} \quad (34)$$

for $\ell \geqq 1$. The following lemma follows entirely the same way as the dynamical construction of Kähler-Einstein metrics.
Lemma 6 \( \lim_{\ell \to \infty} \omega_{m,a,\ell} \) exists in \( C^\infty \)-topology on \( X \). And

\[
\lim_{\ell \to \infty} \omega_{m,a,\ell} = \omega_{m+1,a}
\]  

holds. \( \square \)

We use this auxiliary Ricci iteration to connect \( \omega_{m,a} \) and \( \omega_{m+1,a} \) by a dynamical system of Bergman kernels. This method is exactly the same one in [T7].

### 4.3 Dynamical systems of Bergman kernels

To prove the semipositivity of \( \omega(t) \) on \( X \) for \( t \in [0, T_0] \), it is enough to prove the following lemma.

**Lemma 7** \( h_{m,a} \) has semipositive curvature on \( X \). \( \square \)

We now use the strategy as in [T7]. We shall prove Lemma 7 by induction on \( m \). Since \( h_L \) has positive curvature, \( h_{0,a} = h_L \) has semipositive curvature.

Suppose that we have already proven that \( h_{m-1,a} \) has semipositive curvature.

Let \( A \) be a sufficiently ample line bundle on \( X \) and let \( h_A \) be a \( C^\infty \)-hermitian metric on \( X \) with strictly positive curvature.

Now we shall define the metric on \( L_{m+1} \) by

\[
h_{m,a,\ell}|_{X_s} = h_{m,a,\ell,s}(s \in S).
\]

By induction on \( \ell \), we shall prove the following lemma.

**Lemma 8** \( h_{m,a,\ell} \) has semipositive curvature on \( X \) for every \( \ell \geq 0 \). \( \square \)

**Proof of Lemma 8.** By the construction (cf. (32)), \( h_{m,a,0} \) has semipositive curvature.

Suppose that we have already proven that \( h_{m,a,\ell-1} \) is a hermitian metric with semipositive curvature on \( X \). For every \( s \in S \), we shall consider the dynamical system of Bergman kernels as follows. We set

\[
K_{1,s} := K \left( X_s, A + K_{X_s} + (q - a - 1) L_{m+1} + a L_m |_{X_s}, h_A \cdot h_{m,a,\ell-1}^{q-a-1} \cdot h_{m,a} |_{X_s} \right)
\]

and

\[
h_{1,s} := K_{1,s}^{-1}.
\]

Suppose that we have already constructed \( K_{p-1,s} \) and \( h_{p-1,s} \) for some \( p \geq 2 \). Then we define \( K_{p,s} \) and \( h_{p,s} \) by

\[
K_{p,s} := K \left( X_s, A + p(K_{X_s} + (q - a - 1) L_{m+1} + a L_m |_{X_s}), h_{m,\ell-1}^{q-a-1} \cdot h_{m,a} \cdot h_{p-1} |_{X_s} \right)
\]

and

\[
h_{p,s} := \frac{1}{K_{p,s}}.
\]

Similarly as in [T4, T7] we have the following lemma.
Lemma 9

\[ K_{\infty,s} := \lim_{p \to \infty} \sup_{\infty} ((p!)^{-n} h_{A} \cdot K_{p,s})^{\frac{1}{pq}} \]  

exists in \( L^1 \)-topology and

\[ h_{m,a,\ell,s} := K_{\infty,s}^{-1} \]  

is a \( C^\infty \)-hermitian metric on \( L_{m+1}|X_s \). And the curvature

\[ \omega_{m,a,\ell,s} := \sqrt{-1} \Theta_{h_{m,a,\ell,s}} \]  

satisfies the differential equation:

\[ -\text{Ric}_{\omega_{m,a,\ell,s}} + (q - a - 1)\omega_{m,a,\ell-1,s} + a\omega_{m,a,s} = q\omega_{m,a,\ell,s} \]

on \( X \). \( \square \)

We define the relative Bergman kernel \( K_p \) on \( X \) by

\[ K_p|X_s = K_{p,s} \]

Then \( h_p = K_{p}^{-1} \) is a hermitian metric with semipositive curvature on \( A + p(K_{X/S} + (q - a - 1)L_{m+1} + aL_m) \) by induction on \( p \) by the following theorem mainly due to B. Berndtsson.

**Theorem 9** ([B1, B2, B3, B-P]) Let \( f : X \to S \) be a projective family of projective varieties over a complex manifold \( S \). Let \( S^0 \) be the maximal nonempty Zariski open subset such that \( f \) is smooth over \( S^0 \).

Let \( (L, h_L) \) be a pseudo-effective singular hermitian line bundle on \( X \).

Let \( K_s := K(X_s, K_X + L|X_s, h|X_s) \) be the Bergman kernel of \( K_{X,s} + (L|X_s) \) with respect to \( h|X_s \) for \( s \in S^0 \). Then the singular hermitian metric \( h \) of \( K_{X/S} + L|f^{-1}(S^0) \) defined by

\[ h|X_s := K_{s}^{-1}(s \in S^0) \]

has semipositive curvature on \( f^{-1}(S^0) \) and extends to \( X \) as a singular hermitian metric on \( K_{X/S} + L \) with semipositive curvature in the sense current. \( \square \)

Now we prove the semipositivity of \( \sqrt{-1}\Theta_{h_p} \) by induction on \( p \). First the semipositivity of \( \sqrt{-1}\Theta_{h_1} \) follows from Theorem 9 by the assumption that \( \sqrt{-1}\Theta_{h_{m,a,\ell-1}} \) and \( \sqrt{-1}\Theta_{h_{m-1,a}} \) are semipositive. Suppose that we have already proven the semipositivity of \( h_{p-1} \) for some \( p \geq 2 \). We note that \( h_{p-1}, h_{m,a,\ell-1} \) and \( h_{m,a} \) has semipositive curvature on \( X \) by the induction assumption. Then by the inductive definition of \( h_p \) (cf. (39) and (40)) and Theorem 9, we see that \( \sqrt{-1}\Theta_{h_p} \) is also semipositive.

Hence by induction, we see that \( \{h_p\}_p^\infty \) has semipositive curvature on \( X \). Then by Lemma 9, we see that \( h_{m,a,\ell} \) has semipositive curvature. This completes the proof of Lemma 8. \( \square \)

By Lemmas 6 and 8, we see that \( h_{m+1} \) is a metric on \( L_{m+1} \) with semipositive curvature. Hence by induction on \( m \), we complete the proof of Lemma 7. \( \square \)
Now by Lemma 7 and the proof of Theorem 1, we see that $\omega(t)$ is semipositive on $X$ for $t \in [0, T_0]$.

Now we complete the proof of Theorem 7 by repeating the similar argument inductively for $t \in [T_j, T_{j+1}](j \geq 0)$. This completes the proof of Theorem 7. □

References


Authors’ address
S. Boucksom, Department of Mathematics, University of Paris VII, Jusseu, Paris, France
H. Tsuji, Department of Mathematics, Sophia University, 7-1, Kioicho, Chiyoda-ku, Tokyo, 102-8554, Japan