<table>
<thead>
<tr>
<th>Title</th>
<th>RELATIVE PROJECTIVE COVERS AND THE BRAUER CONSTRUCTION OVER FINITE GROUP ALGEBRAS (Cohomology Theory of Finite Groups and Related Topics)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>OKUYAMA, TETSURO</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2012), 1784: 77-95</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2012-03</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/172719">http://hdl.handle.net/2433/172719</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
RELATIVE PROJECTIVE COVERS AND THE BRAUER CONSTRUCTION OVER FINITE GROUP ALGEBRAS

北海道教育大学旭川校 奥山 哲郎 (TETSURO OKUYAMA)
HOKKAIDO UNIVERSITY OF EDUCATION, ASAHIKAWA CAMPUS

Some properties of relative projective covers of modules in the modular representation theory of finite groups will be discussed. Especially, we study effects of the Brauer constructions for relative projective covers of $p$-permutation modules. We also discuss some use of our results to investigate derived equivalences in the principal block algebras of finite groups with Sylow $p$-subgroup isomorphic to $M_{n+1}(p)$, $p$ odd.

In my lecture, we only talked on the sections 1 and 2 below. We include the sections 3 and 4 which provide some results for proofs of theorems in sections 2. Section 5 is also included to give another examples with finite groups with Sylow $p$-subgroups $M_{n+1}(p)$.

Let $k$ be an algebraically closed field of characteristic $p > 0$.

1. $M_{n+1}(p)$

Let $p$ be odd and $n$ be an integer with $n \geq 2$. The $p$-group $M_{n+1}(p) = P$ of order $p^{n+1}$ is presented by

$$M_{n+1}(p) = P := \langle x, y \mid y^{p^n} = 1 = x^p, xyx^{-1} = y^{1+p^{n-1}} \rangle$$

$P$ has a unique maximal elementary abelian $p$-subgroup $\langle x, y^{p^{n-1}} \rangle$. Set

$$Q = \langle y \rangle, \quad R = \langle x \rangle$$

We fix an integer $s \in \mathbb{Z}$ which has multiplicative order $p - 1$ in the residue ring $\mathbb{Z}/p^n\mathbb{Z}$. Notice then that $s$ has multiplicative order $p - 1$ in $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ also. $P$ has an automorphism $t_0$ of order $p - 1$ which sends $x \mapsto x$, $y \mapsto y^s$ so that we have a group.

$$P \rtimes \langle t_0 \rangle = \langle x, y, t_0 \mid y^{p^n} = 1 = x^p, xyx^{-1} = y^{1+p^{n-1}}, t_0^{p-1} = 1, t_0^{-1}xt_0 = x, t_0^{-1}yt_0 = y^s \rangle$$

In the following discussion, fix a positive divisor $e \geq 2$ of $p - 1$ and set $t = t_0^\ell$ where $\ell = \frac{p - 1}{e}$. And set

$$H = P \rtimes \langle t \rangle \cong P \rtimes \mathbb{Z}_e$$

1.1. Some Complexes of $kH$-modules. $kH$ has $e$ simple modules $S(i)$, $i \in \mathbb{Z}/e\mathbb{Z}$ (all of dimension 1). We can name the simples so that the following facts hold.

$$\text{Ext}^1_{kH}(S(i), S(i + 1)) \neq 0, \quad S(0) = k_H$$

Let denote a projective cover of $S(i)$ by $P(i)$. 
By a result of Okuyama and Sasaki [7], we have a (chain) complex $X(1)$ of $kH$-modules

\[ X_{k}^{*}(1) : \cdots \rightarrow S(1) \rightarrow P(1) \rightarrow (P(1) \oplus P(1) \rightarrow \Omega^{-2k}(S(1)) \rightarrow 0 \rightarrow \cdots \]

satisfying $H_{1}(X_{k}^{*}(1)) \cong H_{2}(X_{k}^{*}(1)) = 0$ and $H_{0}(X_{k}^{*}(1)) \cong H_{3}(X_{k}^{*}(1)) = 0$ for each $1 \leq k \leq \ell$ where the last nonzero terms are in degree $0$.

1.2. **Richard's Tilting.** Let $A$ be an arbitrary symmetric algebra over $k$ and let $\{ S(i) \mid i \in I \}$ be the set of simple $A$-modules. Let $P(i)$ be a projective cover of $S(i)$.

Take a (nonempty proper) subset $I_{0}$ of $I$. For each $i \in I$, construct a complex $P^{*}(i) \in C^{b}(P(A))$ of projective $A$-modules as follows.

\[ P^{*}(j) : \cdots \rightarrow 0 \rightarrow R(j) \stackrel{\lambda_{j}}{\rightarrow} P(j) \rightarrow 0 \rightarrow \cdots \quad j \notin I_{0} \]

\[ P^{*}(i) : \cdots \rightarrow 0 \rightarrow P(i) \rightarrow 0 \rightarrow 0 \rightarrow \cdots \quad i \in I_{0} \]

where for $j \notin I_{0}$, $R(j) \stackrel{\lambda_{j}}{\rightarrow} P(j)$ is a minimal one satisfying that

1. $R(j)$ is a direct sum of $P(i), i \in I_{0}$
2. Composition factors of $\text{Cok} \ \lambda_{j}$ are $S(k)$ for some $k \notin I_{0}$

Set

\[ P^{*}(I_{0}) = \bigoplus_{i \in I} P^{*}(i) \]

Then in the homotopy category $K^{b}(P(A))$ of complexes of projective $A$-modules,

$P^{*}(I_{0})$ is a tilting complex for $A$

Set

\[ B = \text{End}_{K^{b}(P(A))}(P^{*}(I_{0})) \]

so that $B$ is a derived equivalent algebra to $A$. $B$ is also a symmetric algebra and simple $B$-modules are also parametrized by the set $I$. Let $Q(i)$ be a projective indecomposable $B$-module corresponding to the summand $P^{*}(i)$ of $P^{*}(I_{0})$. Let $T(i)$ be the simple $B$-module corresponding to $T(i)$. There is a $(A, B)$-bimodule $M(I_{0})$ (with no bimodule projective summand) constructed from the complex $P^{*}(I_{0})$ satisfying the following.

1. Both of $_{A}M(I_{0})$ and $M(I_{0})_{B}$ are projective and a functor

\[ F_{0} = F(I_{0}) : \text{mod-} A \rightarrow \text{mod-} B, \quad V \mapsto V \otimes_{A} M(I_{0}) \]

gives a stable equivalence of Morita type between $	ext{mod-} A$ and $	ext{mod-} B$.

2.1. For $j \notin I_{0}$, $F_{0}(S(j)) = T(j)$ for $j \notin I_{0}$.

2.2. For $i \in I_{0}$, let $\text{Soc} \ P(i) \subset W(i) \subset P(i)$ be the largest submodule of $P(i)$ such that all the composition factors of $W(i)/S(i)$ are $S(k)$ for some $k \notin I_{0}$. Then $F_{0}(P(i)/W(i)) = T(i)$ for $i \in I_{0}$. 


We call the procedure above the Richard's Tilting with respect to the set $I_0 \subset I$. The functor $F_0$ given above is called the associated functor of the tilting. The dual argument to the above discussion is also valid which we call the dual Richard's Tilting.

1.2.1. Examples. For $kH$, apply Richard's tiltings with respect to the set $\{1\}$ twice.

First do the Richard's tilting with respect to the set $\{1\}$. And then for the resulting new algebra, do the Richard's tilting with respect to the set $\{1\}$.

Let $A_2$ be the resulting algebra and let $S(i)_2$ (resp. $P(i)_2$) be a simple (resp. projective) $A_2$-module corresponding to $S(i), i \in I$. Let $F^2 : \text{mod}-kH \rightarrow \text{mod}-A_2$ be the associated functor. Then by the existence of the complex $X_i(1)$ in (1.1), we have

**Lemma 1.1.**

$$F^2(S(i)) = S(i)_2, \quad i \neq 1, \quad F^2(S(1)) = \Omega^{2e}(S(1)_2)$$

The existence of complexes $X^*_k(1)$ $(1 \leq k \leq \ell)$ implies the following. For each $k$ with $1 \leq k \leq \ell$, do the Richard's tiltings with respect to the set $\{1\}$ $2k$ times.

Let $A_{2k}$ be the resulting algebra and let $S(i)_{2k}$ (resp. $P(i)_{2k}$) be a simple (resp. projective) $A_{2k}$-module corresponding to $S(i), i \in I$. Let $F^{2k} : \text{mod}-kH \rightarrow \text{mod}-A_{2k}$ be the associated functor. Then

**Lemma 1.2.**

$$F^{2k}(S(i)) = S(i)_{2k}, \quad i \neq 1, \quad F^{2k}(S(1)) = \Omega^{2ke}(S(1)_{2k})$$

The discussion above is valid for any fixed $i_0 \in I$.

**Lemma 1.3.** Let $i_0 \in I$ and $k$ be an integer with $1 \leq k \leq \ell$.

1. There exists an algebra $B$ derived equivalent to $kH$ satisfying the following. Let $T(i), i \in I$ be the set of simple $B$-modules and $F^* : \text{mod}-kH \rightarrow \text{mod}-B$ be the associated stable equivalence. Then

$$F^*(S(i)) = T(i), \quad i \neq i_0, \quad F^*(S(i_0)) = \Omega^{2ke}(T(i_0))$$

2. There exists an algebra $C$ derived equivalent to $kH$ satisfying the following. Let $U(i), i \in I$ be the set of simple $C$-modules and $F_* : \text{mod}-kH \rightarrow \text{mod}-C$ be the associated stable equivalence. Then

$$F_*(S(i)) = U(i), \quad i \neq i_0, \quad F_*(S(i_0)) = \Omega^{-2ke}(U(i_0))$$

1.3. Relative Projective Covers. Set

$$K = R \times \langle t \rangle \times \langle x \rangle \times \langle t \rangle \subset H$$

and

$$P_R(i) = (S(i) \downarrow_{K}) \uparrow^H = P_R(0) \otimes S(i)$$

Then we have a canonical surjection and a canonical injection

$$P_R(0) \mu \rightarrow S(0) \rightarrow 0, \quad 0 \rightarrow S(0) \nu \rightarrow P_R(0)$$

$\mu$ is so called an (relative) $R$-projective cover of $S(0) = k_H$ and $\nu$ is an (relative) $R$-injective hull of $S(0) = k_H$. 
For any $kH$-module $V$, an $R$-projective cover ($R$-injective hull) of $V$ is obtained as a summand of the sequence obtained by tensoring with the above sequences. Let $\Omega_R(V)$ (resp. $\Omega_R^{-1}(V)$) be the kernel (resp. cokernel) of an $R$-projective cover (resp. $R$-injective hull) of $V$. In particular, we have the following short exact sequences,

$$0 \rightarrow \Omega_R(S(0)) \rightarrow P_R(0) \xrightarrow{\mu} S(0) \rightarrow 0, \quad 0 \rightarrow S(1) \xrightarrow{\nu} P_R(0) \rightarrow \Omega_R^{-1}(S(0)) \rightarrow 0$$

The **heart** $H_R(0)$ of $P_R(S(0)) = P_R(k_H)$ is defined by

$$H_R(0) = \text{Ker} \mu / \text{Im} \nu$$

1.3.1. **Examples.** By a result of Okuyama and Sasaki [7], we have

$$\Omega_R^2(S(0)) \cong \Omega^{-2(p-1)}(S(1))$$

Actually, we can show that an $R$-projective cover of $\Omega_R(S(0))$ has the form

$$0 \rightarrow \Omega^{-2(p-1)}(S(1)) \rightarrow P(1) \oplus P_R(1) \rightarrow \Omega_R(S(0)) \rightarrow 0$$

so that we have the complex $X^\bullet_0$ of $kH$-modules of the form

$$X^\bullet_0 : \cdots \rightarrow 0 \rightarrow \Omega^{-2(p-1)}(S(1)) \rightarrow P(1) \oplus P_R(1) \rightarrow H_R(0) \rightarrow 0 \rightarrow \cdots$$

which satisfies that

$$H_1(X^\bullet_0) = S(0), \quad H_2(X^\bullet_0) = 0 = H_0(X^\bullet_0)$$

where $H_R(0)$ is in degree 0 term. Set

$$F_*(1) = \Omega^{-1}\Omega_R(H_R(0))$$

Then by the sequence (1.2), we have the complex $X^\bullet$ of $kH$-modules of the form

$$X^\bullet : \cdots \rightarrow 0 \rightarrow \Omega^{-2(p-1)}(S(1)) \rightarrow P(1) \oplus P(1) \rightarrow F_*(1) \rightarrow 0 \rightarrow \cdots$$

which satisfies that

$$H_1(X^\bullet) = S(0), \quad H_2(X^\bullet) = 0 = H_0(X^\bullet) \quad \text{and} \quad F_*(1) \subset P(-1) \oplus P(1)$$

where $F_*(1)$ is in degree 0 term.

Assume that $e = 2$. Do the Richard’s tiltings with respect to the set \{ 1 \} $p = 2k + 1$ times.

Let $A_0$ be the resulting algebra and let $S(i)_0$ (resp. $P(i)_0$) be a simple (resp. projective) $A_0$-module corresponding to $S(i)$, $i \in I$. Let $F^0 : \text{mod-}kH \rightarrow \text{mod-}A_0$ be the associated functor. Then by the existence of complexes $X^\bullet_0(1)$ in (1.1) and $X^\bullet$ in (1.3), we have the following lemma.

**Lemma 1.4.** Assume that $e = 2$. Then in the notations above, we have

$$F^0(S(0)) = S(0)_0, \quad i \neq 1, \quad F^0(F_*(1)) = S(1)_0$$
2. Example $SL(2, q)$

The example here is one discussed by Holloway-Koshitani-Kunugi [4].

Let $q_1$ be a prime power and $p$ be an odd prime such that $p$ divides $q_1 + 1$. Write $q_1 + 1 = p^{n-1} \ell'$, $(p, \ell') = 1, n \geq 2$. Set $q = q_1^p$. Then $q + 1 = p^n \ell$ for some positive integer $\ell$ with $(p, \ell) = 1$.

Set

$G_0 = SL(2, q), \ C_0 = SL(2, q_1), \ R = \mathcal{G}(GF(q)/GF(q_1)) = \langle x \rangle, \ G = R \ltimes G_0$

Let $B_0 = T_0 \ltimes U_0$ be a Borel subgroup of $G_0$ where $|T_0| = (q - 1)$ and $|U_0| = q$. We have an $R$-invariant subgroup $F_0 \supset Z(G_0)$ of order $q + 1$ such that $F_0 \cap C_0$ is of order $q_1 + 1$ and $B_0 \cap F_0 = Z(G_0)$.

Let $P_0 \subset F_0$ be a Sylow $p$-subgroup of $G_0$ and set $P = R \ltimes P_0$. We have that $P \cong M_{n+1}(p)$. We use notations introduced in the beginning of the talk.

So $Q = P_0$. Let $H = N_C(Q) = N_C(P_0)$. Then $H/O_P(H)$ is our $H$ with $e = 2$. Set $H_0 = N_{C_0}(P_0) = H \cap G_0$.

2.1. $B_0(kG_0)$. The principal block algebra $B_0(kG_0)$ of $kG_0$ has a cyclic defect group and is well understood. It is known that $B_0(kG_0)$ and the principal block $B_0(kH_0)$ are derived equivalent. A two sided tilting complex for $B_0(kG_0)$ and $B_0(kH_0)$ due to Rouquier is given as follows. Set

$A = B_0(kG_0), \ B = B_0(kH_0)$

$B_0(kG_0)$ and $B_0(kH_0)$ have two simple modules

$B_0(kG_0) : \phi_0 = k_{G_0}, \ \phi_1, \ \dim_k \phi_1 = q - 1$

$B_0(kH_0) : T_0 = k_{H_0}, \ T_1, \ \dim_k T_1 = 1$

$B = B_0(kH_0)$ is a symmetric Nakayama algebra of length $p^n$.

Let $P(\phi_i)$ ($i = 0, 1$) be a projective cover of $\phi_i$ and $P(T_i)$ ($i = 0, 1$) be a projective cover of $T_i$.

$A$ is a $(A, B)$-bimodule (a $(kG_0, kH_0)$-bimodule). As usual, we can regard $A$ as $k[G \times H]$-module. Let $M_0$ be a Broué-Puig indecomposable $(A, B)$-summand of $A$. As a $k[G \times H]$-module, $M_0$ is a Scott module with vertex $\Delta P_0 = \{ (a, a) ; a \in P_0 \} \subset G_0 \times H_0$.

Actually, for the group $GL(2, q)$, $M_0 = A$. Notice also that $A M_0, \ M_B$ are both projective.

A functor

$F : \text{mod-} A \rightarrow \text{mod-} B, \ V \mapsto V \otimes_A M_0$

gives a stable equivalence of Morita type between $\text{mod-} A$ and $\text{mod-} B$.

We can see that a $\Delta P_0$-projective cover of $k = k_{G \times H}$ has the form

$P(T_1) \xrightarrow{\lambda} K \rightarrow 0$

and $\text{Top Ker } \pi = \phi_1^* \otimes_k T(1)$ where $\phi_1^* = \text{Hom}_k(\phi_1, k)$ is a left $kG_0$-module. Let $P(\phi_1)^* \otimes_k P(T_1) \xrightarrow{\lambda} \text{Ker } \pi \rightarrow 0$ be a projective cover of $\text{Ker } \pi$ and consider the following complex $M^*$ of $(A, B)$-bimodules.

$M^* : \cdots \rightarrow 0 \rightarrow P(\phi_1)^* \otimes_k P(T_1) \xrightarrow{\lambda} M_0 \rightarrow 0 \rightarrow \cdots$
The complex $M^\bullet$ satisfies the following conditions.

\[ M^\bullet \otimes_B M^\bullet \cong A[0] \oplus Z^\bullet, \quad M^\bullet \otimes_A M^\bullet \cong B[0] \oplus W^\bullet \]

in $\text{C}^b(\text{mod- } A^\oplus \otimes A)$ and $\text{C}^b(\text{mod- } B^\oplus \otimes B)$, respectively where $Z^\bullet$ is a contractible complex of projective $(A, A)$-bimodules and $W^\bullet$ is a contractible complex of projective $(B, B)$-bimodules.

\[
\begin{align*}
T_1 \\
T_0 \\
F(\phi_0) = T_0, \quad F(\phi_1) = \phi_1 \otimes_A M_0 = : \text{of length } p^n - 2 \\
T_0 \\
T_1
\end{align*}
\]

and

\[
\begin{align*}
\phi_0 \otimes_A M^\bullet : \cdots & \to 0 \to 0 \to T_0 \to 0 \to \cdots \\
\phi_1 \otimes_A M^\bullet : \cdots & \to 0 \to P(T_1) \overset{\pi_1}{\to} F(T_1) \to 0 \to \cdots
\end{align*}
\]

where $P(T_1) \overset{\pi_1}{\to} F(T_1) \to 0$ is a projective cover of $F(T_1)$.

As a complex of projective $B$-modules, $M^\bullet_B$ is a complex obtained by the Richard's tilting for the algebra $B$ with respect to the set $I_0 = 1 \subset I = \{0, 1\}$. We have

\[
\begin{align*}
P(\phi_0) \otimes_A M^\bullet & : \cdots \to 0 \to P(T_1) \to P(T_0) \to 0 \to \cdots \\
P(\phi_1) \otimes_A M^\bullet & : \cdots \to 0 \to P(T_1) \to 0 \to 0 \to \cdots
\end{align*}
\]

and

$M^\bullet_B \cong P^\bullet(0) \oplus (q - 1)P^\bullet(1)$

2.2. Let $\Gamma = (G_0 \times H_0)\Delta R \subset G \times H$. $M^\bullet_B = M_0$ so that $M_0$ is a $k\Gamma$-module and has a vertex $\Delta P$. $M = M_0 \uparrow^{G \times H}$ is a Broué-Puig indecomposable $(B_0(kG), B_0(kH))$-module. There exists a $p$-permutation $k\Gamma$-module $X_0$ with vertex $\Delta R$ such that $X_0 \downarrow^{G \times H} = P(\phi_1)^* \otimes_k P(T_1)$. So it is natural to ask whether we can construct a complex $X^\bullet$ of $k\Gamma$-modules of the form

$X^\bullet : \cdots \to 0 \to X_0 \overset{\mu}{\to} M_0 \to 0 \to \cdots$

such that $X^\bullet \downarrow^{G_0 \times H_0} \cong M^\bullet$. If such a complex exits, then $X^\bullet \uparrow^{G \times H}$ gives a two-sided tilting complex for $B_0(kG)$ and $B_0(kH)$.

However, we can no have such a complex.

2.3. Recall that $C_0 = SL(2, q_1) = C_{G_0}(R)$ and $N_G(R) = R \times C_0$. The principal block algebra $B_0(kC_0)$ has a cyclic defect group $Q_0 = C_G(R)$ and the structure of $B_0(kC_0)$ is described in the entirely same way as in $B_0(kG_0)$. $B_0(kC_0)$ and $B_0(kN_{C_0}(Q_0))$ have two simple modules

$B_0(kC_0) : \theta_0 = k_{C_0}, \quad \theta_1, \quad \dim_k \theta_1 = q_1 - 1$

$B_0(kN_{C_0}(Q_0)) : T'_0 = k_{N_{C_0}(Q_0)}, \quad T_1', \quad \dim_k T_1' = 1$

$B_0(kC_0)$ and $B_0(kN_{C_0}(Q_0))$ are derived equivalent. Let $N_0$ be a Broué-Puig indecomposable module for them and let $N_0^\bullet$ be the two-sided tilting complex for them so that $N_0^\bullet$ has the form

$N_0^\bullet : \cdots \to 0 \to P(\theta_1)^* \otimes P(T_1') \to N_0 \to 0 \to \cdots$
Using the isomorphism \((C_0 \times N_{C_0}(Q_0))\Delta(R)/\Delta R = C_0 \times N_{C_0}(Q_0)\), we can lift \(N_0^*\) to a twosided tilting complex \(N^*\) for \(B_0(kN_G(R))\) and \(B_0(N_H(R))\).

\[(2.2)\]
\[N^* : \cdots \to 0 \to Y \to N \to 0 \to \cdots\]

\(N\) is a Broué-Puig indecomposable \((B_0(kN_G(R)), B_0(kN_H(R)))\)-module. The \textbf{Brauer constructions} for \(M\) with respect to \(\Delta R\) is \(N\). If we set \(X = X_1 \uparrow^{G \times H}\), then \(X(\Delta R) = Y\). And we can construct a complex of \((B_0(kG), B_0(kH))\)-bimodules \(X^*\) of the form

\[(2.3)\]
\[X^* : \cdots \to 0 \to X \xrightarrow{\mu} M \to 0 \to \cdots\]

such that \(X^*(\Delta R) \cong N^*\). \(X^*(\Delta R)\) satisfies the following conditions.

\[X^* \otimes_{B_0(kH)} X^* \cong B_0(kG)[0] \oplus Z^*\]

\[X^* \otimes_{B_0(kG)} X^* \cong B_0(kH)[0] \oplus W^*\]

in \(C^b(mod-B_0(kG)^{op} \otimes B_0(kG))\) and \(C^b(mod-B_0(kH)^{op} \otimes B_0(kH))\), respectively where \(Z^*\) is a complex of projective \((B_0(kG), B_0(kG))\)-bimodules and \(W^*\) is a contractible complex of projective \((B_0(kH), B_0(kH))\)-bimodules. A way of construction of \(X^*\) by \(Y^*\) is a (very special type of) \textit{gluing methods of Rouquier}.

If we take a suitable projective \((B_0(kG), B_0(kH))\)-bimodule \(X'\) and a map \(X' \xrightarrow{\nu} M\) such that

\[X \oplus X' \xrightarrow{\mu \oplus \nu} M \to 0\] (exact)

Then the complex

\[(2.4)\]
\[X^* : \cdots \to 0 \to X \oplus X' \xrightarrow{\mu \oplus \nu} M \to 0 \to \cdots\]

has the same properties as for \(X^*\) where the complexes \(Z^*\) and \(W^*\) have homologies concentrated in degree 0. In particular, if we set

\[M_1 = \Omega^{-1}(\text{Ker}(\mu \oplus \nu))\]

, then a functor

\[F_1 : mod-B_0(kG) \to mod-B_0(kH), \quad V \mapsto V \otimes_{B_0(kG)} M_1\]

gives a stable equivalence of Morita type between \(mod-B_0(kG)\) and \(mod-B_0(kH)\). We have the following lemma.

\textbf{Lemma 2.1}. \[F_1(\varphi_0) = S(0), \quad F_1(\varphi_1) = \Omega^{-1}\Omega_R(S(0))\]

Thus by Lemma 1.4, the following result follows.

\textbf{Corollary 2.2} (Holloway-Koshitani-Kunugi [4]). \(B_0(kG)\) and \(B_0(kH)\) are derived equivalent.

The procedure of Richard’s tilting in the previous section implies that the resulting twosided tilting complex has the following form

\[\cdots \to 0 \to X_p \to X_{p-1} \to \cdots \to X_2 \to X_1 \oplus X_1' \to M \to 0 \to \cdots\]

The results in this section are obtained through the discussions with Koshitani and Kunugi.
3. Relative Projective Covers and Brauer Construction

3.1. Relative Projective Covers. Let $G$ be a finite group and $\mathfrak{X}$ be a nonempty family of subgroups of $G$. For a $kG$-module $M$, a short exact sequence $0 \to N \to X \to M \to 0$ of $kG$-modules is called $\mathfrak{X}$-projective cover of $M$ if it satisfies

1. $X$ is $\mathfrak{X}$-projective,
2. the sequence $M$ is $\mathfrak{X}$-split.

For a $kG$-module $M$, a minimal $\mathfrak{X}$-projective cover of $M$ exists and is uniquely determined up to isomorphism of exact sequences. An arbitrary $\mathfrak{X}$-projective cover contains a minimal one as a summand of exact sequences. If the above sequence $M$ is minimal, then we denote $N$ by $\Omega_{\mathfrak{X}}(M)$. $M$ is $\mathfrak{X}$-projective if and only if $\Omega_{\mathfrak{X}}(M) = 0$. For $kG$-modules $M$ and $M'$, $\Omega_{\mathfrak{X}}(M \oplus M') = \Omega_{\mathfrak{X}}(M) \oplus \Omega_{\mathfrak{X}}(M')$.

Let $H$ be a subgroup of $G$ and set $\mathfrak{Y} = X^{G} \cap H = \{ A^{g} \cap H ; g \in G, A \in \mathfrak{X} \}$. Then the short exact sequence of $kH$-module $M \downarrow H : 0 \to N \downarrow H \to X \downarrow H \to M \downarrow H \to 0$ is a $\mathfrak{Y}$-projective presentation of a $kH$-module $M \downarrow H$, not necessarily minimal even if $M$ is minimal.

3.2. Brauer Construction.

3.3. Let $G$ be a finite group and $Q$ be a $p$-subgroup of $G$. Then a functor called the Brauer construction with respect to $Q$;

$$-(Q) : \text{mod}(kG) \to \text{mod}(kN_{G}(Q)/Q)$$

is defined by

$$M(Q) = M^{Q}/(\sum_{R \subsetneq Q} \text{Tr}_{R,Q}(M^{R}))$$

The canonical epimorphism from $M^{Q} \to M(Q)$ is denoted by $\text{Br}_{Q}$ and is called the Brauer homomorphism with respect to $Q$.

If $M$ and $N$ are $kG$-modules and $f : M \to N$ is a $kG$-homomorphism, $f$ induces a $kN_{G}(Q)/Q$-homomorphism $f(Q) : M(Q) \to N(Q)$. We denote $f(Q)$ by $\text{Br}_{Q}(f)$. The Green correspondence with respect to $(G, N_{G}(Q), Q)$ gives a bijection between the set of isomorphism classes of indecomposable $p$-permutation $kG$-modules with vertex $Q$ and the set of isomorphism classes of indecomposable projective $kN_{G}(Q)/Q$-modules. If $X$ is an indecomposable $p$-permutation $kG$-modules with vertex $Q$, then the corresponding indecomposable projective $kN_{G}(Q)/Q$-module is the Brauer construction $X(Q)$.

**Lemma 3.1.** Assume that $M \downarrow Q$ is a permutation $kQ$-module. If $M(Q)$ has a projective $kN_{G}(Q)/Q$-summand $U$, then $M$ has a $Q$-projective summand $V$ with vertex $Q$ such that $V(Q) = U$.

**Proof.** $kN_{G}(Q)$-module $M \downarrow N_{G}(Q)$ satisfies the assumption in the lemma for the group $N_{G}(Q)$ and a $p$-subgroup $Q$ of $N_{G}(Q)$. Thus by a theorem of Burry-Carlson, we may assume that $Q$ is normal in $G$. Let

$$X \xrightarrow{f} M \to 0 \quad 0 \to M \xrightarrow{g} Y$$

be a $Q$-projective cover and a $Q$-injective hull of $M$, respectively. As $M \downarrow Q$ is a permutation module, $X$ and $Y$ are $Q$-projective, $p$-permutation $kG$-modules. In particular,
$X(Q)$ and $Y(Q)$ are projective $kG/Q$-modules. As the sequence above are $Q$-split, we have exact sequences,

$$X(Q) \xrightarrow{f(Q)} M(Q) \to 0, \quad 0 \to M(Q) \xrightarrow{g(Q)} Y(Q)$$

There exists a primitive idempotent $e \in kG$ such that $e|Q|kG \cong U$. Thus there exists an element $m \in M^Q$ such that $me = m$ and $\overline{m}kG = U$ where $\overline{m} \in M(Q)$ is the image of $m \in M^Q$ in $M(Q)$. We can take an element $x \in X^Q$ such that $f(x) = m$ and $xe = x$.

Write $X = X_0 \oplus X_1$ where $X_0$ is a projective $kG/Q$-module and each indecomposable summand of $X_1$ has a vertex properly contained in $Q$. And write $x = x_0 + x_1$ with $x_i \in X_i$. Then $x_0 e = x, x_1 e = x_1$ and $x_1 \in \sum_{R \subseteq Q} Tr_R(X_i^R)$. Thus $\overline{m} = f(x_0)$ and $f(x_0)kG \cong U$.

As $X_0$ is a $kG/Q$-module and $x_0 e = e$, $x_0 kG$ is a homomorphic image of $[Q]ekG$ and we can conclude that $x_0 kG \cong [Q]ekG \cong U$. Set $V = x_0 kG$. Then $V$ is a direct summand of $X_0$ (and of $X$). Thus we have proved that we have a direct sum decomposition of $kG$-modules

$$X = V \oplus V'$$

such that $V \cong U$, $f(Q)(V(Q)) = U \subset M(Q)$ and $f(Q) \downarrow_{V(Q)}: V(Q) \to M(Q)$ induces isomorphisms

$$f(Q) \downarrow_{V(Q)}: V(Q) \to U$$

Write $Y = Y_0 \oplus Y_1$ where $Y_0$ is a projective $kG/Q$-module and each indecomposable summand of $Y_1$ has a vertex properly contained in $Q$. And write $g(m) = y_0 + y_1$ with $y_i \in Y_i$. Then $y_0 e = y_0$ and $g(Q)(\overline{m}) = \overline{y_0} \in Y_0(Q)$. By the similar argument as above, it follows that $y_0 kG \cong [Q]ekG \cong U$ and we have a direct sum decomposition of $kG$-modules

$$Y = W \oplus W'$$

such that $W \cong U$, $g(Q)(U) = W(Q) \subset Y(Q)$.

Let $\lambda : V \to X$, $\mu : Y \to W$ be the injection and projection with respect to the above decompositions and consider the maps $f' = f \circ \lambda : V \to M$ and $g' = \mu \circ g : M \to W$. Then $f'(Q) = f(Q) \circ \lambda(Q)$ and $g'(Q) = \mu(Q) \circ g(Q)$. By the discussions above, the composite $g'(Q) \circ f'(Q) : V(Q) \to M(Q) \to W(Q)$ is an isomorphism. As $(g' \circ f')(Q) = g'(Q) \circ f'(Q)$, it follows that the map $g' \circ f : V \to M \to W$ is an isomorphism and that $V$ is isomorphic to a summand of $M$.

**Lemma 3.2.** Assume that $M \downarrow Q$ is a permutation $kQ$-module and let $0 \to N \to X \to M \to 0$ be a $Q$-projective cover of $M$. Then $0 \to N(Q) \to X(Q) \to M(Q) \to 0$ is a minimal projective cover of a $kN_G(Q)/Q$-module $M(Q)$.

**Proof.** As the sequence $0 \to N \to X \to M \to 0$ is $Q$-split, the resulting sequence $0 \to N(Q) \to X(Q) \to M(Q) \to 0$ is exact and a projective presentation of a $kN_G(Q)/Q$-module $M(Q)$. We also have that $N \downarrow Q$ is a permutation module. By Lemma 3.1, $N(Q)$ has no projective $kN_G(Q)/Q$-summand and the lemma follows.

The results in this section are obtained by joint works with Kunugi.
4. **Finite Groups with Sylow $p$-Subgroup $M_{n+1}(p)$**

Let $p$ be an odd prime and $n \geq 2$ be an integer. Consider the $p$-group $M_{n+1}(p) = P$ of order $p^{n+1}$ given in Section 1. We use notations in Section 1. And set

$$ P_0 = \langle y \rangle, \quad z = y^{n-1} = [x, y], \quad Z = \langle z \rangle $$

$$ R = \langle x \rangle, \quad Z(P) = \langle y^p \rangle, \quad Q = C_P(R) = R \times Z(P) $$

For an integer $i$,

$$ (y^i x)^p = y^{ip} $$

Thus

$$ \Omega_p(P) = \langle x, z \rangle, \quad x \sim_P xz^i, \quad 0 \leq i \leq p-1, \quad Z(P) = \langle y^p \rangle $$

and it follows that a nontrivial subgroup $S$ of $P$ contains $Z$ or is conjugate to $R$ in $P$.

Let $G$ be a finite group with Sylow subgroup $P = M_{n+1}(p)$ such that there exists a normal subgroup $G_0$ satisfying that

$$ G = R \ltimes G_0, \quad G_0 \cap P = P_0 $$

Set

$$ H = N_G(P_0) = R \ltimes N_{G_0}(P_0), \quad N_{G_0}(P_0) = H_0 $$

Then $N_G(P) \subset N_G(P_0) = H$ and $H/O'_{p'}(C_G(P))$ isomorphic to a subgroup of $\langle t_0 \rangle \ltimes P$. Set

$$ e = |H/PC_G(P)|, \quad H = \langle t, PC_G(P) \rangle $$

so that $t^e \in O'_{p'}(C_G(P))$ and $H/O'_{p'}(H)$ is the group $H$ in Section 1.

In this section, we shall be concerned with the principal block algebras $B_0(kG)$ and $B_0(kH)$ of $kG$ and $kH$.

Notice that $G$ and $H$ have the same $p$-local structure.

**4.1. $p$-Locals.** Let $M$ be a Broué-Puig indecomposable $k[G \times H]$-direct summand of $B_0(kG)$ with vertex $\Delta P$. As we are working on the principal block case, $M$ is a Scott $k[G \times H]$-module with vertex $\Delta P$. We investigate Brauer constructions $M(\Delta S)$ for nontrivial subgroups $S$ of $P$.

**4.1.1. $Z$.** By a theorem of Burnside, $C_{G_0}(Z)$ is $p$-nilpotent and therefore so is $C_G(Z)$. In particular,

$$ N_G(Z) = O'_{p'}(C_G(Z))N_H(Z) $$

and

$$ M(\Delta Z) = B_0(kC_G(Z)) = B_0(kC_H(Z)). $$

**4.1.2. $S \supset Z$.** Let $S \supset P$ with $S \supset Z$. Then $N_G(S) \subset N_G(Z)$ because $Z \subset S \cap G_0$ and $S \cap G_0$ is cyclic.

Thus

$$ M(\Delta S) = B_0(kC_G(S)) = B_0(kC_H(S)). $$
4.1.3. $R$. We can see that

$$N_G(R) = R \ltimes N_{G_0}(R) = R \times C_{G_0}(R) = C_G(R)$$

Set $C = C_G(R)$ and $C_0 = C_{G_0}(R)$. Then $Q = R \times Z(P)$ is a Sylow $p$-subgroup of $C$ and $Z(P)$ is a Sylow $p$-subgroup of $C_0$. We also have that

$$N_H(R) = C_H(R) = R \times C_{H_0}(R)$$

Set $K_0 = C_{H_0}(R)$. Then $N_{C_0}(Z) = K_0 O_{p'}(C_{C_0}(Z))$ by the following facts.

$$C_{H_0}(R) \subset N_{C_0}(Z(P)), \quad N_{C_0}(Z(P)) = C_{H_0}(R) O_{p'}(N_{C_0}(Z(P)))$$

$$N_{C_0}(Z(P)) \subset N_{C_0}(Z) = \langle t, C_{C_0}(Z) \rangle, \quad N_{C_0}(Z) = N_{C_0}(Z(P)) O_{p'}(C_{C_0}(Z))$$

As $(kC_0, kN_{C_0}(Z))$-module, $B_0(kC_0) = N' \oplus$ proj. where $N'$ is a Broué-Puig module for $B_0(kC_0)$ and $B_0(kK_0)$.

Thus by the result above, as $(kC_0, kK_0)$-module,

$$B_0(kC_0) = N_0 \oplus \text{proj.}$$

where $N_0$ is indecomposable and has a vertex $\Delta Z(P)$ (Actually, in the situation here, $N_0 = N'$). $N_0$ gives a stable equivalence between $B_0(kC_0)$ and $B_0(kK_0)$. By a result of Rouquier, there exists two terms Rickard complex $Y_0$ for $B_0(kC_0)$ and $B_0(kK_0)$ of the following form,

$$Y_0 : \cdots \rightarrow 0 \rightarrow Y_0 \xrightarrow{\nu_0} N_0 \rightarrow 0 \rightarrow \cdots$$

where $Y_0$ is a projective $k[C_0 \times K_0]$-module. If $Q_0 \xrightarrow{\nu_0} N_0 \rightarrow 0$ is a projective cover of $N_0$, then $Y_0$ can be taken from a direct summand of $Q_0$ and $\nu_0 = \nu_0 \downarrow Y_0$. We know that

$$B_0(kC) = B_0(kC_G(R)) = k R \otimes_k B_0(kC_0), \quad B_0(kC_H(R)) = k R \otimes_k B_0(kK_0)$$

Thus as $k[C \times C_H(R)]$-module,

$$B_0(kC) = N \oplus \Delta R \text{-proj.}$$

where $N$ is a Broué-Puig module for $B_0(kC)$ and $B_0(kC_H(R))$. As $N_{G \times H}(\Delta R) = C_G(R) \times C_H(R)$, we have $M(\Delta R) = N$.

The complex $Y_0$ can be lifted to a Rickard complex for $B_0(kC_G(R))$ and $B_0(kC_H(R))$ as follows. By the canonical epimorphism $\Delta R(C_0 \times K_0)/\Delta R \cong C_0 \times K_0$, the inflated complex $\overline{Y_0}$ of $k[\Delta R(C_0 \times K_0)]$-modules of $Y_0$ can be constructed.

$$\overline{Y_0} : \cdots \rightarrow 0 \rightarrow \overline{Y_0} \xrightarrow{\overline{\nu_0}} \overline{N_0} \rightarrow 0 \rightarrow \cdots$$

Then the induced complex $Y = \overline{Y_0} \uparrow^{C_G(R) \times C_H(R)}$ is the desired Rickard complex for $B_0(C_G(R))$ and $B_0(C_H(R))$. The degree 0 term of $Y$ is $\overline{N_0} \uparrow^{C_G(R) \times C_H(R)}$ and is isomorphic to $N = M(\Delta R)$. Thus $Y$ has the form

$$Y : \cdots \rightarrow 0 \rightarrow Y \xrightarrow{\nu} M(\Delta R) \rightarrow 0 \rightarrow \cdots$$

where $Y = \overline{Y_0} \uparrow^{C_G(R) \times C_H(R)}$.

Let $Q_0 \xrightarrow{\nu_0} N_0 \rightarrow 0$ be a projective cover of $N_0$ as before so that $Q_0 = Y_0 \oplus Z_0$ for some projective $k[C_0 \times K_0]$-module and $\nu_0 = \nu_0 \downarrow Y_0$. Set

$$Q = \overline{Q_0} \uparrow^{C_G(R) \times C_H(R)}, \quad Y = \overline{Y_0} \uparrow^{C_G(R) \times C_H(R)}, \quad Z = \overline{Z_0} \uparrow^{C_G(R) \times C_H(R)}$$
Then the resulting sequence $Q \to N \to 0$ is a $\Delta R$-projective cover of $N = M(\Delta R)$, $Q = Y \oplus Z$ and $v = \nu' \downarrow Y$.

By our construction, each indecomposable summand of $Y$ has a vertex $\Delta R$. Let $X' \to M \to 0$ be a $\Delta R$-projective cover of $M$. Then its Brauer construction $X'(\Delta R) \to M(\Delta R) \to 0$ is a $\Delta R$-projective cover of $kN_{G \times H}(\Delta R)$-module $M(\Delta R)$. Thus we have a decomposition $X' = X \oplus W$ of $\Delta R$-modules such that each indecomposable summand of $X$ has a vertex $\Delta R$ and $X(\Delta R) = Y$. Now set $\mu = \mu' \downarrow X$ and set

$$X; \cdots \to 0 \to X \to M \to 0 \to \cdots$$

Then by our construction we have $X(\Delta R) = Y$. And for $S \subset P$ with $S \supset Z$, we have $X(S) = M(\Delta S)$.

Now a result of Rouquier says the following fact.

**Lemma 4.1.** The complex $X$ induces a stable equivalence of Rickard type between $B_0(kG)$ and $B_0(kH)$.

*4.1.4. Stable Equivalence.* Let $W$ be the $(B_0(kG), B_0(kH))$-bimodule given in the previous subsections. And let $P \to W \to 0$ be a projective cover of $W$ so that we have an exact sequence of $(B_0(kG), B_0(kH))$-bimodule

$$X \oplus P \to M \to 0 \text{ (exact)}$$

where $\lambda = (\mu, \nu \circ \lambda')$ with $\nu = \mu' \downarrow W$. Set $M_0 = \Omega^{-1}(\text{Ker} \lambda)$ so that we have an exact sequence of $(B_0(kG), B_0(kH))$-modules of the form

$$0 \to X \to M \oplus P_0 \to M_0 \to 0$$

where $P_0$ is a projective $(B_0(kG), B_0(kH))$-bimodule.

$$
\begin{array}{cccccc}
0 & \longrightarrow & \text{Ker} \lambda & \longrightarrow & X \oplus P & \longrightarrow & M \\
& & \text{id} & \downarrow & \lambda & \downarrow \\
0 & \longrightarrow & \text{Ker} \lambda & \longrightarrow & P_0 \oplus P & \longrightarrow & M_0 & \longrightarrow & 0
\end{array}
$$

**Lemma 4.2.** The functor $- \otimes_{B_0(kG)} M_0 : \text{mod-} B_0(kG) \to \text{mod-} B_0(kH)$ gives a stable equivalence of Morita type between $B_0(kG)$ and $B_0(kH)$.

Set $A = B_0(kG)$ and $B = B_0(kH)$.

For a nonprojective indecomposable $A$-module $V$, Let $F(V)$ be a nonprojective $B$-summand of $V \otimes_A M_0$ so that $F(V)$ is indecomposable and $V \otimes_A M_0 = F(V) \oplus \text{proj}$.

Assume that $V \downarrow R$ is a permutation $kR$-module and is not $R$-projective. Notice that a $kC_0$-module $V(R)$ has no projective summand by Lemma 3.1. Assume, furthermore that $V(R)$ is simple. Then a $B$-module $F(V)$ is obtained by the following way.

Set $A_0 = B_0(kC_G(R)/R) = B_0(kC_0)$ and $B_0 = B_0(kK_0)$. Then by a result of Puig-Rickard [11],

$$(V \otimes_A M)(R) \cong V(R) \otimes_{A_0} M(\Delta R) = V(R) \otimes_{A_0} N_0$$

and

$$(V \otimes_A X)(R) \cong V(R) \otimes_{A_0} X(\Delta R) = V(R) \otimes_{A_0} Y_0$$
as $B_0(kK_0)$-modules. Thus

$$(V \otimes_A X)(R) \cong V(R) \otimes_{A_0} Y_0$$

By the discussion on $p$-locals, we can write $V \otimes_A M = F'(V) \oplus U'$ where $F'(V)$ is indecomposable and $U'$ is $R$-projective. Then

$$F'(V)(R) \oplus U'(R) = (V \otimes_A M)(R) \cong V(R) \otimes_{A_0} N_0$$

As we are assuming that $V(R)$ is simple, $V(R) \otimes_{A_0} N_0$ is indecomposable. In particular, $U'(R) = 0$ and $U'$ is projective. If we set $U = V(R) \otimes_{A_0} N_0$, then by a result of Rouquier [13, 14], one of the following occurs.

\begin{align*}
V(R) \otimes_{A_0} Y_0 & : \cdots \to 0 \to 0 \to U \to 0 \to \cdots \quad (*.1) \\
V(R) \otimes_{A_0} Y_0 & : \cdots \to 0 \to Q(U) \xrightarrow{\rho} U \to 0 \to \cdots \quad (*.2)
\end{align*}

where $Q(U) \xrightarrow{\rho} U \to 0$ is a projective cover of a $B_0(kK_0)$-module $U$.

We have proved the following lemma.

**Lemma 4.3.** Let $V$ be an indecomposable $B_0(kG)$-module such that $V \downarrow R$ is a permutation $kR$-module and is not $R$-projective. Assume, furthermore that $V(R)$ is simple. Then $F'(V) = V \otimes_{B_0(kG)} M$ or $F'(V) = \Omega^{-1}(V \otimes_{B_0(kG)} M)$ according to the case $(*.1)$ occurs or the case $(*.2)$ occurs.

**Corollary 4.4.** Let $V$ be a $B_0(kG)$-module satisfying the conditions in the Lemma and assume that $\text{Hom}_k(V, k) \otimes_k V = k_G \oplus V'$ for some $R$-projective $kG$-module $V'$.

Then

$$\text{Hom}_k(F(V), k) \otimes_k F(V) = k_H \oplus V_0$$

for some projective $kH$-module $V_0$. In particular, $F(V) \downarrow P$ is an endo-trivial $kP$-module.

**Proof.** By our construction of the functor $F$, we can write

$$\text{Hom}_k(F(V), k) \otimes_k F(V) = k_H \oplus V_0$$

where $V_0$ is an $R$-projective $p$-permutation $kH$-module. Thus it suffices to show that $V_0(R) = 0$. We use the notations in the discussion before the lemma.

By a result of Puig-Richard,

$$k_{C_0} \oplus V_0(R) = (\text{Hom}_k(V, k) \otimes_k V)(R) \cong \text{Hom}_k(V(R), k) \otimes_k V(R) \quad (*)$$

as $kC_0$-modules. Then for $U = V(R) \otimes_{A_0} N_0$,

$$k_{C_0} \oplus U_0 \cong \text{Hom}_k(U, k) \otimes_k U$$

as $kC_0$-modules where $U_0$ is a projective $kK_0$-module. As $kC_H(R)$-modules, we also have

$$k_{C_H(R)} \oplus U_1 \cong \text{Hom}_k(\Omega_R(U), k) \otimes_k \Omega_R(U)$$

where $U_1$ is an $R$-projective $kC_H(R)$-module. By $(*)$, we can see that a source of $B_0(kC_0)$-module $V(R)$ is $k_{Z(P)}$ or $\Omega(k_{Z(P)})$. By properties of Roquier’s complex $Y_0$, the case $(*.1)$ occurs if a source of $V(R)$ is $k_{Z(P)}$ and the case $(*.2)$ occurs if a source of $V(R)$ is $\Omega(k_{Z(P)})$. 


If the case \((\ast.1)\) occurs, then \(U\) is a simple \(B_0(kC_H(R))\)-module. If the case \((\ast.2)\) occurs, then an \(R\)-projective cover \(\Omega_R(U)\) of \(U\) as \(kC_H(R)\)-module is a simple \(B_0(kC_H(R))\)-module. Notice that simple \(B_0(kC_H(R))\)-modules are one dimensional. Thus \(U_0 = 0\) in the case \((\ast.1)\) and \(U_1 = 0\) in the case \((\ast.2)\).

5. Examples

We shall give some examples of groups \(G\) with Sylow \(p\)-subgroup \(M_{n+1}(p)\) where we could check that simple \(B_0(kG)\)-modules \(V\) satisfy the assumption in Lemma 4.3.

Our groups \(G\) are constructed from \(G_0\) isomorphic to \(SL(2, q), SU(3, q^2)\) and \(Sp(4, q)\) for suitably chosen prime power \(q\) such that \(p \mid q + 1, p \mid q^2 - q + 1\) and \(p \mid q^2 + 1\), respectively. These groups \(G_0\) have cyclic Sylow \(p\)-subgroups and the Brauer trees of \(B_0(kG_0)\) are the following shapes. In the figures, \(\chi_k\) is an ordinary irreducible characters of degree \(k\). See the paper by Fong and Srinivasan [3].

\[
\begin{align*}
SL(2, q), & \quad p \mid q + 1 \\
& \quad \chi_1 \quad \chi_q \quad \phi_1 \quad \chi_q^{-1} \\
& \quad \phi_0 \quad \phi_1 \\
SU(3, q^2), & \quad p \mid q^2 - q + 1 \\
& \quad \chi_1 \quad \chi_{q^3} \quad \phi_1 \quad \chi_{q^2(q-1)} \quad \chi_{q^{(q-1)}} \\
& \quad \phi_0 \quad \phi_1 \quad \phi_2 \\
Sp(4, q), & \quad p \mid q^2 + 1 \\
& \quad \chi_1 \quad \chi_{q^2} \quad \phi_1 \quad \chi_q \quad \chi_q \phi_3 \quad \chi_{qt} \\
& \quad \phi_0 \quad \phi_1 \quad \phi_2 \quad \phi_3 \\
& \quad (s = \frac{1}{2}(q + 1)^2, t = \frac{1}{2}(q - 1)^2, u = (q^2 - 1))
\end{align*}
\]

5.1. \(SL(2, q)\). Let \(r\) be a prime power and \(p\) be an odd prime such that \(p\) divides \(r + 1\). Write \(r + 1 = p^n \ell, \ (p, \ell) = 1, n \geq 2\). Set \(q = r^p\). Then \(q + 1 = p^n \ell\) for some positive integer \(\ell\) with \((p, \ell) = 1\).

Set

\[
G_0 = SL(2, q), \quad C_0 = SL(2, r). \quad R = G(GF(q)/GF(r)) = \langle x \rangle, \quad G = R \rtimes G_0
\]

Let \(B_0 = T_0 \rtimes U_0\) be a Borel subgroup of \(G_0\) where \(|T_0| = q - 1\) and \(|U_0| = q\). We have an \(R\)-invariant subgroup \(F_0 \supseteq Z(G_0)\) of order \(q + 1\) such that \(F_0 \cap C_0\) is of order \(r + 1\) and \(B_0 \cap F_0 = Z(G_0)\).

Let \(P_0 \subset F_0\) be a Sylow \(p\)-subgroup of \(G_0\) and set \(P = R \rtimes P_0\). We have that \(P \cong M_{n+1}(p)\).

\(B_0(kG_0)\) and \(B_0(kC_0)\) have two simple modules

\[
B_0(kG_0) : \phi_0 = k_{G_0}, \quad \phi_1, \quad \dim_k \phi_1 = q-1, \quad B_0(kC_0) : \theta_0 = k_{C_0}, \quad \theta_1, \quad \dim_k \theta_1 = r-1
\]
A simple module $\phi_1$ is the heart of a projective cover $P(\phi_0) = P(k_{G_0})$.

$$P(\phi_0) = k_{B_0} \uparrow^{G_0}$$

and is uniserial of the form $P(\phi_0) = \phi_0$

The entirely same thing occurs for a projective cover $Q(\theta_0)$ of $\theta_0$.

Set $B = R \ltimes B_0$ and $P_R(k_{G_0}) = k_{B} \uparrow^{G}$. $P_R(k_{G_0})$ is an extension of $P(k_{G_0})$ and therefore is uniserial of length 3 with form

$$P_R(k_{G_0}) = \varphi_0$$

where $\varphi_0 = k_{G}$ and $\varphi_1 \downarrow_{G_0} = \phi_1$. It is not hard to see that $\varphi_1 \downarrow_{R}$ is a permutation $k_{R}$-module and

$$P_R(k_{G})(R) = Q(k_{C_0}), \quad \varphi_1(R) = \theta_1$$

5.2. $SU(3, q^2)$. Let $r$ be a prime power and $p$ be a prime with $p \geq 5$ such that $p$ divides $r^2 - r + 1$. Write $r^2 - r + 1 = p^{n-1}\ell'$, $(p, \ell') = 1$, $n \geq 2$. Set $q = r^p$. Then $q^2 - q + 1 = p^n\ell$ for some positive integer $\ell$ with $(p, \ell) = 1$.

Set $G_0 = SU(3, q^2)$, $C_0 = SU(3, r^2)$. $R = \mathcal{G}(GF(q)/GF(r)) = \langle x \rangle$, $G = R \ltimes G_0$.

Let $B_0 = T_0 \ltimes U_0$ be a Borel subgroup of $G_0$ where $|T_0| = (q + 1)(q - 1)$ and $|U_0| = q^3$.

We have an $R$-invariant subgroup $F_0 \supseteq Z(G_0)$ of order $q^2 - q + 1$ such that $F_0 \cap C_0$ is of order $r^2 - r + 1$ and $B_0 \cap F_0 = Z(G_0)$.

Let $P_0 \subseteq F_0$ be a Sylow $p$-subgroup of $G_0$ and set $P = R \ltimes P_0$. We have that $P \cong M_{n+1}(p)$.

$B_0(k_{G_0})$ and $B_0(k_{C_0})$ have three simple modules

$B_0(k_{G_0}) : \phi_0 = k_{G_0}$, $\phi_1$, $\dim_k \phi_1 = q^3 - 1$, $\phi_2$, $\dim_k \phi_2 = q(q - 1)$

$B_0(k_{C_0}) : \theta_0 = k_{C_0}$, $\theta_1$, $\dim_k \theta_1 = r^3 - 1$, $\theta_2$, $\dim_k \theta_2 = r(r - 1)$

Simple modules $\phi_1$ and $\phi_2$ are described as follows.

5.2.1. $\phi_1$. A simple module $\phi_1$ is the heart of a projective cover $P(\phi_0) = P(k_{G_0})$.

$$P(\phi_0) = k_{B_0} \uparrow^{G_0}$$

and is uniserial of the form $P(\phi_0) = \phi_0$

The same thing occurs for a projective cover $Q(\theta_0)$ of $\theta_0$.

Set $B = R \ltimes B_0$ and $P_R(k_{G_0}) = k_{B} \uparrow^{G}$. $P_R(k_{G_0})$ is an extension of $P(k_{G_0})$ and therefore is uniserial of length 3 with form

$$P_R(k_{G}) = \varphi_0$$

where $\varphi_0 = k_{G}$ and $\varphi_1 \downarrow_{G_0} = \phi_1$. It is not hard to see that $\varphi_1 \downarrow_{R}$ is a permutation $k_{R}$-module and

$$P_R(k_{G})(R) = Q(k_{C_0}), \quad \varphi_1(R) = \theta_1$$
5.2.2. $\phi_2$. Set $B = R \times B_0$. By the knowledge of the character tables of $G_0$ and $B_0$, we can see that there exists a simple $B_0(kG_0)$-module $\phi_2$ of dimension $q(q - 1)$ and the restriction $\phi' = \phi_2 \downarrow_{B_0}$ is a simple $kB_0$-module which is $R$-invariant. $\phi' \downarrow_{Z(U_0)}$ does not contain $kZ(U_0)$. The block of $kB$ which covers $\phi'$ has a cyclic defect group $R$ and $B_0$ is a $p'$-group. Thus Alperin-Brauer-Dade-Glauberman theory can be applied. Notice that $C_{B_0}(R)$ is a Borel subgroup of $C_0 = SU(3, r^2)$.

$\phi'$ has a unique extension $\varphi'$ to $B$ as $|B : B_0|$ is a $p$-group. The Brauer-Glauberman correspondent $\theta'$ of $\phi'$ does not contain $Z(C_{U_0}(R))$ in its kernel. Thus $\dim_k \theta' = r(r - 1)$.

We see that $\dim_k \phi' - \dim_k \theta' = q(q - 1) - r(r - 1) \equiv 0 \pmod{p}$. Thus the extension $\varphi'$ of $\phi'$ has a trivial source module and

$$\varphi'(R) = \theta'$$

$\phi_2$ also has a unique extension $\varphi_2$ to $G$. Then $\varphi_2 \downarrow_B$ is an extension of $\phi'$. Thus $\varphi_2 \downarrow_B = \varphi'$. $\varphi_2(R)$ is a $B_0(kC_0)$-module and $\varphi_2(R) \downarrow_{B_0C_0} = \theta'$. Such a $B_0(kC_0)$-module must be simple and

$$\varphi_2(R) = \theta_2$$

The simple $B_0(kG_0)$-module $\phi_2$ is self-dual and we can see that $\phi_2 \otimes \phi_2 = kG_0 \oplus \text{defect } 0 \text{ blocks. Thus}$

$$\varphi_2^* \otimes \varphi_2 = kG \oplus \text{R-projective}$$

5.3. $Sp(4, q)$. Let $r$ be a prime power and $p$ be a prime with $p \geq 5$ such that $p$ divides $r^2 + 1$. Write $r^2 + 1 = p^{n-1}\ell'$, $(p, \ell') = 1$, $n \geq 2$. Set $q = r^p$. Then $q^2 + 1 = p^n\ell$ for some positive integer $\ell$ with $(p, \ell) = 1$.

Set $G_0 = Sp(4, q)$, $C_0 = Sp(4, r)$. $R = G(GF(q)/GF(r)) = \langle x \rangle$, $G = R \rtimes G_0$.

Let $B_0 = T_0 \rtimes U_0$ be a Borel subgroup of $G_0$ where $|T_0| = (q - 1)^2$ and $|U_0| = q^4$. Let $W = N_G(T)/T = \langle w_a, w_b \rangle$ be the Weyl group of $G$ where $w_a$ is a reflection corresponding to a long root. We have an $R$-invariant subgroup $F_0$ of order $q^2 + 1$ such that $F_0 \cap C_0$ is of order $r^2 + 1$ and $B_0 \cap F_0 = Z(G_0)$.

Let $P_0 \subset F_0$ be a Sylow $p$-subgroup of $G_0$ and set $P = R \rtimes P_0$. We have that $P \cong M_{n+1}(p)$.

$B_0(kG_0)$ and $B_0(kC_0)$ have four simple modules

$$B_0(kG_0) : \phi_0 = kG_0, \quad \phi_1, \quad \dim_k \phi_1 = \frac{1}{2}q(q + 1)^2 - 1,$$

$$\phi_2, \quad \dim_k \phi_2 = q^4 - \frac{1}{2}q(q + 1)^2 + 1, \quad \phi_3, \quad \dim_k \phi_3 = \frac{1}{2}q(q - 1)^2$$

$$B_0(kC_0) : \theta_0 = kC_0, \quad \theta_1, \quad \dim_k \theta_1 = \frac{1}{2}r(r + 1)^2 - 1,$$

$$\theta_2, \quad \dim_k \theta_2 = r^4 - \frac{1}{2}r(r + 1)^2 + 1, \quad \theta_3, \quad \dim_k \theta_3 = \frac{1}{2}r(r - 1)^2$$

Simple modules $\phi_1, \phi_2$ and $\phi_3$ are described as follows.
5.3.1. $\phi_1$, $\phi_2$. Let $B_0 \subset K_0 = \langle w_a, B_0 \rangle = L_0 \ltimes V_0$ be a maximal parabolic subgroup of $G_0$. A simple module $\phi_1$ is the heart of a projective cover $P(\phi_0) = P(k_{G_0})$. We have

$$k_{K_0} \uparrow^G = P(k_{G_0}) \oplus P_0'$$

where $P'$ is a simple projective $kG_0$-module of dimension $\frac{1}{2}q(q^2 + 1)$. $P(\phi_0)$ is uniserial of the form

$$P(\phi_0) = \phi_0$$

The same thing occurs for a projective cover $Q(\theta_0)$ of $\theta_0$.

Set $K = R \ltimes K_0$. Then

$$k_K \uparrow^G = P_R(k_G) \oplus P'$$

where $P_R(k_G)$ is an extension of $P(k_G)$. In particular, $P_R(k_G)$ is uniserial of length 3 with form

$$P_R(k_G) = \phi_0$$

where $\phi_0 = k_G$ and $\varphi_1 \downarrow_{C_0} = \phi_1$. It is not hard to see that $\varphi_1 \downarrow R$ is a permutation $kR$-module and

$$P_R(k_G)(R) = Q(k_{C_0}), \quad \varphi_1(R) = \theta_1$$

Write $k_{B_0} \uparrow^{K_0} = k_{K_0} \oplus \rho_0$. $\rho_0$ is the Steinberg module of $K_0/V_0 = L_0 = GL(2, q)$. We have

$$\rho_0 \uparrow^G = P(\phi_1) \oplus P''$$

where $P''$ is a simple projective $kG_0$-module of dimension $\frac{1}{2}q(q^2 + 1)$. $P(\phi_1)$ has the form

$$P(\phi_1) = \phi_1 \oplus \phi_2$$

for some simple $kG_0$-module $\phi_2$. The same thing occurs for a projective cover $Q(\theta_1)$ of $\theta_1$ and we have a simple $kC_0$-module $\theta_2$.

It is not hard to see that $\rho_0$ has a unique extension $\rho$ to $K$ and $\rho$ is a $p$-permutaion module. And we have

$$\rho \uparrow^G = P_R(\phi_1) \oplus P''$$

where $P_R(\phi_1)$ is an extension of $P(\phi_1)$. In particular, $P_R(\phi_1)$ has the form

$$P_R(\phi_1) = \varphi_0 \oplus \varphi_2$$

where $\varphi_2 = \phi_2$. It is not hard to see that $\varphi_2 \downarrow R$ is a permutation $kR$-module and

$$P_R(\phi_1)(R) = Q(\theta_1), \quad \varphi_2(R) = \theta_2$$
5.3.2. $\phi_3$. By the knowledge of the character tables of $G_0$ and $K_0$, we can see that there exists a simple $B_0(kG_0)$-module $\phi_3$ of dimension $\frac{1}{2}q(q - 1)^2$ and the restriction $\phi' = \phi_3 \downarrow_{K_0}$ is a simple $kK_0$-module which is $R$-invariant. $\phi' \downarrow_{Z(U_0)}$ does not contain $kZ(U_0)$. The block of $kK$ which covers $\phi'$ has a cyclic defect group $R$ and $K_0$ is a $p'$-group. Thus Alperin-Brauer-Dade-Glauberman theory can be applied. Notice that $C_{K_0}(R)$ is a maximal parabolic subgroup of $C_0 = Sp(4, r)$.

$\phi'$ has a unique extension $\varphi'$ to $K$ as $|K : K_0|$ is a $p$-group. The Brauer-Glauberman correspondent $\theta'$ of $\phi'$ does not contain $Z(C_{U_0}(R))$ in its kernel. Thus $\dim_k \theta' = \frac{1}{2}r(r - 1)^2$.

We see that $\dim_k \phi' - \dim_k \theta' = \frac{1}{2}(q(q - 1)^2 - r(r - 1)^2) \equiv 0 \pmod{p}$. Thus the extension $\varphi'$ of $\phi'$ has a trivial source module and

$$\varphi'(R) = \theta'$$

$\phi_3$ also has a unique extension $\varphi_3$ to $G$. Then $\varphi_3 \downarrow_K$ is an extension of $\phi'$. Thus $\varphi_3 \downarrow_K = \varphi'$. $\varphi_3(R)$ is a $B_0(kC_0)$-module and $\varphi_3(R) \downarrow_{K \cap C_0} = \theta'$. Such a $B_0(kC_0)$-module must be simple and

$$\varphi_3(R) = \theta_3$$

The simple $B_0(kG_0)$-module $\phi_3$ is self-dual and we can see that

$$\phi_3 \otimes \phi_3 = kG_0 \oplus \text{defect 0 blocks}$$

Thus

$$\varphi_3^* \otimes \varphi_3 = kG \oplus \text{R-projective}$$

REFERENCES

