

## EXAMPLES OF MOTIVIC COHOMOLOGY OF CLASSIFYING SPACES (II)

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### 1. INTRODUCTION

Let  $G$  be a compact Lie group. Taking complexification, we can identify the group  $G = G_{\mathbb{C}}$  as the reductive algebraic group over the complex number field  $\mathbb{C}$ . The main result of this paper is the computation of the mod  $p$  motivic cohomology  $H^{*,*'}(BG; \mathbb{Z}/p)$  of the classifying spaces of algebraic groups (over  $\mathbb{C}$ ) corresponding Lie groups  $G$ . We compute  $H^{*,*'}(BG; \mathbb{Z}/2)$  for  $G = O_n, SO_4, Q_8$  and  $D_8$ .

### 2. THE MOTIVIC COHOMOLOGY OF $B\mathbb{Z}/p$

In this section we consider the relation between the motivic and the usual ordinary cohomologies. Let  $R$  be  $\mathbb{Z}$  or  $\mathbb{Z}/p$ . The motivic cohomology has the following properties ([6],[7],[9]).

(C1)  $H^{*,*'}(X; R)$  is a bigraded multiplicative cohomology theory in (some good) category  $Spc$  of pointed (algebraic) spaces (the cohomology of a space means the reduced cohomology of a pointed space); For any map  $f : X \rightarrow Y$  in the category  $Spc$ , we have the cofiber sequence  $X \rightarrow Y \rightarrow Y/X$ , which induces the long exact sequence

$$\leftarrow H^{*,*'}(X; R) \leftarrow H^{*,*'}(Y; R) \leftarrow H^{*,*'}(Y/X; R) \leftarrow H^{*-1,*'}(X; R) \leftarrow \dots$$

(In particular, we get the Mayer-Vietoris, Gysin and blow up long exact sequences.)

(C2) There are maps (realization maps)

$$t_{\mathbb{C}}^{m,n} : H^{m,n}(X; R) \rightarrow H^m(X(\mathbb{C}); R)$$

which sum up  $t_{\mathbb{C}}^{*,*'} = \bigoplus_{m,n} t_{\mathbb{C}}^{m,n}$  the natural ring homomorphism.

(C3) There are ( the Bockstein, the reduced powers ) operations

$$\beta : H^{*,*'}(X; \mathbb{Z}/p) \rightarrow H^{*+1,*'}(X; \mathbb{Z}/p)$$

$$P^i : H^{*,*'}(X; \mathbb{Z}/p) \rightarrow H^{*+2(p-1)i,*'+(p-1)i}(X; \mathbb{Z}/p)$$

which commutes with the realization map  $t_{\mathbb{C}}$ .

(C4) For the projective space  $\mathbb{P}^n$ , there is an isomorphism

$$H^{*,*'}(X \wedge (\mathbb{P}^n/\mathbb{P}^{n-1}); R) \cong H^{*,*'}(X; R)\{y'\}$$

with  $\deg(y') = (2n, n)$  and  $t_{\mathbb{C}}(y') \neq 0$ .

(C5) For a smooth  $X$ , if  $H^{m,n}(X; R) \neq 0$ , then

$$m \leq n + \dim(X), \quad m \leq 2n \quad \text{and} \quad m \geq 0.$$

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For an element  $x \in H^{m,n}(X; \mathbb{Z}/p)$ , we define the (weight and difference degree)

$$w(x) = 2n - m, \quad d(x) = m - n$$

Hence for smooth  $X$ ,  $w(x) \geq 0$ , and  $d(x) \leq \dim(X)$  for nonzero  $x \in H^{*,*'}(X; \mathbb{Z}/p)$ .

**Remark.** For  $x' \in H^*(X(\mathbb{C}); \mathbb{Z}/p)$ , we can define the weight degree  $w(x')$  which is the least number of  $w(x)$  such that  $t_{\mathbb{C}}(x) = x'$ .

Lichtenbaum defined the similar cohomology  $H_L^{*,*'}(X; R)$  by using the étale topology, while  $H^{*,*'}(X; R)$  is defined by using Nisnevich topology. Since Nisnevich covers are some restricted étale covers, there is the natural (cyclic) map  $cl^{*,*'} : H^{*,*'}(X; R) \rightarrow H_L^{*,*'}(X; R)$ . We say that the condition  $BL(n, p)$  holds if

$$BL(n, p) : H^{m,n}(X; \mathbb{Z}_{(p)}) \cong H_L^{m,n}(X; \mathbb{Z}_{(p)}) \quad \text{for all } m \leq n + 1$$

$$(\text{hence } H^{m,n}(X; \mathbb{Z}/p) \cong H_L^{m,n}(X; \mathbb{Z}/p) \quad \text{for all } m \leq n)$$

and all smooth  $X$ . The Beilinson-Lichtenbaum conjecture is that  $BL(n, p)$  holds for all  $n, p$ . It is proved that the  $BL(n, p)$  condition is equivalent the Bloch-Kato conjecture (BK) for degree  $n$  and prime  $p$ . Recently, V. Voevodsky proved the Bloch-Kato conjecture [10]. Hence  $BL(n, p)$  holds for all  $n$  and  $p$ .

Moreover Suslin-Voevodsky proves  $H_L^{m,n}(X; \mathbb{Z}/p) \cong H_{et}^m(X; \mu_p^{\otimes n})$ .

From the dimensional condition (C5) and the above isomorphism, we have isomorphisms

$$H^{m,n}(\text{Spec}(k); \mathbb{Z}/p) = H^{m,n}(pt; \mathbb{Z}/p) \cong H_{et}^m(pt; \mu_p^{\otimes n}) \cong H_{et}^m(pt; \mathbb{Z}/p) \quad \text{if } m \leq n$$

and  $H^{m,n}(pt; \mathbb{Z}/p) \cong 0$  otherwise. Let  $\tau \in H^{0,1}(pt; \mathbb{Z}/p)$  be the element corresponding a generator of  $H_{et}^0(pt; \mu_p) \cong \mathbb{Z}/p$ . Then we get the isomorphism

$$H^{*,*'}(\text{Spec}(k); \mathbb{Z}/p) \cong H_{et}^*(\text{Spec}(k); \mathbb{Z}/p) \otimes \mathbb{Z}/p[\tau]$$

since  $\tau : H_{et}^m(pt; \mu_p^{\otimes n}) \cong H_{et}^m(pt; \mu_p^{\otimes(n+1)})$ . For examples, with  $\deg(\rho) = (1, 1)$ ,

$$H^{*,*'}(\text{Spec}(\mathbb{R}); \mathbb{Z}/2) \cong \mathbb{Z}/2[\rho, \tau], \quad H^{*,*'}(\text{Spec}(\mathbb{C}); \mathbb{Z}/p) \cong \mathbb{Z}/p[\tau].$$

Next we compute the motivic cohomology of  $\mathbb{P}^\infty$  and  $B\mathbb{Z}/p$ . By the cofiber map  $\mathbb{P}^{n-1} \rightarrow \mathbb{P}^n \rightarrow \mathbb{P}^n/\mathbb{P}^{n-1}$  and (C4), we can inductively prove that

$$H^{*,*'}(\mathbb{P}^\infty; \mathbb{Z}/p) \cong H^{*,*'}(pt.; \mathbb{Z}/p) \otimes \mathbb{Z}/p[y]$$

with  $\deg(y) = (2, 1)$ . The Lens space is identified with the sphere bundle associated with the line bundle. This induces the ring isomorphism for  $p = \text{odd}$

$$H^{*,*'}(B\mathbb{Z}/p; \mathbb{Z}/p) \cong \mathbb{Z}/p[y] \otimes \Lambda(x) \otimes H^{*,*'}(pt; \mathbb{Z}/p)$$

with  $\deg(x) = (1, 1)$ . However note that when  $p = 2$ , we see ([8])

$$x^2 = y\tau + x\rho$$

where  $\rho \in H^{1,1}(pt; \mathbb{Z}/p) \cong k^*/k^{2*}$  represents  $-1$ .

By the above cofiber sequence, we can easily see that  $\mathbb{P}^\infty$  and  $B\mathbb{Z}/p$  satisfy the Kunneth formula for all spaces (while Kunneth formula does not hold for general  $X, Y$  in the mod  $p$  motivic cohomology). In particular, we have the ring isomorphisms

$$H^{*,*'}((\mathbb{P}^\infty)^n; \mathbb{Z}/p) \cong \mathbb{Z}/p[y_1, \dots, y_n] \otimes H^{*,*'}(pt; \mathbb{Z}/p)$$

$$H^{*,*'}((B\mathbb{Z}/p)^n; \mathbb{Z}/p) \cong \mathbb{Z}/p[y_1, \dots, y_n] \otimes \Lambda(x_1, \dots, x_n) \otimes H^{*,*'}(pt; \mathbb{Z}/p)$$

(when  $p = 2$ ,  $x_i^2 = y_i\tau + x_i\rho$ ).

This fact is used to define the reduced power operation  $P^i$  in (C3). Since a Sylow  $p$  subgroup of the symmetric group  $S_p$  of  $p$ -letters is isomorphic to  $\mathbb{Z}/p$ , we know the isomorphism

$$H^{*,*'}(BS_p; \mathbb{Z}/p) \cong H^{*,*'}(B\mathbb{Z}/p; \mathbb{Z}/p)^{F_p^*} \cong \mathbb{Z}/p[Y] \otimes \Lambda(W) \otimes H^{*,*'}(pt; \mathbb{Z}/p)$$

with identifying  $Y = y^{p-1}$  and  $W = xy^{p-2}$ . If  $X$  is smooth (and suppose  $p$  is odd to simplify arguments), we can define the reduced powers (of Chow rings) as follows. Consider maps

$$H^{2*,*}(X; \mathbb{Z}/p) \xrightarrow{i_!} H^{2p^*,p^*}(X^p \times_{S_p} ES_p) \xrightarrow{\Delta^*}$$

$$H^{2*,*}(X \times BS_p; \mathbb{Z}/p) \cong H^{2*,*}(X; \mathbb{Z}/p) \otimes_{H^{2*,*}(pt; \mathbb{Z}/p)} H^{2*,*}(BS_p; \mathbb{Z}/p)$$

where  $i_!$  is the Gysin map for  $p$ -th external power, and  $\Delta$  is the diagonal map. For  $deg(x) = (2n, n)$ , the reduced powers are defined as

$$\Delta^* i_!(x) = \sum P^i(x) \otimes Y^{n-i} + \beta P^i(x) \otimes WY^{n-i-1}.$$

Hence note  $deg(P^i) = deg(Y^i) = deg(y^{i(p-1)}) = (2i(p-1), i(p-1))$ .

Voevodsky defined  $i_!$  for non smooth  $X$  also. By using suspensions maps, he defined reduced powers for all degree elements in  $H^{*,*}(X; \mathbb{Z}/p)$  for all  $X$  [8]. Thus we get the operations in (C3).

Moreover Voevodsky defined the motivic Milnor operation such that  $Q_i = [Q_{i-1}, P^{p^{i-1}}] \text{ mod}(\rho)$  (for details see [8])

$$Q_i : H^{*,*'}(X; \mathbb{Z}/p) \rightarrow H^{**+2p^i-1, *'+p^i-1}(X; \mathbb{Z}/p)$$

which is derivative,  $Q_i(xy) = Q_i(x)y + xQ_i(y)$  if  $\rho = 0$ . For the case  $\rho \neq 0$  see [14] or [7].

### 3. MOTIVIC COHOMOLOGY OF $BO_n$ AND $BSO_4$ .

The motivic cohomology of the classifying space is defined as follows. Let  $G$  be a linear algebraic group over  $k$ . Let  $V$  be a representation of  $G$  such that  $G$  acts freely on  $V - S$  for some closed subset  $S$ . Then  $(V - S)/G$  exists as a quasi-projective variety over  $k$ . According to Totaro ([?]) and V.Voevodsky ([6]), we define

$$H^{*,*'}(BG; \mathbb{Z}/p) = \lim_{dim(V), codim(S) \rightarrow \infty} H^{*,*'}((V - S)/G; \mathbb{Z}/p).$$

We still know the motivic cohomologies of  $B\mathbb{G}_m$  and  $B\mathbb{Z}/p$ . Since  $BGL_n$  is cellular, we have

$$H^{*,*'}(BGL_n; \mathbb{Z}/p) \cong \mathbb{Z}/p[c_1, \dots, c_n] \otimes H^{*,*'}(pt; \mathbb{Z}/p)$$

where the Chern class  $c_i$  with  $deg(c_i) = (2i, i)$  are identified with the elementary symmetric polynomial in  $H^{2*,*}((\mathbb{P}^\infty)^n; \mathbb{Z}/p)$ . So we can define the Chern class  $\rho^*(c_i) \in H^{2*,*}(BG; \mathbb{Z}/p)$  for each representation  $\rho : G \rightarrow GL_n$ .

Hereafter we assume  $k = \mathbb{C}$  throughout this paper.

The mod 2 cohomology of the classifying space  $BO_n$  of the  $n$ -th orthogonal group is

$$H^*(BO_n; \mathbb{Z}/2) \cong H^*((B\mathbb{Z}/2)^n; \mathbb{Z}/2)^{S_n} \cong \mathbb{Z}/2[w_1, \dots, w_n]$$

where  $S_n$  is the  $n$ -th symmetry group,  $w_i$  is the Stiefel-Whitney class which restricts the elementary symmetric polynomial in  $\mathbb{Z}/2[x_1, \dots, x_n]$ . Each element  $w_i^2$  is represented by Chern class  $c_i$  of the induced representation  $O(n) \subset U(n)$ . Let us write  $w_i^2$  by  $c_i$ .

Since  $Q_{i-1} \dots Q_0(w_i) \neq 0$ , we see each  $w(w_i) = i$ . However even the module structure of  $gr^* H^*(BO_n; \mathbb{Z}/2)$  seems complicated. W.S.Wilson ([11],[1]) found a good  $Q(i) = \Lambda(Q_0, \dots, Q_i)$ -module decomposition for  $BO_n$ , namely,

$$H^*(BO_n; \mathbb{Z}/2) = \bigoplus_{i=-1} Q(i)G_i \quad \text{with } Q_0 \dots Q_i G_i \in \mathbb{Z}/2[c_1, \dots, c_n].$$

Here  $G_{k-1}$  is quite complicated, namely, it is generated by symmetric functions

$$\Sigma x_1^{2i_1+1} \dots x_k^{2i_k+1} x_{k+1}^{2j_1} \dots x_{k+q}^{2j_q}, \quad k+q \leq n,$$

with  $0 \leq i_1 \leq \dots \leq i_k$  and  $0 \leq j_1 \leq \dots \leq j_q$ ; and if the number of  $j$  equal to  $j_u$  is odd, then there is some  $s \leq k$  such that  $2i_s + 2^s < 2j_u < 2i_s + 2^{s+1}$ . We can prove  $w(G_i) = i + 1$  and hence ;

**Theorem 3.1.** *An element  $x \in H^*(BO_n; \mathbb{Z}/2)$  is  $w(x) = s$  if and only if  $s$  is the maximal number such that  $Q_{i_1} \dots Q_{i_s}(x) \neq 0$  for some  $(i_1, \dots, i_s)$ . Moreover we have the isomorphisms*

$$H^{*,*'}(BO_n; \mathbb{Z}/2) \cong H^{*,*'}((B\mathbb{Z}/2); \mathbb{Z}/2)^{S_n} \cong \mathbb{Z}/2[\tau] \otimes (\bigoplus Q(i)G_i).$$

When  $n = \text{odd}$ , it is well known that there is the isomorphism  $O_n \cong SO_n \times \mathbb{Z}/2$ . Hence we have the isomorphism

$$H^{*,*'}(BSO_{2m+1}; \mathbb{Z}/2) \cong \mathbb{Z}/2[\tau] \otimes (\bigoplus Q(j)G'_j) \quad \text{with } G'_j = i^*G_j$$

where  $i : SO_n \rightarrow O_n$  is the inclusion. (Note  $p^*i^*(w_n) \neq w_n \in H^*(BO_n; \mathbb{Z}/2)$  for the projection  $p : O_n \rightarrow SO_n$ .)

Since the direct decomposition of  $BO_3$  is complicated to write, we only write here that of  $SO_3$  (note  $O_3 \cong SO_3 \times \mathbb{Z}/2$ ).

$$\begin{aligned} H^*(BSO_3; \mathbb{Z}/2) &\cong \mathbb{Z}/2[w_1, w_2, w_3]/(w_1) \cong \mathbb{Z}/2[w_2, w_3] \\ &\cong \mathbb{Z}/2[c_2, c_3]\{1, w_2, w_3 = Q_0 w_2, w_2 w_3 = Q_1 w_2\} \\ &\cong \mathbb{Z}/2[c_2, c_3]\{w_2, Q_0 w_2, Q_1 w_2, c_3 = Q_0 Q_1 w_2\} \oplus \mathbb{Z}/2[c_2] \\ &\cong \mathbb{Z}/2[c_2, c_3] \otimes Q(1)\{w_2\} \oplus \mathbb{Z}/2[c_2]. \end{aligned}$$

Of course, this case  $w(w_2) = 2$  and we have

$$H^{*,*'}(BSO_3; \mathbb{Z}/2) \cong \mathbb{Z}/2[\tau] \otimes (\mathbb{Z}/2[c_2, c_3] \otimes Q(1)\{w_2\} \oplus \mathbb{Z}/2[c_2]).$$

For  $n = \text{even}$ ,  $O_n \not\cong SO_n \times \mathbb{Z}/2$ . The motivic cohomology seems difficult to compute. Even  $n = 4$  it seems complicated. In fact, the realization map  $t_{\mathbb{C}}$  is not injective (i.e.,  $\tau \times y_2 = t_{\mathbb{C}}(y_2) = 0$  in the following theorem).

**Theorem 3.2.** *The motivic cohomology  $H^{*,*'}(BSO_4; \mathbb{Z}/2)$  is isomorphic to*

$$\begin{aligned} &\mathbb{Z}/2[c_2, c_4]\{y_2\} \oplus \mathbb{Z}/2[\tau, c_2] \otimes (\mathbb{Z}/2[c_4]\{1\} \\ &\oplus \mathbb{Z}/2[c_3] \otimes Q(1)\{w_2\} \oplus \mathbb{Z}/2[c_4] \otimes (\mathbb{Z}/2[c_3]Q(2) - \mathbb{Z}/2\{1\})\{a\}) \end{aligned}$$

where  $a$  is a virtual element so that  $t_{\mathbb{C}}(c_3 a) = w_2 w_3 w_4$ ,  $t_{\mathbb{C}}(Q_0 a) = w_4$ ,  $t_{\mathbb{C}}(Q_1 a) = w_2 w_4$  and  $t_{\mathbb{C}}(Q_2 a) = c_4 w_2$ .

4. MOTIVIC COHOMOLOGY OF  $BD_8$  AND  $BQ_8$ 

In this section, we compute the  $mod(2)$  motivic cohomology of  $BD_8$  and  $BQ_8$ .

At first, we consider the case  $Q_8$ . The mod 2 (usual) cohomology is well known (see Theorem 2.7)

$$H^*(BQ_8; \mathbb{Z}/2) \cong \mathbb{Z}/2\{1, x_1, y_1, x_2, y_2, w\} \otimes \mathbb{Z}/2[c_2]$$

where  $x_i^2 = \beta x_i = y_i$  and  $|w| = 3$ . The graded algebra  $gr^{*'} H^*(BQ_8; \mathbb{Z}/2)$  is given by letting the weight degree by

$$w(y_i) = w(c_2) = 0, \quad w(x_i) = w(w) = 1.$$

The facts  $w(y_i) = w(c_2) = 0$  follows from that they are Chern classes. We can prove that  $w(w) = 1$  (in fact, we can take  $w \in H^{3,2}(BQ_8; \mathbb{Z}/2)$ .)

**Theorem 4.1.** *We have the bidegree isomorphism*

$$H^{*,*'}(BQ_8; \mathbb{Z}/2) \cong \mathbb{Z}/2[\tau] \otimes gr^{*'} H^*(BQ_8; \mathbb{Z}/2).$$

Now we consider the case  $G = D_8$ . We recall the  $mod(2)$  cohomology.

$$H^*(BD_8; \mathbb{Z}/2) \cong (\mathbb{Z}/2[x_1, x_2]/(x_1x_2)) \otimes \mathbb{Z}/2[u] \cong (\bigoplus_{i=1}^2 \mathbb{Z}/2[y_i]\{y_i, x_i, y_iu, x_iu\} \oplus \mathbb{Z}/2\{1, u\}) \otimes \mathbb{Z}/2[c_2]$$

Here we identify,  $y_i = x_i^2$  and  $c_2 = u^2$ . The cohomology operations on  $H^*(BD_8; \mathbb{Z}/2)$  is well known, e.g., (see [Te-Ya])

$$Q_0(u) = (x_1 + x_2)u = e, \quad Q_1Q_0(u) = (y_1 + y_2)c_2.$$

**Lemma 4.2.** *There exist  $u'_1, u'_2 \in H^{3,2}(BD_8; \mathbb{Z}/2)$  with  $\tau u'_i = x_iu \in H^{3,3}(BD_8; \mathbb{Z}/2)$  (so  $u'_i = \tau^{-1}x_iu$ ).*

Therefore we get  $gr^{*'} H^*(BD_8; \mathbb{Z}/2)$  which is isomorphic to

$$(\bigoplus_{i=1}^2 \mathbb{Z}/2[y_i]\{y_i, x_i, x_iu'_i, u'_i\} \oplus \mathbb{Z}/2\{1, u\}) \otimes \mathbb{Z}/2[c_2]$$

with  $w(y_i) = w(c_2) = 0$ ,  $w(x_i) = w(u'_i) = 1$  and  $w(u) = w(x_iu'_i) = 2$ . (Note  $u, x_iu'_i \notin CH^*(BG)/2$ , and  $x_iu'_i = y_iu$ ).

**Theorem 4.3.** *We have the the bidegree module isomorphism*

$$H^{*,*'}(BD_8; \mathbb{Z}/2) \cong \mathbb{Z}/2[\tau] \otimes gr^{*'} H^*(BD_8; \mathbb{Z}/2).$$

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