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Kyoto University
Hochschild cohomology of quiver algebras defined by two cycles and a quantum-like relation

(2 サイクルを持つ擬量子多元環のホッホシルトコモロジー)

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Abstract

This paper is based on my talk given at the Symposium on Cohomology Theory of Finite Groups and Related Topics held at Kyoto University, Japan, 29 August to 2 September 2011. In this paper, we consider quiver algebras $A_q$ over a field $k$ defined by two cycles and a quantum-like relation depending on a non-zero element $q$ in $k$. We determine the ring structure of the Hochschild cohomology ring of $A_q$ modulo nilpotence and give a necessary and sufficient condition for $A_q$ to satisfy the finiteness conditions given in [4].

Introduction

Let $A$ be an indecomposable finite dimensional algebra over a field $k$. We denote by $A^e$ the enveloping algebra $A \otimes_k A^{op}$ of $A$, so that left $A^e$-modules correspond to $A$-bimodules. The Hochschild cohomology ring is given by $HH^*(A) = \text{Ext}_{A^e}^*(A, A) = \oplus_{n \geq 0} \text{Ext}_{A^e}^n(A, A)$ with Yoneda product. It is well-known that $HH^*(A)$ is a graded commutative ring, that is, for homogeneous elements $\eta \in HH^m(A)$ and $\theta \in HH^n(A)$, we have $\eta \theta = (-1)^{mn} \theta \eta$. Let $\mathcal{N}$ denote the ideal of $HH^*(A)$ which is generated by all homogeneous nilpotent elements. Then $\mathcal{N}$ is contained in every maximal ideal of $HH^*(A)$, so that the maximal ideals of $HH^*(A)$ are in 1-1 correspondence with those in the Hochschild cohomology ring modulo nilpotence $HH^*(A)/\mathcal{N}$.

Let $q$ be a non-zero element in $k$ and $s, t$ integers with $s, t \geq 1$. We consider the quiver algebra $A_q = kQ/I_q$ defined by the two cycles $Q$ with $s + t - 1$ vertices and $s + t$ arrows as follows:

\[
\begin{array}{cccccccc}
 & a(3) & \xleftarrow{\alpha_2} & a(2) & & b(2) & \xrightarrow{\beta_2} & b(3) & \\
\alpha_3 & & & \beta_1 & & & & & \\
& a(s) \ \\
\alpha_{s-1} & & & \beta_t & & & & & \\
& b(t) & \xleftarrow{\beta_{t-1}} & & & & & & \\
& & l & & & & & & \\
\end{array}
\]

and the ideal $I_q$ of $kQ$ generated by

\[X^{sa} , X^{sb} - qY^t X^{sa} Y^{tb}\]

for $a, b \geq 2$ where we set $X := \alpha_1 + \alpha_2 + \cdots + \alpha_s$ and $Y := \beta_1 + \beta_2 + \cdots + \beta_t$. We denote the trivial path at the vertex $a(i)$ and at the vertex $b(j)$ by $e_{a(i)}$ and by $e_{b(j)}$ respectively. We regard the numbers $i$ in the subscripts of $e_{a(i)}$ modulo $s$ and $j$ in the subscripts of $e_{b(j)}$ modulo $t$. In this paper, we describe the ring structure of $HH^*(A_q)/\mathcal{N}$. 

\[= a(3) \xleftarrow{\alpha_2} a(2) \quad b(2) \xrightarrow{\beta_2} b(3) \quad a(s) \xleftarrow{\alpha_{s-1}} l \xrightarrow{\beta_t} b(t) \xleftarrow{\beta_{t-1}} \]

and the ideal $I_q$ of $kQ$ generated by

\[X^{sa} , X^{sb} - qY^t X^{sa} Y^{tb}\]
In [19], Snashall and Solberg used the Hochschild cohomology ring modulo nilpotence $\text{HH}^*(A)/\mathcal{N}$ to define a support variety for any finitely generated module over $A$. This led us to consider the structure of $\text{HH}^*(A)/\mathcal{N}$. In [19], Snashall and Solberg conjectured that $\text{HH}^*(A)/\mathcal{N}$ is always finitely generated as a $k$-algebra. But a counterexample to this conjecture was given by Snashall [18] and Xu [23]. This example makes us consider whether we can give necessary and sufficient conditions on a finite dimensional algebra $A$ for $\text{HH}^*(A)/\mathcal{N}$ to be finitely generated as a $k$-algebra.

On the other hand, in the theory of support varieties, it is interesting to know when the variety of a module is trivial. In [4], Erdmann, Holloway, Snashall, Solberg and Taillefer gave the necessary and sufficient conditions on a module for it to have trivial variety under some finiteness conditions on $A$. In the paper, we show that $A_q$ satisfies the finiteness conditions given in [4] if and only if $q$ is a root of unity.

The content of the paper is organized as follows. In Section 1 we deal with the definition of the support variety given in [19] and precedent results about the Hochschild cohomology ring modulo nilpotence. In Section 2, we describe the finiteness conditions given in [4] and introduce precedent results about these conditions. In Section 3, we determine the Hochschild cohomology ring of $A_q$ modulo nilpotence and show that $A_q$ satisfies the finiteness conditions if and only if $q$ is a root of unity.

1 Support variety

In [19], Snashall and Solberg defined the support variety of a finitely generated $A$-module $M$ over a noetherian commutative graded subalgebra $H$ of $\text{HH}^*(A)$ with $H^0 = \text{HH}^0(A)$. In this paper, we consider the case $H = \text{HH}^*(A)$.

**Definition 1.1 ([19]).** The support variety of $M$ is given by

$$V(M) = \{m \in \text{MaxSpec } \text{HH}^*(A)/\mathcal{N} | \text{AnnExt}^*_A(M, M) \subseteq m'\}$$

where $\text{AnnExt}^*_A(M, M)$ is the annihilator of $\text{Ext}^*_A(M, M)$, $m'$ is the pre-image of $m$ for the natural epimorphism of $\text{Ext}^*_A(A, A)$ is given by the graded algebra homomorphism $\text{HH}^*(A) \xrightarrow{-\otimes M} \text{Ext}^*_A(M, M)$.

Since $A$ is indecomposable, we have that $H^0(A)$ is a local ring. Thus $\text{HH}^*(A)/\mathcal{N}$ has a unique maximal graded ideal $m_{gr} = (\text{rad } \text{HH}^*(A), \text{HH}^{\geq 1}(A))/\mathcal{N}$. We say that the variety of $M$ is trivial if $V(M) = \{m_{gr}\}$.

In [18], Snashall gave the following question.

**Question ([18]).** Whether we can give necessary and sufficient conditions on a finite dimensional algebra for the Hochschild cohomology ring modulo nilpotence to be finitely generated as a $k$-algebra.

With respect to sufficient condition, it is shown that $\text{HH}^*(A)/\mathcal{N}$ is finitely generated as a $k$-algebra for various classes of algebras by many authors as follows:

1. In [6], [22], Evens and Venkov showed that $\text{HH}^*(A)/\mathcal{N}$ is finitely generated for any block of a group ring of a finite group.

2. In [7], Friedlander and Suslin showed that $\text{HH}^*(A)/\mathcal{N}$ is finitely generated for any block of a finite dimensional cocommutative Hopf algebra.
(3) In [11], Green, Snashall and Solberg showed that $\text{HH}^*(A)/\mathcal{N}$ is finitely generated for finite dimensional self-injective algebras of finite representation type over an algebraically closed field.

(4) In [12], Green, Snashall and Solberg showed that $\text{HH}^*(A)/\mathcal{N}$ is finitely generated for finite dimensional monomial algebras.

(5) In [13], Happel showed that $\text{HH}^*(A)/\mathcal{N}$ is finitely generated for finite dimensional algebras of finite global dimension.

(6) In [17], Schroll and Snashall showed that $\text{HH}^*(A)/\mathcal{N}$ is finitely generated for the principal block of the Hecke algebra $H_q(S_5)$ with $q = -1$ defined by the quiver

\[ Q : \varepsilon \xrightarrow{\alpha} 1 \xleftarrow{\overline{\alpha}} 2 \xrightarrow{\varepsilon} \]

and the ideal $I$ of $kQ$ generated by

\[ \alpha \varepsilon, \overline{\alpha} \varepsilon, \varepsilon \overline{\alpha}, \varepsilon^2 - \alpha \overline{\alpha}, \varepsilon^2 - \overline{\alpha} \alpha. \]

(7) In [20], Snashall and Taillefer showed that $\text{HH}^*(A)/\mathcal{N}$ is finitely generated for a class of special biserial algebras.

(8) In [14], Koenig and Nagase produced many examples of finite dimensional algebras with a stratifying ideal for which $\text{HH}^*(A)/\mathcal{N}$ is finitely generated as a $k$-algebra.

(9) In [18] and [23], Snashall and Xu gave the example of a finite dimensional algebra for which $\text{HH}^*(A)/\mathcal{N}$ is not a finitely generated $k$-algebra.

**Example 1.2.** ([18, Example 4.1]) Let $A = kQ/I$ where $Q$ is the quiver

\[ \xymatrix{ & 2 \
1 \ar@(ul,ur)^a \ar@(dl,dr)^b & \varepsilon 
} \]

and $I = \langle a^2, b^2, ab - ba, ac \rangle$. Then Snashall showed the following in [18, Theorem 4.5].

(a) $\text{HH}^*(A)/\mathcal{N} \cong \begin{cases} k \oplus k[a, b]b & \text{if char } k = 2, \\ k \oplus k[a^2, b^2]b^2 & \text{if char } k \neq 2. \end{cases}$

(b) $\text{HH}^*(A)/\mathcal{N}$ is not finitely generated as a $k$-algebra.

Xu showed this in the case char $k = 2$ in [23].

## 2 Finiteness conditions

In [4], Erdmann, Holloway, Snashall, Solberg and Taillefer gave the following two conditions (Fg1) and (Fg2) for an algebra $A$ and a graded subalgebra $H$ of $\text{HH}^*(A)$.

(Fg1) $H$ is a commutative Noetherian algebra with $H^0 = \text{HH}^0(A)$. 

(Fg2) $\text{Ext}^*_A(A/\text{rad} A, A/\text{rad} A)$ is a finitely generated $H$-module.

In [4], under the finiteness conditions above, some geometric properties of the support variety and some representation theoretic properties are related. In particular, the following theorem hold.

**Theorem 2.1** ([4, Theorem 2.5]). Suppose that $A$ satisfies the finiteness conditions.

(a) $A$ is Gorenstein, that is, $A$ has finite injective dimension both as a left $A$-module and as a right $A$-module.

(b) The following are equivalent for an $A$-module $M$.

(i) The variety of $M$ is trivial.

(ii) The projective dimension of $M$ is finite.

(iii) The injective dimension of $M$ is finite.

There are some papers which deal with the finiteness conditions as follows.

(1) In [2], Bergh and Oppermann show that a codimension $n$ quantum complete intersection satisfies the finiteness conditions if and only if all the commutators $q_{ij}$ are roots of unity.

**Definition 2.2.** A codimension $n$ quantum complete intersection is defined by

$$k\langle x_1, \ldots, x_n\rangle/I$$

where $I$ generated by

$$x_i^{a_i}, x_j x_i - q_{ij} x_i x_j \quad \text{for} \ 1 \leq i < j \leq n, a_i \geq 2, q_{ij} \in k.$$

(2) In [5], Erdmann and Solberg gave the necessary and sufficient conditions on a Koszul algebra for it to satisfy the finiteness conditions.

**Theorem 2.3** ([5, Theorem 1.3]). Let $A$ be a finite dimensional Koszul algebra over an algebraically closed field, and let $E(A) = \text{Ext}^*_A(A/\text{rad} A, A/\text{rad} A)$. $A$ satisfies the finiteness conditions if and only if $Z_{gr}(E(A))$ is Noetherian and $E(A)$ is a finitely generated $Z_{gr}(E(A))$-module.

(3) In [9], Furuya and Snashall provided examples of $(D, A)$-stacked monomial algebras which are not self-injective but satisfy the finiteness conditions.

**Example 2.4.** ([9, Example 3.2]) Let $Q$ be the quiver

\[
\begin{array}{c}
1 \\
\downarrow \alpha \\
4 \\
\downarrow \delta \\
\downarrow \gamma \\
3
\end{array}
\]

and $I$ the ideal of $kQ$ generated by

$$\alpha \beta \gamma \delta, \alpha \gamma \delta \beta \gamma \delta.$$

Then, $A = kQ/I$ is not self-injective but satisfies the finiteness conditions.

(4) In [17], Schroll and Snashall show that the finiteness conditions hold for the principal block of the Hecke algebra $H_q(S_3)$ with $q = -1$. 

3 Hochschild cohomology ring of quiver algebras defined by two cycles and a quantum-like relation

In this section, we consider the quiver algebras $A_q = kQ/I_q$ defined by the quiver $Q$ as follows:

and the ideal $I_q$ of $kQ$ generated by

$$X^{sa}, X^sY^t - qY^tX^s, Y^a$$

for $a, b \geq 2$ where we set $X := \alpha_1 + \alpha_2 + \cdots + \alpha_s$ and $Y := \beta_1 + \beta_2 + \cdots + \beta_t$, and $q$ is non-zero element in $k$. Paths are written from right to left. We will determine the Hochschild cohomology ring of $A_q$ modulo nilpotence $\text{HH}^*(A_q)/\mathcal{N}$ and show that $A_q$ satisfies the finiteness conditions if and only if $q$ is a root of unity.

First, we note that the following elements in $A_q$ form a $k$-basis of $A_q$.

So we have $\dim_k A_q = ab(s+t-1)^2$.

3.1 Projective resolution of $A_q$

For $n \geq 0$, we define left $A_q^n$-modules, equivalently $A_q$-bimodules

$$P_{2n} = \prod_{l=0}^{2n} A_q e_1 \otimes e_1 A_q \oplus \prod_{i=2}^{s} A_q e_{a(i)} \otimes e_{a(i)} A_q \oplus \prod_{j=2}^{t} A_q e_{b(j)} \otimes e_{b(j)} A_q,$$

$$P_{2n+1} = \prod_{l=1}^{2n} A_q e_1 \otimes e_1 A_q \oplus \prod_{i=1}^{s} A_q e_{a(i+1)} \otimes e_{a(i)} A_q \oplus \prod_{j=1}^{t} A_q e_{b(j+1)} \otimes e_{b(j)} A_q,$$

where $\prod_{l=1}^{0} A_q e_1 \otimes e_1 A_q = 0$. The generators $e_1 \otimes e_1, e_{a(i)} \otimes e_{a(i)}$ and $e_{b(j)} \otimes e_{b(j)}$ of $P_{2n}$ are labeled $\epsilon_l^{2n}$ for $0 \leq l \leq 2n$, $\epsilon_{a(i)}^{2n}$ for $2 \leq i \leq s$, and $\epsilon_{b(j)}^{2n}$ for $2 \leq j \leq t$ respectively.
Similarly, we denote the generators $e_1 \otimes e_1, e_{a(i+1)} \otimes e_{a(i)}$ and $e_{b(j+1)} \otimes e_{b(j)}$ of $P_{2n+1}$ by $\epsilon^{2n+1}_l$ for $1 \leq l \leq 2n$, $\epsilon^{2n+1}_{a(i)}$ for $1 \leq i \leq s$, and $\epsilon^{2n+1}_{b(j)}$ for $1 \leq j \leq t$ respectively. In [15], we give the minimal projective bimodule resolution of $A_q$ as follows.

**Theorem 3.1** ([15, Theorem 1.1]). The following sequence $\mathbb{P}$ is a minimal projective resolution of the left $A_q^e$-module $A_q$:

$$
\mathbb{P} : \cdots \to P_{2n+1} \xrightarrow{d_{2n+1}} P_{2n} \xrightarrow{d_{2n}} P_{2n-1} \to \cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\pi} A_q \to 0.
$$

where $\pi : P_0 \to A_q$ is the multiplication map, and we define left $A^e$-homomorphisms $d_{2n+1}$ and $d_{2n+2}$ by

$d_{2n+1} :$

$$
\begin{align*}
\epsilon^{2n+1}_{b(j)} &\mapsto \epsilon^{2n+1}_{b(j+1)} Y - Y \epsilon^{2n+1}_{b(j)}, & &1 \leq j \leq t, \\
\epsilon^{2n+1}_l &\mapsto -\sum_{j=0}^{b-1} q^{-b(l+1)} Y^{d_{2l+1}} Y^{t(b-l-j)} - X^s \epsilon^{2n+1}_{2l} + q^{b(n-l)} \epsilon^{2n+1}_{2l+1} X^s \\
& & & & & & & & & & & \text{if } l'' = 2l + 1 \text{ for } 0 \leq l \leq n-1, \\
-q^{-l' l''} Y^{t} \epsilon^{2n+1}_{2l} + \epsilon^{2n+1}_{2l'} Y^{t} + \sum_{i=0}^{a-1} q^{-i(b(n-l')-1)} X^s \epsilon^{2n+1}_{2l-1} X^{si} &\mapsto \sum_{l=0}^{a-1} q^{-i(b(n-l')-1)} X^s \epsilon^{2n+1}_{2l-1} X^{si} \\
& & & & & & & & & & & \text{if } l'' = 2l' \text{ for } 1 \leq l' \leq n,
\end{align*}
$$

$d_{2n+2} :$

$$
\begin{align*}
\epsilon^{2n+2}_0 &\mapsto \sum_{l=0}^{b-1} \epsilon^{2n+1}_{b(l-1)} Y^{t(l+1)} + \epsilon^{2n+1}_{b(l)} Y^{t(l)} - \epsilon^{2n+1}_{b(l+1)} Y^{t(l-1)}, \\
\epsilon^{2n+2}_i &\mapsto \sum_{l=0}^{b-1} \epsilon^{2n+1}_{b(l+1)} Y^{t(l+1)} + \epsilon^{2n+1}_{b(l+1)} Y^{t(l)} & &1 \leq j \leq t, \\
\epsilon^{2n+2}_1 &\mapsto \sum_{j=0}^{b-1} q^{-b(j+1)} Y^{d_{2l+1}} Y^{t(b-l-j)} + \sum_{i=0}^{a-1} q^{-i(b(n-l)-1)} X^s \epsilon^{2n+1}_{2l-1} X^{si} \\
& & & & & & & & & & & \text{if } l'' = 2l \text{ for } 1 \leq l \leq n, \\
\left(q^{-l'+1} Y^t \epsilon^{2n+1}_{2l+1} - \epsilon^{2n+1}_{2l+1} Y^t - X^s \epsilon^{2n+1}_{2l+1} + q^{-h(n-l')} \epsilon^{2n+1}_{2l'} X^sight) &\mapsto \sum_{l=0}^{a-1} q^{-i(b(n-l')-1)} X^s \epsilon^{2n+1}_{2l-1} X^{si} \\
& & & & & & & & & & & \text{if } l'' = 2l' + 1 \text{ for } 1 \leq l' \leq n-1,
\end{align*}
$$

for $n \geq 0$, where in the case $n = 0$, $\epsilon^1_{b}, \epsilon^2_{b}, \epsilon^1_{a}$, and $\epsilon^2_{a}$ vanish, and the image of $\epsilon^2_t$ by $d_2$ is

$$
-\sum_{j=1}^{t} X^s Y^{t-j} \epsilon^1_{b(j)} Y^{j-1} + q \sum_{j=1}^{t} Y^{t-j} \epsilon^1_{b(j)} Y^{j-1} - \sum_{i=1}^{s} X^s \epsilon^2_{a(i)} X^{i-1} - \sum_{i=1}^{s} X^s \epsilon^2_{a(i)} X^{i-1} Y^t.
$$

### 3.2 Hochschild cohomology group of $A_q$

Next, we give a basis of the $n$-th Hochschild cohomology group $HH^n(A_q) := \text{Ext}^n_{A_q}(A_q, A_q)$ for $n \geq 0$, using the minimal projective $A_q^e$-resolution given in Theorem 3.1. Now we
consider the case $s, t \geq 2$. In the case $s = 1$ or $t = 1$, we can give a basis of $HH^n(A_q)$ by the same method.

Now, we consider the vector space structure of $HH^n(A_q)$ for all $n \geq 0$. By the definition of $P_n$, we have isomorphisms

$$u_{2n} : \text{Hom}_{A_q^e}(P_{2n}, A_q) \xrightarrow{\sim} \prod_{l=1}^{2n} e_{1}A_q e_{1} \oplus \prod_{i=2}^{s} e_{a(i)}A_q e_{a(i)} \oplus \prod_{j=2}^{t} e_{b(j)}A_q e_{b(j)},$$

$$u_{2n+1} : \text{Hom}_{A_q^e}(P_{2n+1}, A_q) \xrightarrow{\sim} \prod_{l=1}^{2n} e_{1}A_q e_{1} \oplus \prod_{i=1}^{s} e_{a(i+1)}A_q e_{a(i)} \oplus \prod_{j=1}^{t} e_{b(j+1)}A_q e_{b(j)},$$

where $\prod_{l=1}^{0} e_{1}A_q e_{1} = 0$. We denote the $k$-modules $\text{Im} u_{2n}$ and $\text{Im} u_{2n+1}$ by $P_{2n}^*$ and $P_{2n+1}^*$ respectively. We see that the dimension of $P_n^*$ is given by

$$\dim_k P_n^* = nab + ab(s + t - 1).$$

The elements $e_1, e_{a(i)}$ and $e_{b(j)}$ of $P_{2n}$ are labeled $e_l^{2n}$ for $0 \leq l \leq 2n$, $e_{a(i)}^{2n}$ for $2 \leq i \leq s$ and $e_{b(j)}^{2n}$ for $2 \leq j \leq t$ respectively. Similarly, we denote the elements $e_1, e_{a(i)}$ and $e_{b(j)}$ of $P_{2n+1}$ by $e_l^{2n+1}$ for $1 \leq l \leq 2n$, $e_{a(i)}^{2n+1}$ for $1 \leq i \leq s$ and $e_{b(j)}^{2n+1}$ for $1 \leq j \leq t$ respectively. These labels correspond to that of generators of $P_n$. Using the maps $u_{2n}, u_{2n+1}, d_{2n+1}, d_{2n+2}$ for $n \geq 0$, we obtain the following diagram:

$$0 \xrightarrow{\partial_1} \text{Hom}_{A_q^e}(P_0, A_q) \xrightarrow{d_1} \text{Hom}_{A_q^e}(P_1, A_q) \xrightarrow{d_2} \text{Hom}_{A_q^e}(P_2, A_q) \xrightarrow{d_3} \cdots$$

$$0 \xrightarrow{0^*} P_0^* \xrightarrow{d_1^*} P_1^* \xrightarrow{d_2^*} P_2^* \xrightarrow{d_3^*} \cdots,$$

where we put $d_n = \text{Hom}_{A_q^e}(d_n, A_q)$, $d_n^* = u_n d_n u_{n-1}^{-1}$ for $n \geq 1$. Hence we have the complex

$$P^* : 0 \rightarrow P_0^* \xrightarrow{d_1^*} P_1^* \rightarrow \cdots \rightarrow P_{n-1}^* \xrightarrow{d_n^*} P_n^* \rightarrow \cdots.$$ See [16] for the homomorphism $d_n^*$. Now, we denote some elements of $P_n^*$ as follows:

- For $n = 0$:

  $$T_{l,0} := X^s Y^{1-t} e_0^0 + \sum_{j=2}^{t} Y^{j-1} X^s Y^{t-1} e_{b(j)}^0 + \sum_{i=2}^{s} X^{s(l-1)+i-1} Y^{t-l} X^{s-i+1} e_{a(i)}^0$$

  for $1 \leq l \leq a$ and $1 \leq l' \leq b$,

- For $n$ odd, $n \geq 1$:

  $$U_{l,0} := XY^{l-t} e_0^n,$$

  $$U_{m,0} := X^m X^l e_m^n,$$

  $$U_{m,0} := X^{l+1} Y^{m-t} e_0^n$$

  for $0 \leq l \leq a - 1$ and $0 \leq l' \leq b - 1$,
• For $n$ even, $n \geq 2$:

$$W_{0,l,l'}^{n} := X^{sl}Y^{tl'}e_{0}^{n} + \sum_{j=2}^{l} Y^{j-1}X^{sl}Y^{t(j-1)+t-j+1}e_{b(j)}^{n}$$

for $0 \leq l \leq a - 1$ and $0 \leq l' \leq b$,

$$W_{0,l,b-1}^{n*} := bX^{sl} Y^{t(b-1)}e_{0}^{n} + b\sum_{j=2}^{l} Y^{j-1}X^{sl}Y^{t(b-2)+t-j+1}e_{b(j)}^{n} + (q^{b(n/2-1)+1}-1)X^{s(l+1)}e_{1}^{n}$$

for $0 \leq l \leq a - 1$,

$$W_{m,l,l'}^{n} := X^{sl}Y^{tl'}e_{m}^{n}$$

for $1 \leq m \leq n-1, 0 \leq l \leq a - 1$ and $0 \leq l' \leq b - 1$,

$$W_{n,l,l'}^{n} := X^{sl}Y^{tl'}e_{n}^{n} + \sum_{i=2}^{s} X^{s(l-1)+i-1}Y^{tl'}X^{s-i+1}e_{a(i)}^{n}$$

for $0 \leq l \leq a$ and $0 \leq l' \leq b - 1$,

$$W_{n,a-1,l'}^{n*} := aX^{s(a-1)} Y^{tl'}e_{n}^{n} + a\sum_{i=2}^{s} X^{s(a-2)+i-1}Y^{tl'}X^{s-i+1}e_{a(i)}^{n} + (q^{a(n/2-1)+1}-1)Y^{t(l'+1)}e_{n-1}^{n}$$

for $0 \leq l' \leq b - 1$.

In the following results we use the complex $\mathbb{P}^{*}$ to compute a basis of the Hochschild cohomology group $HH^{n}(A_{q}) = \text{Kerd}_{n+1}^{*} / \text{Im} d_{n}^{*}$ of our algebra $A_{q}$ for $n \geq 0$.

Theorem 3.2 ([15, Proposition 3.3] and [16, Theorem 2.1]). If $q$ is an $r$-th root of unity for integer $r \geq 1$. Now, we set $\overline{z}$ is the remainder when we divide $z$ by $r$ for any integer $z$. Then we have $0 \leq \overline{z} \leq r - 1$.

(1) Basis of $HH^{0}(A_{q})$:

(a) $1_{A} = e_{0}^{0} + \sum_{j=2}^{t} e_{b(j)}^{0} + \sum_{i=2}^{s} e_{a(i)}^{0}$,

(b) $T_{l,l'}$ for $1 \leq l \leq a - 1, 1 \leq l' \leq b - 1$ if $\overline{l} = \overline{l'} = 0$,

(c) $T_{l,b}$ for $1 \leq l \leq a - 1$,

(d) $T_{a,l'}$ for $1 \leq l' \leq b - 1$,

(2) Basis of $HH^{2n}(A_{q})$ for $n \geq 1$:

(a) $W_{0,0,l}^{2n}$ for $0 \leq l' \leq b - 1$ if $\overline{l} = \overline{b(n-1)}$,

(b) $W_{0,l,l'}^{2n}$ for $1 \leq l \leq a - 1, 1 \leq l' \leq b - 1$ if $\overline{l} = 0, \overline{l'} = \overline{b(n-1)}$,

(c) $W_{0,l,b}^{2n}$ for $1 \leq l \leq a - 1$ if $\overline{l} = 0, \overline{b(n-1)} = 0, \text{char } k | b$,

(d) $W_{0,a-1,l'}^{2n}$ for $0 \leq l' \leq b - 1$ if $\overline{a} = 1, \overline{b(n-1)} \neq 0$,

(e) \[
W_{0,l,b-1}^{2n} \text{ for } 0 \leq l \leq a - 1 \text{ if } \overline{l} = 0, \overline{b(n-1)+1} \neq 0, \text{char } k \nmid b,
\]

$f_{0,l,1}^{2n} \text{ for } 0 \leq l \leq a - 2 \text{ if } \overline{l} = 0, \overline{b(n-1)+1} \neq 0, \text{char } k | b$,

(f) $W_{1,l+1,1}^{2n}$ for $1 \leq l \leq a - 2$ if $\overline{l} = 0, \overline{b(n-1)} \neq 0$,

(g) \[
W_{2l-1,l,l}^{2n} \text{ for } 1 \leq l \leq a - 1, 1 \leq l' \leq b - 1 \text{ if } \text{char } k \nmid a, \text{char } k \nmid b,
\]

$W_{2l-1,l,l'}^{2n} \text{ for } 1 \leq l \leq a - 1, 0 \leq l' \leq b - 1 \text{ if } \text{char } k \nmid a, \text{char } k | b$,

$W_{2l-1,l,l'}^{2n} \text{ for } 0 \leq l \leq a - 1, 1 \leq l' \leq b - 1 \text{ if } \text{char } k | a, \text{char } k \nmid b$,

$W_{2l-1,l,l'}^{2n} \text{ for } 0 \leq l \leq a - 1, 0 \leq l' \leq b - 1 \text{ if } \text{char } k | a, \text{char } k | b$,

for $1 \leq l' \leq n \text{ if } \overline{l} = a(l''-1)+1, \overline{l'} = b(n-l'')+1$.
\[
\begin{align*}
\{W_{2l,l,l}^{2n}, \text{for } 0 \leq l \leq a-2, 0 \leq l' \leq b-2 \text{ if } \text{char } k|a, \text{ char } k|b, \\
W_{2l,l,l}^{2n}, \text{for } 0 \leq l \leq a-2, 0 \leq l' \leq b-1 \text{ if } \text{char } k|a, \text{ char } k|b, \\
W_{2l,l,l}^{2n}, \text{for } 0 \leq l \leq a-1, 0 \leq l' \leq b-2 \text{ if } \text{char } k|a, \text{ char } k|b, \\
W_{2l,l,l}^{2n}, \text{for } 0 \leq l \leq a-1, 0 \leq l' \leq b-1 \text{ if char } k|a, \text{ char } k|b,
\end{align*}
\]
for \(1 \leq l'' \leq n-1\) if \(\overline{l} = al^{l'}, \overline{l'} = b(n-l'')\),

\[
\begin{align*}
W_{2n-1,1,l+1}^{2n} & \text{ for } 1 \leq l' \leq b-2 \text{ if } \overline{l'} = 0, \overline{a(n-l')-1} \neq 0, \\
W_{2n-1,0,l+1}^{2n} & \text{ for } 0 \leq l' \leq b-2 \text{ if } \overline{l'} = 0, \overline{a(n-1)+l} \neq 0, \text{ char } k|a,
\end{align*}
\]

\[
\begin{align*}
W_{2n,a-1,l'}^{2n} & \text{ for } 0 \leq l' \leq b-1 \text{ if } \overline{l'} = 0, \\
W_{2n,a,l}^{2n} & \text{ for } 1 \leq l' \leq b-1 \text{ if } \overline{l'} = 0, \overline{a(n-1)} = 0,
\end{align*}
\]

\[
\begin{align*}
W_{2n,l,0}^{2n} & \text{ for } 0 \leq l \leq a-1 \text{ if } \overline{b} = 1, \overline{an-l} \neq 0,
\end{align*}
\]

Additionally in the case of \(q = -1\):

\[(i)\] \(W_{0,l,b}^{2n+1}\) for \(0 \leq l \leq a-1\) if \(\overline{an} = 0, \overline{b} = 0, \overline{l} = 1\),

\[(ii)\] \(W_{1,l,0}^{2n+1}\) for \(1 \leq l \leq a-1\) if \(\overline{b} = 0, \overline{l} = 0\),

\[(iii)\] \(W_{2l-1,0,0}^{2n+1}\) for \(1 \leq l'' \leq n\) if \(\overline{a} = \overline{b} = 0\),

\[(iv)\] \(W_{2l,a-1,b-1}^{2n+1}\) for \(1 \leq l'' \leq n-1\) if \(\overline{a} = \overline{b} = 0\),

\[(v)\] \(W_{2n-1,0,l}^{2n+1}\) for \(1 \leq l' \leq b-1\) if \(\overline{a} = 0, \overline{l'} = 0\),

\[(vi)\] \(W_{2n,a,l}^{2n+1}\) for \(0 \leq l' \leq b-1\) if \(\overline{a} = 0, \overline{bn} = 0, \overline{l'} = 1\),

\[(3)\] Basis of \(HH^{2n+1}(A_q)\) for \(n \geq 0\):

\[(a)\] \(U_{0,l,l}^{2n+1}\) for \(0 \leq l \leq a-1, 1 \leq l' \leq b-1\) if \(\overline{l'} = \overline{bm}, \overline{l} = b(n-1) + 1 = 0, \overline{l} = 0\),

\[(b)\] \(U_{0,a-1,l}^{2n+1}\) for \(1 \leq l' \leq b-1\) if \(\overline{a} = 1, \overline{bn-l'} \neq 0\),

\[(c)\] \(U_{0,l,b-1}^{2n+1}\) for \(0 \leq l \leq a-1\) if \(\overline{l} = 0, \overline{bn} = 0, \text{ char } k|b\),

\[(d)\] \(U_{0,0,0}^{2n+1}\) if \(\overline{bn} = 0, \text{ char } k|b\),

\[(e)\] \(U_{0,l,0}^{2n+1}\) for \(0 \leq l \leq a-2\) if \(\overline{l} = 0, \overline{bn} \neq 0\),

\[(f)\] \(U_{2l+1,l',l}^{2n+1}, \text{ for } 1 \leq l \leq a-1, 0 \leq l' \leq b-2 \text{ if char } k|a, \text{ char } k|b, \overline{l} = 0, \overline{bn-l'} = b(n-1) + 1 = 1, \overline{bn} = 0, \overline{l} = 0, \overline{l'} = \overline{a(n-l')-1} = 0\),

\[(g)\] \(U_{2l+1,l',l}^{2n+1}, \text{ for } 0 \leq l \leq a-2, 0 \leq l' \leq b-1 \text{ if char } k|a, \text{ char } k|b, \overline{l} = 0, \overline{bn-l'} = b(n-1) + 1 = 1, \overline{bn} = 0, \overline{l} = 0, \overline{l'} = \overline{a(n-l')-1} = 0\),
(h) $U_{2n,0,l+1}^{2n+1}$ for $0 \leq l' \leq b - 2$ if $\overline{l} = 0$, $\overline{a} \neq 0$,

(i) $U_{2n+1,1,l}^{2n+1}$ for $1 \leq l \leq a - 1$, $0 \leq l' \leq b - 1$ if $\overline{l} = 0$, $\overline{a(n-1) + 1} \neq 0$,

(j) $U_{2n+1,0,l+1}^{2n+1}$ for $0 \leq l' \leq b - 1$ if $\overline{a} \neq 0$, $\overline{an} = 0$,

(k) $U_{2n+1,l,b-1}^{2n+1}$ for $1 \leq l \leq a - 1$ if $\overline{b} = 1$, $\overline{an-l} = 0$,

(l) \[
\begin{cases} 
U_{2n+1,0,l}^{2n+1} & \text{if } \overline{an} = 0, \text{ char } k \mid a, \\
U_{2n+1,0,l}^{2n+1} & \text{if } 0 \leq l' \leq b - 1 \text{ if } \overline{l'} = 0, \overline{an} = 0, \text{ char } k \mid a,
\end{cases}
\]

(m) Additionally in the case of $q = -1$:

i. $U_{2n+1,0,0}^{2n+1}$ for $0 \leq l \leq a - 1$ if $\overline{a(n-1)} = 0$, $\overline{b} = 0$, $\overline{l} = 1$,

ii. $U_{2n+1,1,b-1}^{2n+1}$ for $1 \leq l \leq a - 1$ if $\overline{a} = 0$, $\overline{b} = 0$, $\overline{l'} = 0$,

iii. $U_{2n+1,0,0}^{2n+1}$ for $0 \leq l' \leq b - 1$ if $\overline{a} = 0$, $\overline{b} = 0$,

iv. $U_{2n+1,1,b-1}^{2n+1}$ for $1 \leq l \leq b - 1$ if $\overline{a} = 0$, $\overline{b} = 0$,

v. $U_{2n+1,0,0}^{2n+1}$ for $0 \leq l' \leq b - 1$ if $\overline{a} = 0$, $\overline{b} = 0$, $\overline{l'} = 1$.

In the case $q = 1$, $q$ is a first root of unity. Then $\overline{z} = 0$ for any integer $z$. Hence if $q = 1$ then a basis of $HH^n(A_q)$ is formed by the elements of (1), (2), (a), (b), (c), (g), (h), (k), (l), (m) and (3)(a), (d), (f), (g), (i), (l).

Next, in the case where $q$ is not a root of unity, we give a basis of $HH^n(A_q)$ for $n \geq 0$.

**Theorem 3.3** ([16, Theorem 2.2]). If $q$ is not a root of unity and $s, t \geq 2$, then the following elements form a basis of $HH^n(A_q)$.

1. **Basis of $HH^0(A_q)$:**
   
   (a) $1_{A_q} = e_0^0 + \sum_{j=2}^{t} e_{b(j)}^0 + \sum_{i=2}^{s} e_{a(i)}^0$, 
   
   (b) $T_{l,b}$ for $1 \leq l \leq a - 1$,
   
   (c) $T_{a,l'}$ for $1 \leq l \leq b - 1$,

2. **Basis of $HH^{2n+1}(A_q)$ for $n \geq 0$:**
   
   (a) $U_{0,l+1}^{1,b-1}$ for $0 \leq l \leq a - 2$ if $n = 0$,
   
   (b) $U_{0,b-1}^{2n+1}$,
   
   (c) $U_{0,0}^{2n+1}$ if $n = 0$,
   
   (d) $U_{1,a-1,l+1}^{1,b-1}$ for $0 \leq l \leq b - 3$ if $n = 0$,
   
   (e) $U_{2n+1,a-1,0}^{2n+1}$,
   
   (f) $U_{1,0,0}^{2n+1}$ if $n = 0$,

3. **Basis of $HH^{2n+2}(A_q)$ for $n \geq 0$:**
   
   (a) $W_{2n+2}^{1,1,1}$ if $n = 0$,
   
   (b) $W_{2n+2}^{2n+2, 0, b-1}$,
   
   (c) $W_{2n+2,2,a-1,0}$.
3.3 Hochschild cohomology ring of $A_q$

In this section, we determin the Hochschild cohomology ring of $A_q$ modulo nilpotence. Now we recall the Yoneda product in $HH^*(A)$ (see [8]). For homogeneous elements $\eta \in HH^m(A)$ and $\theta \in HH^n(A)$ represented by $\eta: P_m \rightarrow A$ and $\theta: P_n \rightarrow A$ respectively, the Yoneda product $\eta\theta \in HH^{m+n}(A)$ is given as follows: There exists a commutative diagram of $A$-bimodules

\[
\begin{array}{ccccccccc}
\cdots & \rightarrow & P_{m+n} & d_{m+n} & \rightarrow & P_{n+1} & d_{n+1} & \rightarrow & P_n \\
\downarrow \sigma_m & & \downarrow \sigma_1 & & \downarrow \sigma_0 & & \downarrow \theta & & \\
\cdots & \rightarrow & P_m & d_m & \rightarrow & P_1 & d_1 & \rightarrow & P_0 & \rightarrow & A & \rightarrow & 0,
\end{array}
\]

where $\sigma_i$ are liftings of $\theta$. Here we see that such liftings always exist but are not unique. Then we have $\eta\theta = \eta\sigma_m \in HH^{m+n}(A)$. We note that $\eta\theta$ is independent of the choice of representation $\eta, \theta$ and liftings $\sigma_i (0 \leq i \leq m)$. See [16, Proposition 3.1] for the liftings of the basis of $HH^n(A_q) \ (n \geq 0)$. In the case where $q$ is a root of unity, by the liftings given in [16, Proposition 3.1], we see that $HH^{n+2r}(A_q)$ is generated by the elements in $HH^n(A_q)$ and that in $HH^{2r}(A_q)$ for $n \geq 2r$. By corresponding Yoneda product of the basis elements of $HH^*(A_q) := \oplus_{n \geq 0} HH^n(A_q)$ given in Theorem 3.2, we now have the generators of $HH^*(A_q)$. In this paper, we consider the case where $s, t \geq 2$ and $\bar{a}, \bar{b} \neq 0$. In the other cases, we have similar results to the following theorem and corollary. See [16] for the other cases.

**Theorem 3.4** ([16, Theorem 3.2]). In the case where $\bar{a}, \bar{b} \neq 0$, $HH^*(A_q)$ is generated as an algebra by the following generators:

1. The generators of $HH^*(A_q)$ in degree 0:
   
   $1_{A_q}, T_{r,r}, T_{l_1,r}, T_{r,l_1}, T_{l_2,b}, T_{a,l_2'}$
   
   for $1 \leq l_1, l_2 \leq a - 1$, $1 \leq l_1', l_2' \leq b - 1$ if $\bar{l}_1 = \bar{l}_1' = 0$.

2. The generators of $HH^*(A_q)$ in degree 1:
   
   - $U_{0,0,l'}^1, U_{1,l,0}^1 \ for \ 0 \leq l \leq a - 1, 0 \leq l' \leq b - 1$ if $\bar{l} = \bar{l}' = 0$,
   
   - $U_{0,a-1,l'}^1$ for $1 \leq l' \leq b - 1$ if $\bar{a} = 1, \bar{l}' \neq 0$,
   
   - $U_{1,a-1,l'}^1$ for $0 \leq l' \leq b - 1$ if $\bar{a} \neq 1, \bar{l}' = 0$,
   
   - $U_{1,l,b-1}^1$ for $1 \leq l' \leq a - 1$ if $\bar{l} \neq 0, \bar{b} = 1$,
   
   - $U_{0,l,b-1}^1$ for $1 \leq l \leq a - 1$ if $\bar{l} = 0, \bar{b} \neq 1$,
   
   - $U_{1,0,l'}^1$ for $1 \leq l' \leq b - 1$ if $\bar{l}' = 0, \text{char} k | a$,
   
   - $U_{0,0,l}^1$ for $1 \leq l \leq a - 1$ if $\bar{l} = 0, \text{char} k | b$.

3. The generators of $HH^*(A_q)$ in degree 2:
   
   - $W_{0,0,l'}^2, W_{2,l,0}^2, W_{0,2,l_2}^2$ for $1 \leq l_1, l_2 \leq a - 1, 1 \leq l_1', l_2' \leq b - 1$ if $\bar{l}_1 = \bar{a}, \bar{l}_1' = \bar{b}, \bar{l}_2 = \bar{l}_2' = 0$,
   
   - $W_{0,a,b-1}^2$ for $0 \leq l \leq a - 1$ if $\bar{l} = 0, \text{char} k \nmid b$,
   
   - $W_{1,l+1,0}^2$ for $0 \leq l \leq a - 2$ if $\bar{l} = 0, \text{char} k | b$.
\[
W_{2,a-1,l}^{2*} \text{ for } 0 \leq l' \leq b - 1 \text{ if } \overline{l'} = 0, \text{ char } k \nmid a,
\]
\[
W_{2,0,l'+1}^{2} \text{ for } 0 \leq l' \leq b - 2 \text{ if } \overline{l'} = 0, \text{ char } k \nmid a,
\]
\[
W_{0,a-1,l}^{2} \text{ for } 0 \leq l' \leq b - 1 \text{ if } \overline{a} = 1, \overline{l'} \neq \overline{b},
\]
\[
W_{2,b-1}^{2} \text{ for } 0 \leq l \leq a - 1 \text{ if } \overline{b} = 1, l \neq \overline{a}.
\]

(4) The generators of \( HH^*(A_q) \) in degree \( 2n \) for \( 2 \leq n \leq r \):

\[
\begin{cases}
W_{2n-1,0,l}^{2n} \text{ if } \min\{\overline{an'}|1 \leq n' \leq n - 1\} > \overline{an}, \\
W_{2n-1,a-1,l'}^{2n} \text{ for } 0 \leq l' \leq b - 1 \text{ if } \overline{l'} = 0, \overline{a(}0,
\end{cases}
\]

\[
W_{2n}^{2n} \text{ if } \min\{\overline{bn'}|1 \leq n' \leq n - 1\} > \overline{bn},
\]
\[
W_{2n,a-1,l'}^{2n} \text{ for } 0 \leq l' \leq b - 1 \text{ if } \overline{l'} = 0, \overline{a(n-1)+1} \neq 0,
\]
\[
W_{2n}^{2n} \text{ if } \overline{a(n-1)} \neq 0,
\]
\[
W_{2n}^{2n} \text{ if } \overline{bn'} \leq \overline{bn} \text{ for any } 1 \leq n' \leq n - 1,
\]
\[
W_{2n}^{2n} \text{ if } \overline{b(n-l'')+1} \leq \overline{n''}, \text{ char } k \nmid a,
\]
\[
W_{2n}^{2n} \text{ if } \overline{a(l''-1)+1} = \overline{b(n-l'')} + 1 = 0, \text{ char } k|a, \text{ char } k|b,
\]
\[
W_{2n}^{2n} \text{ if } \overline{an''} \neq 0, \overline{b(n-l'')} \neq 0,
\]
\[
W_{2n}^{2n} \text{ if } 1 \leq n'' \leq n - 1.
\]

(5) The generators of \( HH^*(A_q) \) in degree \( 2n + 1 \) for \( 1 \leq n \leq r - 1 \):

\[
U_{0,l,0,0}^{2n+1} \text{ if } 0 \leq l \leq a - 1 \text{ if } \overline{l} = 0, \min\{\overline{bn'}|1 \leq n' \leq n - 1\} \geq \overline{bn},
\]
\[
U_{0,l,b-1}^{2n+1} \text{ if } 0 \leq l \leq a - 1 \text{ if } \overline{l} = 0, b(n-1)+1 \neq 0, \text{ char } k|b,
\]
\[
U_{1,l,0}^{2n+1} \text{ if } 0 \leq l \leq a - 2 \text{ if } \overline{l} = 0, \overline{bn} \neq 0,
\]
\[
U_{2n+1}^{2n+1} \text{ if } 0 \leq l \leq a - 2 \text{ if } \overline{l} = 0, \overline{bn} \neq 0,
\]
\[
U_{2n+1,0}^{2n+1} \text{ for } 2 \leq l'' \leq n, 0 \leq \overline{a^l} \leq \begin{cases} a-2 \text{ if char } k \nmid a, \\
a-1 \text{ if char } k|a,
\end{cases}
\]
\[
U_{2n+1,0}^{2n+1} \text{ if char } k|b,
\]
\[
U_{2n+1,0}^{2n+1} \text{ if char } k|b,
\]
\[
U_{2n+1,0}^{2n+1} \text{ if char } k|b,
\]
\[
U_{2n+1,0}^{2n+1} \text{ if char } k|b,
\]
\[
U_{2n+1,0}^{2n+1} \text{ if char } k|b,
\]
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U_{2n+1,0}^{2n+1} \text{ if char } k|b,
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U_{2n+1,0}^{2n+1} \text{ if char } k|b,
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U_{2n+1,0}^{2n+1} \text{ if char } k|b,
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\[
U_{2n+1,0}^{2n+1} \text{ if char } k|b,
\]
\[
U_{2n+1,0}^{2n+1} \text{ if char } k|b,
\]
\[
U_{2n+1,0}^{2n+1} \text{ if char } k|b,
\[
\begin{align*}
&\begin{cases}
U_{2n+1}^{2n+1,1,0,b(n-l''')} 
\text{for } 0 \leq l'' \leq n-1, 0 \leq b(n-l'') \leq \begin{cases} b-2 & \text{if char } k \nmid b, \\
b-1 & \text{if char } k \mid b,
\end{cases}
\end{cases}
\end{align*}
\]

if \(a'' + 1 = 0\), \(\min\{b(n-l'')l'' + 1 \leq n' \leq n-1\} > b(n-l'')\),

- \(U_{2n+1}^{2n+1,1,0,0} \) for \(0 \leq l' \leq b-2\) if \(a(n-1) + 1 = 0, l = b\),

if \(\text{char } k \mid a\),

- \(U_{2n,0,l'+1}^{2n+1,1,0,0} \) for \(0 \leq l' \leq b-2\) if \(l = 0, a = 0\),

- \(U_{2n+1,0,l'+1}^{2n+1,1,0,0} \) for \(0 \leq l' \leq b-1\) if \(l = 0, a(n-1) + 1 \neq 0\), \(\text{char } k \mid a\).

(6) The generators of \(\text{HH}^*(A_q)\) in degree \(2r + 2n + 1\) for \(0 \leq n \leq r-2\):

- \(U_{2n+1,0,l'}^{2r+2n+1,1,0,0,0} \) for \(n + 1 \leq l'' \leq r, 0 \leq a'' \leq \begin{cases} a-2 & \text{if char } k \nmid a, \\
a-1 & \text{if char } k \mid a,
\end{cases}
\)

if \(\min\{a''l'' | l'' \leq l''-1\} > a'' b(n-l'') = 1, \text{char } k \mid b\),

- \(U_{2n+1,0,l'}^{2n+1,1,0,0,0,0} \) for \(1 \leq l'' \leq r-1, 0 \leq b(n-l'') \leq \begin{cases} b-2 & \text{if char } k \nmid b, \\
b-1 & \text{if char } k \mid b,
\end{cases}
\)

if \(a'' + 1 = 0, \min\{b(n-l''-l''')l'' + 1 \leq n' \leq n-1\} > b(n-l''), \text{char } k \mid a\).

It follows from the Theorem 3.4 that \(1_{A_q}, W_{0,0,0}^{2r}\) and \(W_{2r,0,0,0}^{2r}\) are not nilpotent and the other generators are nilpotent. Thus we have the following corollary.

Corollary 3.5. If \(s, t \geq 2\) and \(a, b \neq 0\), then the quotient of the Hochschild cohomology ring of \(A_q\) modulo nilpotence is isomorphic to the polynomial ring of two variables in all characteristic:

\[ \text{HH}^*(A_q)/\mathcal{N} \cong k[W_{0,0,0}^{2r}, W_{2r,0,0}^{2r}].\]

Finally, we consider the ring structure of \(\text{HH}^*(A_q)\) in the case where \(q\) is not a root of unity. It follows from the liftings given in [16] that all basis elements except \(1_{A_q}\) of \(\text{HH}^*(A_q)\) are nilpotent elements for \(n \geq 0\). Thus we have the following results.

Theorem 3.6. If \(q\) is not a root of unity then \(\text{HH}^*(A_q)\) is not a finitely generated \(k\)-algebra.

Corollary 3.7. If \(q\) is not a root of unity then \(\text{HH}^*(A_q)/\mathcal{N} \cong k\).

In general, our algebra \(A_q\) is not self-injective, monomial or Koszul. Moreover \(A_q\) does not have a stratifying ideal. Therefore \(A_q\) is new example of a class of algebras for which the Hochschild cohomology ring modulo nilpotence is finitely generated as a \(k\)-algebra. For example, in the case where \(s = 2, t = 1\) and \(a = b = 2\), our algebra \(A_q\) is not self-injective, monomial or Koszul. Moreover \(A_q\) does not have a stratifying ideal.

3.4 Finiteness conditions for \(A_q\)

Finally, we show that \(A_q\) satisfies the finiteness conditions in the case where \(q\) is a root of unity.

Now we consider the case where \(q\) is an \(r\)-th root of unity, \(s, t \geq 2\) and \(a, b \neq 0\). In the other case, we see that \(A_q\) satisfies the finiteness conditions by the same method. The Yoneda algebra or Ext algebra of \(A_q\) is given by \(E(A_q) = \oplus_{n \geq 0} \text{Ext}_{A_q}^n(A_q/\tau, A_q/\tau)\).
with the Yoneda product. We use the notation $E(A_{q})^{n} = \text{Ext}^{n}_{A_{q}}(A_{q}/r, A_{q}/r)$ for the $n$-th graded component of $E(A_{q})$. Then it is easy to see that $E(A_{q})^{n} \simeq \prod_{j=2}^{t} k e_{b(j)}^{2n} \oplus \prod_{i=2}^{s} k e_{a(i)}^{2n}$.

Let $\varphi: HH^{*}(A_{q}) \rightarrow E(A_{q})$ be a homomorphism of graded rings given by $\varphi(\eta) = \eta \otimes_{A_{q}} A_{q}/r$. Then it is easy to see that the image of $\varphi$ is precisely the graded ring $k[x, y]$ where $x := e_{0}^{2r} + \sum_{j=2}^{t} e_{b(j)}^{2r}$ and $y := e_{2r}^{0} + \sum_{i=2}^{s} e_{a(i)}^{2r}$ in degree $2r$.

**Proposition 3.8.** $E(A_{q})$ is a finitely generated left $k[x, y]$-module with generators:

- $e_{l}^{2n}$ for $0 \leq l \leq 2n$ in degree $2n$ for $0 \leq n \leq r - 1$,
- $e_{l}^{2n+1}$ for $0 \leq l \leq 2r - 1$ in degree $2r$,
- $e_{l}^{2r}$ for $2n + 1 \leq l \leq 2r$ in degree $2r + 2n + 1$ for $0 \leq n \leq r - 1$,
- $e_{l}^{2r+2}$ for $2n + 1 \leq l \leq 2r - 1$ in degree $2r + 2n + 2$ for $0 \leq n \leq r - 2$.

Now we consider the conditions (Fg1) and (Fg2). The element $W^{2r}_{0,0,0,0} \in HH^{2r}(A_{q})$ is a pre-image of $x$ and the element $W^{2r}_{2r,0,0} \in HH^{2r}(A_{q})$ is a pre-image of $y$. Let $H$ be the graded subalgebra of $HH^{*}(A_{q})$ generated by $HH^{0}(A_{q}), W^{2r}_{0,0,0,0}$ and $W^{2r}_{2r,0,0}$, so that $H$ is a pre-image of $k[x, y]$ in $HH^{*}(A_{q})$. Then we have the following immediate consequence of Proposition 3.8.

**Theorem 3.9.** The conditions (Fg1) and (Fg2) hold for the algebra $A_{q}$ with respect to the subring $H$ of $HH^{*}(A_{q})$.

By [2], Theorem 3.6 and 3.9, we have the necessary and sufficient conditions for $A_{q}$ to satisfy the finiteness conditions.

**Theorem 3.10.** $A_{q}$ satisfies the finiteness conditions if and only if $q$ is a root of unity.

**References**


