Hochschild cohomology of quiver algebras defined by two cycles and a quantum-like relation

(2 サイクルを持つ擬量子多元環のホッシャルトコホモロジー)

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Abstract

This paper is based on my talk given at the Symposium on Cohomology Theory of Finite Groups and Related Topics held at Kyoto University, Japan, 29 August to 2 September 2011. In this paper, we consider quiver algebras \(A_q\) over a field \(k\) defined by two cycles and a quantum-like relation depending on a non-zero element \(q\) in \(k\). We determine the ring structure of the Hochschild cohomology ring of \(A_q\) modulo nilpotence and give a necessary and sufficient condition for \(A_q\) to satisfy the finiteness conditions given in [4].

Introduction

Let \(A\) be an indecomposable finite dimensional algebra over a field \(k\). We denote by \(A^e\) the enveloping algebra \(A \otimes_k A^{op}\) of \(A\), so that left \(A^e\)-modules correspond to \(A\)-bimodules. The Hochschild cohomology ring is given by \(HH^*(A) = \text{Ext}_{A^e}^*(A, A) = \oplus_{n \geq 0} \text{Ext}_{A^e}^n(A, A)\) with Yoneda product. It is well-known that \(HH^*(A)\) is a graded commutative ring, that is, for homogeneous elements \(\eta \in HH^n(A)\) and \(\theta \in HH^m(A)\), we have \(\eta \theta = (-1)^{mn} \theta \eta\). Let \(\mathcal{N}\) denote the ideal of \(HH^*(A)\) which is generated by all homogeneous nilpotent elements. Then \(\mathcal{N}\) is contained in every maximal ideal of \(HH^*(A)\), so that the maximal ideals of \(HH^*(A)\) are in 1-1 correspondence with those in the Hochschild cohomology ring modulo nilpotence \(HH^*(A)/\mathcal{N}\).

Let \(q\) be a non-zero element in \(k\) and \(s, t\) integers with \(s, t \geq 1\). We consider the quiver algebra \(A_q = kQ/I_q\) defined by the two cycles \(Q\) with \(s + t - 1\) vertices and \(s + t\) arrows as follows:

```
      a(3) α_2 a(2) α_1 b(2) β_2 b(3) b(t) β_{t-1} b(s) α_{s-1} a(s) α_3

  a(1) l  α a(2) β_1 b(3) β_3

  a(s) α_s

  b(t)
```

and the ideal \(I_q\) of \(kQ\) generated by

\[X^{sa}\cdot X^tY^t - qY^tX^s, Y^t\]

for \(a, b \geq 2\) where we set \(X := \alpha_1 + \alpha_2 + \cdots + \alpha_s\) and \(Y := \beta_1 + \beta_2 + \cdots + \beta_t\). We denote the trivial path at the vertex \(a(i)\) and at the vertex \(b(j)\) by \(e_{a(i)}\) and by \(e_{b(j)}\) respectively. We regard the numbers \(i\) in the subscripts of \(e_{a(i)}\) modulo \(s\) and \(j\) in the subscripts of \(e_{b(j)}\) modulo \(t\). In this paper, we describe the ring structure of \(HH^*(A_q)/\mathcal{N}\).
In [19], Snashall and Solberg used the Hochschild cohomology ring modulo nilpotence $HH^*(A)/\mathcal{N}$ to define a support variety for any finitely generated module over $A$. This led us to consider the structure of $HH^*(A)/\mathcal{N}$. In [19], Snashall and Solberg conjectured that $HH^*(A)/\mathcal{N}$ is always finitely generated as a $k$-algebra. But a counterexample to this conjecture was given by Snashall [18] and Xu [23]. This example makes us consider whether we can give necessary and sufficient conditions on a finite dimensional algebra $A$ for $HH^*(A)/\mathcal{N}$ to be finitely generated as a $k$-algebra.

On the other hand, in the theory of support varieties, it is interesting to know when the variety of a module is trivial. In [4], Erdmann, Holloway, Snashall, Solberg and Taillefer gave the necessary and sufficient conditions on a module for it to have trivial variety under some finiteness conditions on $A$. In the paper, we show that $A_q$ satisfies the finiteness conditions given in [4] if and only if $q$ is a root of unity.

The content of the paper is organized as follows. In Section 1 we deal with the definition of the support variety given in [19] and precedent results about the Hochschild cohomology ring modulo nilpotence. In Section 2, we describe the finiteness conditions given in [4] and introduce precedent results about these conditions. In Section 3, we determine the Hochschild cohomology ring of $A_q$ modulo nilpotence and show that $A_q$ satisfies the finiteness conditions if and only if $q$ is a root of unity.

1 Support variety

In [19], Snashall and Solberg defined the support variety of a finitely generated $A$-module $M$ over a noetherian commutative graded subalgebra $H$ of $HH^*(A)$ with $H^0 = HH^0(A)$. In this paper, we consider the case $H = HH^*(A)$.

**Definition 1.1 ([19]).** The support variety of $M$ is given by

$$V(M) = \{m \in \text{MaxSpec } HH^*(A)/\mathcal{N} \mid \text{AnnExt}_{A}^{*}(M, M) \subseteq m'\}$$

where $\text{AnnExt}_{A}^{*}(M, M)$ is the annihilator of $\text{Ext}_{A}^{*}(M, M)$, $m'$ is the pre-image of $m$ for the natural epimorphism of $\text{Ext}_{A}^{*}(A, A)$ is given by the graded algebra homomorphism $HH^*(A)\overset{-\otimes_{M}}{\longrightarrow}\text{Ext}_{A}^{*}(M, M)$.

Since $A$ is indecomposable, we have that $HH^0(A)$ is a local ring. Thus $HH^*(A)/\mathcal{N}$ has a unique maximal graded ideal $m_{gr} = (\text{rad } HH^*(A), HH^{2}_{2}(A))/\mathcal{N}$. We say that the variety of $M$ is trivial if $V(M) = \{m_{gr}\}$.

In [18], Snashall gave the following question.

**Question ([18]).** Whether we can give necessary and sufficient conditions on a finite dimensional algebra for the Hochschild cohomology ring modulo nilpotence to be finitely generated as a $k$-algebra.

With respect to sufficient condition, it is shown that $HH^*(A)/\mathcal{N}$ is finitely generated as a $k$-algebra for various classes of algebras by many authors as follows:

1. In [6], [22], Evens and Venkov showed that $HH^*(A)/\mathcal{N}$ is finitely generated for any block of a group ring of a finite group.

2. In [7], Friedlander and Suslin showed that $HH^*(A)/\mathcal{N}$ is finitely generated for any block of a finite dimensional cocommutative Hopf algebra.
(3) In [11], Green, Snashall and Solberg showed that $\text{HH}^*(A)/\mathcal{N}$ is finitely generated for finite dimensional self-injective algebras of finite representation type over an algebraically closed field.

(4) In [12], Green, Snashall and Solberg showed that $\text{HH}^*(A)/\mathcal{N}$ is finitely generated for finite dimensional monomial algebras.

(5) In [13], Happel showed that $\text{HH}^*(A)/\mathcal{N}$ is finitely generated for finite dimensional algebras of finite global dimension.

(6) In [17], Schroll and Snashall showed that $\text{HH}^*(A)/\mathcal{N}$ is finitely generated for the principal block of the Hecke algebra $H_q(S_5)$ with $q = -1$ defined by the quiver

$$Q : \epsilon \xRightarrow{1} \xrightarrow{\alpha} 2 \xRightarrow{\epsilon}$$

and the ideal $I$ of $kQ$ generated by

$$\alpha \epsilon, \alpha \epsilon, \epsilon \alpha, \epsilon^2 - \alpha \epsilon, \epsilon^2 - \alpha \epsilon.$$

(7) In [20], Snashall and Taillefer showed that $\text{HH}^*(A)/\mathcal{N}$ is finitely generated for a class of special biserial algebras.

(8) In [14], Koenig and Nagase produced many examples of finite dimensional algebras with a stratifying ideal for which $\text{HH}^*(A)/\mathcal{N}$ is finitely generated as a $k$-algebra.

(9) In [18] and [23], Snashall and Xu gave the example of a finite dimensional algebra for which $\text{HH}^*(A)/\mathcal{N}$ is not a finitely generated $k$-algebra.

**Example 1.2.** ([18, Example 4.1]) Let $A = kQ/I$ where $Q$ is the quiver

$$\circlearrowleft a \quad 1 \xrightarrow{\epsilon} 2$$

and $I = \langle a^2, b^2, ab - ba, ac \rangle$. Then Snashall showed the following in [18, Theorem 4.5].

(a) $\text{HH}^*(A)/\mathcal{N} \cong \begin{cases} k \oplus k[a, b]b & \text{if } \text{char } k = 2, \\ k \oplus k[a^2, b^2]b^2 & \text{if } \text{char } k \neq 2. \end{cases}$

(b) $\text{HH}^*(A)/\mathcal{N}$ is not finitely generated as a $k$-algebra.

Xu showed this in the case $\text{char } k = 2$ in [23].

2 **Finiteness conditions**

In [4], Erdmann, Holloway, Snashall, Solberg and Taillefer gave the following two conditions (Fg1) and (Fg2) for an algebra $A$ and a graded subalgebra $H$ of $\text{HH}^*(A)$.

(Fg1) $H$ is a commutative Noetherian algebra with $H^0 = \text{HH}^0(A)$. 

(Fg2) $H$ is finitely generated as a $k$-algebra.
$\text{(Fg2)}$ $\text{Ext}_A^*(A/\text{rad } A, A/\text{rad } A)$ is a finitely generated $H$-module.

In [4], under the finiteness conditions above, some geometric properties of the support variety and some representation theoretic properties are related. In particular, the following theorem hold.

**Theorem 2.1** ([4, Theorem 2.5]). Suppose that $A$ satisfies the finiteness conditions.

(a) $A$ is Gorenstein, that is, $A$ has finite injective dimension both as a left $A$-module and as a right $A$-module.

(b) The following are equivalent for an $A$-module $M$.
   
   (i) The variety of $M$ is trivial.
   
   (ii) The projective dimension of $M$ is finite.
   
   (iii) The injective dimension of $M$ is finite.

There are some papers which deal with the finiteness conditions as follows.

(1) In [2], Bergh and Oppermann show that a codimension $n$ quantum complete intersection satisfies the finiteness conditions if and only if all the commutators $q_{ij}$ are roots of unity.

**Definition 2.2.** A codimension $n$ quantum complete intersection is defined by

\[ k\langle x_1, \ldots, x_n \rangle/I \]

where $I$ generated by

\[ x_i^{a_i}x_j - q_{ij}x_i x_j \quad \text{for } 1 \leq i < j \leq n, a_i \geq 2, q_{ij} \in k. \]

(2) In [5], Erdmann and Solberg gave the necessary and sufficient conditions on a Koszul algebra for it to satisfy the finiteness conditions.

**Theorem 2.3** ([5, Theorem 1.3]). Let $A$ be a finite dimensional Koszul algebra over an algebraically closed field, and let $E(A) = \text{Ext}_A^*(A/\text{rad } A, A/\text{rad } A)$. $A$ satisfies the finiteness conditions if and only if $Z_{gr}(E(A))$ is Noetherian and $E(A)$ is a finitely generated $Z_{gr}(E(A))$-module.

(3) In [9], Furuya and Snashall provided examples of $(D, A)$-stacked monomial algebras which are not self-injective but satisfy the finiteness conditions.

**Example 2.4.** ([9, Example 3.2]) Let $Q$ be the quiver

\[
\begin{array}{cccc}
1 & \overset{\alpha}{\longrightarrow} & 2 \\
\downarrow & \delta & \downarrow & \beta \\
4 & \overset{\gamma}{\longleftarrow} & 3
\end{array}
\]

and $I$ the ideal of $kQ$ generated by

\[ \alpha\beta\gamma\delta\alpha\beta, \gamma\delta\alpha\beta\gamma. \]

Then, $A = kQ/I$ is not self-injective but satisfies the finiteness conditions.

(4) In [17], Schroll and Snashall show that the finiteness conditions hold for the principal block of the Hecke algebra $H_q(S_3)$ with $q = -1$. 
3 Hochschild cohomology ring of quiver algebras defined by two cycles and a quantum-like relation

In this section, we consider the quiver algebras $A_q = kQ/I_q$ defined by the quiver $Q$ as follows:

$$A_q = kQ/I_q$$

where $Q$ is a quiver and $I_q$ is the ideal generated by certain relations. The relations are determined by the quantum-like relation:

$$X^{s}Y^{t} - qY^{t}X^{s}$$

for $a, b \geq 2$ where we set $X := \alpha_1 + \alpha_2 + \cdots + \alpha_s$ and $Y := \beta_1 + \beta_2 + \cdots + \beta_t$, and $q$ is a non-zero element in $k$. Paths are written from right to left. We will determine the Hochschild cohomology ring of $A_q$ modulo nilpotence $\text{HH}^*(A_q)/\mathcal{N}$ and show that $A_q$ satisfies the finiteness conditions if and only if $q$ is a root of unity.

First, we note that the following elements in $A_q$ form a $k$-basis of $A_q$.

$$\begin{align*}
X^{si+l}Y^{tj+l'}e_{a(i)} & \quad \text{for } 0 \leq i \leq a-1, 0 \leq l \leq s-1, 0 \leq l' \leq t-1, \\
Y^{l'}X^{si}Y^{tj+l'}e_{a(i)} & \quad \text{for } 1 \leq i \leq a-1, 0 \leq l' \leq s-1, 0 \leq l \leq t-1, \\
X^{si+l}Y^{tj} & \quad \text{for } 0 \leq i \leq a-1, 0 \leq j \leq b-1, 0 \leq l \leq s-1, \\
Y^{l'}X^{si}Y^{tj+l'} & \quad \text{for } 1 \leq i \leq a-1, 0 \leq l' \leq s-1, 0 \leq l \leq t-1, \\
X^{si+l}Y^{tj}X^{l'} & \quad \text{for } 0 \leq i \leq a-1, 0 \leq j \leq b-1, 1 \leq l \leq s-1, 1 \leq l' \leq t-1, \\
Y^{l'}X^{si}Y^{tj+l'}X^{l'} & \quad \text{for } 1 \leq i \leq a-1, 1 \leq j \leq b-1, 1 \leq l' \leq s-1, 1 \leq l \leq t-1.
\end{align*}$$

So we have $\dim_k A_q = ab(s + t - 1)^2$.

3.1 Projective resolution of $A_q$

For $n \geq 0$, we define left $A_q^{e}$-modules, equivalently $A_q$-bimodules

$$
\begin{align*}
P_{2n} &= \prod_{l=0}^{2n} A_q e_1 \otimes e_1 A_q \oplus \prod_{i=2}^{s} A_q e_{a(i)} \otimes e_{a(i)} A_q \oplus \prod_{j=2}^{t} A_q e_{b(j)} \otimes e_{b(j)} A_q, \\
P_{2n+1} &= \prod_{i=1}^{2n} A_q e_1 \otimes e_1 A_q \oplus \prod_{i=1}^{s} A_q e_{a(i)} \otimes e_{a(i)} A_q \oplus \prod_{j=1}^{t} A_q e_{b(j)} \otimes e_{b(j)} A_q,
\end{align*}
$$

where $\prod_{l=1}^{0} A_q e_1 \otimes e_1 A_q = 0$. The generators $e_1 \otimes e_1, e_{a(i)} \otimes e_{a(i)}$, and $e_{b(j)} \otimes e_{b(j)}$ of $P_{2n}$ are labeled $e_{l}^{2n}$ for $0 \leq l \leq 2n$, $e_{a(i)}^{2n}$ for $2 \leq i \leq s$, and $e_{b(j)}^{2n}$ for $2 \leq j \leq t$ respectively.
Similarly, we denote the generators $e_1 \otimes e_1, e_{a(i+1)} \otimes e_{a(i)}$ and $e_{b(j+1)} \otimes e_{b(j)}$ of $P_{2n+1}$ by $\varepsilon_{1}^{2n+1}$ for $1 \leq l \leq 2n$, $\varepsilon_{a(i)}^{2n+1}$ for $1 \leq i \leq s$, and $\varepsilon_{b(j)}^{2n+1}$ for $1 \leq j \leq t$ respectively. In [15], we give the minimal projective bimodule resolution of $A_q$ as follows.

**Theorem 3.1** ([15, Theorem 1.1]). The following sequence $\mathcal{P}$ is a minimal projective resolution of the left $A_q$-module $A_q$:

$$
\mathcal{P} : \cdots \rightarrow P_{2n+1} \xrightarrow{d_{2n+1}} P_{2n} \xrightarrow{d_{2n}} P_{2n-1} \xrightarrow{d_{2n-1}} \cdots \rightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{\pi} A_q \rightarrow 0.
$$

where $\pi : P_0 \rightarrow A_q$ is the multiplication map, and we define left $A^e$-homomorphisms $d_{2n+1}$ and $d_{2n+2}$ by

$$
d_{2n+1} : \\
\begin{align*}
\varepsilon_{b(j)}^{2n+1} &\mapsto \varepsilon_{b(j+1)}^{2n}Y - Ye_{b(j)}^{2n}, \\
\varepsilon_{a(i)}^{2n+1} &\mapsto -\sum_{j=0}^{t} q^i \varepsilon_{a(i)}^{2n}X - Xe_{a(i)}^{2n}, \\
\varepsilon_{l}^{2n+1} &\mapsto \sum_{j=0}^{s} q^j \varepsilon_{l}^{2n}Y - Ye_{l}^{2n}.
\end{align*}
$$

$$
d_{2n+2} : \\
\begin{align*}
\varepsilon_{b(j)}^{2n+1} &\mapsto \varepsilon_{b(j+1)}^{2n}Y - Ye_{b(j)}^{2n}, \\
\varepsilon_{a(i)}^{2n+1} &\mapsto \varepsilon_{a(i+1)}^{2n}X - Xe_{a(i)}^{2n}, \\
\varepsilon_{l}^{2n+1} &\mapsto \sum_{j=0}^{t} q^j \varepsilon_{l}^{2n}Y - Ye_{l}^{2n}.
\end{align*}
$$

for $n \geq 0$, where in the case $n = 0$, $\varepsilon_{a(i)}^{1}$ and $\varepsilon_{b(j)}^{1}$ vanish, and the image of $\varepsilon_{l}^{2}$ by $d_2$ is

$$
-\sum_{j=1}^{t} X^s Y^t \varepsilon_{b(j)}^{1} Y^{j-1} + q \sum_{j=1}^{t} X^s Y^t \varepsilon_{a(i)}^{1} Y^{j-1} X^s + q \sum_{i=1}^{s} X^s Y^t \varepsilon_{a(i)}^{1} X^{i-1} Y^{t-1} X^s.
$$

### 3.2 Hochschild cohomology group of $A_q$

Next, we give a basis of the $n$-th Hochschild cohomology group $HH^n(A_q) := \text{Ext}^n_{A_q}(A_q, A_q)$ for $n \geq 0$, using the minimal projective $A_q$-resolution given in Theorem 3.1. Now we
consider the case $s, t \geq 2$. In the case $s = 1$ or $t = 1$, we can give a basis of $\text{HH}^n(A_q)$ by the same method.

Now, we consider the vector space structure of $\text{HH}^n(A_q)$ for all $n \geq 0$. By the definition of $P_n$, we have isomorphisms

$$u_{2n} : \text{Hom}_{A_q^e}(P_{2n}, A_q) \sim \prod_{l=0}^{2n} e_1 A_q e_1 \oplus \prod_{i=1}^{s} e_{a(i)} A_q e_{a(i)} \oplus \prod_{j=1}^{t} e_{b(j)} A_q e_{b(j)},$$

$$u_{2n+1} : \text{Hom}_{A_q^e}(P_{2n+1}, A_q) \sim \prod_{l=1}^{2n} e_1 A_q e_1 \oplus \prod_{i=1}^{s} e_{a(i+1)} A_q e_{a(i)} \oplus \prod_{j=1}^{t} e_{b(j+1)} A_q e_{b(j)},$$

where $\prod_{l=0}^{0} e_1 A_q e_1 = 0$. We denote the $k$-modules $\text{Im} u_{2n}$ and $\text{Im} u_{2n+1}$ by $P_{2n}^*$ and $P_{2n+1}^*$ respectively. We see that the dimension of $P_n^*$ is given by

$$\dim_k P_n^* = nab + ab(s + t - 1).$$

The elements $e_1, e_{a(i)}$ and $e_{b(j)}$ of $P_{2n}^*$ are labeled $e_l^{2n}$ for $0 \leq l \leq 2n,$ $e_{a(i)}^{2n}$ for $2 \leq i \leq s$ and $e_{b(j)}^{2n}$ for $2 \leq j \leq t$ respectively. Similarly, we denote the elements $e_1, e_{a(i)}$ and $e_{b(j)}$ of $P_{2n+1}^*$ by $e_l^{2n+1}$ for $1 \leq l \leq 2n,$ $e_{a(i)}^{2n+1}$ for $1 \leq i \leq s$ and $e_{b(j)}^{2n+1}$ for $1 \leq j \leq t$ respectively. These labels correspond to that of generators of $P_n$. Using the maps $u_{2n}, u_{2n+1}, d_{2n+1}, d_{2n+2}$ for $n \geq 0$, we obtain the following diagram:

$$0 \rightarrow \text{Hom}_{A_q^e}(P_0, A_q) \xrightarrow{d_1} \text{Hom}_{A_q^e}(P_1, A_q) \xrightarrow{d_2} \text{Hom}_{A_q^e}(P_2, A_q) \xrightarrow{d_3} \cdots$$

where we put $d_n = \text{Hom}_{A_q^e}(d_n, A_q), d_n^* = u_n d_n u_{n-1}^{-1}$ for $n \geq 1$. Hence we have the complex

$$\mathbb{P}^* : 0 \rightarrow P_0^* \xrightarrow{d_1^*} P_1^* \rightarrow \cdots \rightarrow P_{n-1}^* \xrightarrow{d_n^*} P_n^* \rightarrow \cdots.$$
For $n$ even, $n \geq 2$:

$$W_{0,l,l'}^{n} = X^{sl} Y^{tl'} e_{0}^{n} + \sum_{j=2}^{t} Y^{j-1} X^{sl} Y^{t(l'-1)+t-j+1} e_{b(j)}^{n}$$

for $0 \leq l \leq a - 1$ and $0 \leq l' \leq b$,

$$W_{0,l,b-1}^{n*} := bX^{sl} Y^{t(b-1)} e_{0}^{n} + b \sum_{j=2}^{t} Y^{j-1} X^{sl} Y^{t(b-2)+t-j+1} e_{b(j)}^{n} + (q^{b(n/2-1)+1}-1)X^{s(l+1)} e_{1}^{n}$$

for $0 \leq l \leq a - 1$,

$$W_{m,l,l'}^{n} = X^{sl} Y^{tl'} e_{m}^{n}$$

for $1 \leq m \leq n-1, 0 \leq l \leq a - 1$ and $0 \leq l' \leq b - 1$,

$$W_{n,l,l'}^{n} := X^{sl} Y^{tl'} e_{n}^{n} + \sum_{i=2}^{s} X^{s(l-1)+i-1} Y^{tl'} X^{s-i+1} e_{a(i)}^{n}$$

for $0 \leq l \leq a$ and $0 \leq l' \leq b - 1$,

$$W_{n,a-1,l'}^{n*} := aX^{s(a-1)} Y^{tl'} e_{n}^{n} + a \sum_{i=2}^{s} X^{s(a-2)+i-1} Y^{tl'} X^{s-i+1} e_{a(i)}^{n} + (q^{a(n/2-1)+1}-1)Y^{t(l'+1)} e_{n-1}^{n}$$

for $0 \leq l' \leq b - 1$.

In the following results we use the complex $\mathbb{P}^{*}$ to compute a basis of the Hochschild cohomology group $HH^{n}(A_{q}) = \text{Kerd}_{n+1}^{*} / \text{Im} d_{n}^{*}$ of our algebra $A_{q}$ for $n \geq 0$.

First, we consider the case where $q$ is an $r$-th root of unity for integer $r \geq 1$. Now, we set $\overline{z}$ is the remainder when we divide $z$ by $r$ for any integer $z$. Then we have $0 \leq \overline{z} \leq r - 1$.

**Theorem 3.2** ([15, Proposition 3.3] and [16, Theorem 2.1]). If $q$ is an $r$-th root of unity for $r \geq 1$ and $s, t \geq 2$, then the following elements form a basis of $HH^{n}(A_{q})$.

1. **Basis of $HH^{0}(A_{q})$:**
   
   (a) $1_{A} = e_{0}^{0} + \sum_{j=2}^{t} e_{b(j)}^{0} + \sum_{i=2}^{s} e_{a(i)}^{0}$,
   
   (b) $T_{l,l'}$ for $1 \leq l \leq a - 1, 1 \leq l' \leq b - 1$ if $\overline{l} = \overline{l'} = 0$,
   
   (c) $T_{l,b}$ for $1 \leq l \leq a - 1$,
   
   (d) $T_{a,l'}$ for $1 \leq l' \leq b - 1$,

2. **Basis of $HH^{2n}(A_{q})$ for $n \geq 1$:**

   (a) $W_{0,0,l}^{2n}$ for $0 \leq l' \leq b - 1$ if $\overline{l} = \overline{b(n-1)+1}$, $\overline{b(n-1)+1} \neq 0$,
   
   (b) $W_{0,l,l}^{2n}$ for $1 \leq l \leq a - 1, 1 \leq l' \leq b - 1$ if $\overline{l} = 0, \overline{l'} = \overline{b(n-1)} = 0, \text{char } k|b$,
   
   (c) $W_{0,a-1,l'}^{2n}$ for $0 \leq l' \leq b - 1$ if $\overline{a} = 1, \overline{b(n-1)+1} \neq 0$,
   
   (d) $W_{0,a-1,l'}^{2n*}$ for $0 \leq l' \leq b - 1$ if $\overline{a} = 1, \overline{b(n-1)+1} \neq 0$,
   
   (e) $W_{0,l,b-1}^{2n}$ for $0 \leq l \leq a - 1$ if $\overline{l} = 0, b(n-1) + 1 \neq 0, \text{char } k \uparrow b$,
   
   (f) $W_{0,l,0}^{2n}$ for $0 \leq l \leq a - 2$ if $\overline{l} = 0, b(n-1) + 1 \neq 0, \text{char } k|b$,
   
   (g) $W_{1-l',l'}^{2n}$ for $0 \leq l \leq a - 1, 1 \leq l' \leq b - 1$ if $\text{char } k|a, \text{char } k \uparrow b$,
\[
\begin{align*}
\text{(h)} & \quad \begin{cases} 
W_{2l,l,l}^{2n}, & \text{for } 0 \leq l \leq a-2, 0 \leq l' \leq b-2 \text{ if } \char k \nmid a, \char k \nmid b, \\
W_{2l+1,l,l}^{2n}, & \text{for } 0 \leq l \leq a-2, 0 \leq l' \leq b-1 \text{ if } \char k \nmid a, \char k \nmid b, \\
W_{2l-1,l,l}^{2n}, & \text{for } 0 \leq l \leq a-1, 0 \leq l' \leq b-2 \text{ if } \char k |a, \char k | b, \\
W_{2l-1,l,l}^{2n+1}, & \text{for } 0 \leq l \leq a-1, 0 \leq l' \leq b-1 \text{ if } \char k |a, \char k | b, \\
\end{cases} \\
\text{for } 1 \leq l'' \leq n-1 \text{ if } \overline{l} = \overline{al''}, \overline{v} = b(n-l''), \\
\text{(i)} & \quad \begin{cases} 
W_{2n-1,1,l+1}^{2n}, & \text{for } 0 \leq l' \leq b-2 \text{ if } \overline{l'} = 0, \overline{b(n-1)} \neq 0, \char k | a, \\
W_{2n-1,0,l+1}^{2n} & \text{for } 0 \leq l' \leq b-1 \text{ if } \overline{l'} = 0, \overline{b(n-1)} + 1 \neq 0, \char k | a, \\
\end{cases} \\
\text{(j)} & \quad \begin{cases} 
W_{2n,a-1,l'}^{2n*}, & \text{for } 0 \leq l' \leq b-2 \text{ if } \overline{l'} = 0, \overline{a(n-1)} \neq 0, \char k | a, \\
W_{2n,a-1,l'}^{2n} & \text{for } 0 \leq l' \leq b-1 \text{ if } \overline{l'} = 0, \overline{a(n-1)} + 1 \neq 0, \char k | a, \\
\end{cases} \\
\text{(k)} & \quad W_{2n,l,0}^{2n} \text{ for } 0 \leq l \leq a-1 \text{ if } \overline{l} = \overline{an}, \\
\text{(l)} & \quad W_{2n,l,l}^{2n} \text{ for } 1 \leq l \leq a-1, 1 \leq l' \leq b-1 \text{ if } \overline{l} = \overline{al'}, \overline{v} = b(n-l'), \\
\text{(m)} & \quad W_{2n,a,l}^{2n} \text{ for } 1 \leq l' \leq b-1 \text{ if } \overline{l'} = 0, \overline{a(n-1)} = 0, \char k | a, \\
\text{(n)} & \quad W_{2n,l,b-1}^{2n} \text{ for } 0 \leq l \leq a-1 \text{ if } \overline{b} = 1, \overline{an-l'} \neq 0, \\
\text{(o)} & \quad \text{Additionally in the case of } q = -1: \\
\text{ i. } & \quad W_{2n-1,1,l}^{2n} \text{ for } 0 \leq l \leq a-1 \text{ if } \overline{an}=0, \overline{b}=0, \overline{l}=1, \\
\text{ ii. } & \quad W_{2n,a,l}^{2n} \text{ for } 1 \leq l \leq a-1 \text{ if } \overline{b}=0, \overline{l}=0, \\
\text{ iii. } & \quad W_{2n,a-1,b-1}^{2n} \text{ for } 1 \leq l' \leq b-1 \text{ if } \overline{a}=0, \overline{b}=0, \overline{bn-l'} \neq 0, \\
\text{ iv. } & \quad W_{2n-1,0,l}^{2n} \text{ for } 0 \leq l' \leq b-1 \text{ if } \overline{a}=0, \overline{bn}=0, \overline{l'}=1, \\
\text{ v. } & \quad W_{2n,a,l}^{2n} \text{ for } 0 \leq l' \leq b-1 \text{ if } \overline{a}=0, \overline{bn}=0, \overline{l'}=1, \\
\text{(3) Basis of } & \quad \HH^{2n+1}(A_q) \text{ for } n \geq 0: \\
\text{ (a)} & \quad U_{0,l,l}^{2n+1} \text{ for } 0 \leq l \leq a-1, 1 \leq l' \leq b-2 \text{ if } \overline{bn}=0, \overline{l}=0, \\
\text{ (b)} & \quad U_{0,a-1,l}^{2n+1} \text{ for } 1 \leq l' \leq b-1 \text{ if } \overline{a}=1, \overline{bn-l'} \neq 0, \\
\text{ (c)} & \quad U_{0,l,b-1}^{2n+1} \text{ for } 0 \leq l \leq a-1 \text{ if } \overline{l}=0, \overline{bn}=0, \overline{l}=0, \\
\text{ (d)} & \quad \begin{cases} 
U_{0,0,0}^{2n+1} & \text{if } \overline{bn}=0, \char k \nmid b, \\
U_{0,l,l}^{2n+1} & \text{for } 0 \leq l \leq a-1, 1 \leq l' \leq b-2 \text{ if } \overline{bn}=0, \overline{l}=0, \\
\end{cases} \\
\text{ (e)} & \quad \begin{cases} 
U_{1,l+1,0}^{2n+1} & \text{for } 0 \leq l \leq a-2 \text{ if } \overline{l}=0, \overline{bn}=0, \\
\end{cases} \\
\text{ (f)} & \quad \begin{cases} 
U_{2l,l,l}^{2n+1}, & \text{for } 0 \leq l \leq a-2, 1 \leq l' \leq b-1 \text{ if } \char k \nmid a, \char k \nmid b, \\
U_{2l+1,l,l}^{2n+1}, & \text{for } 0 \leq l \leq a-2, 0 \leq l' \leq b-1 \text{ if } \char k \nmid a, \char k \nmid b, \\
U_{2l+1,l,l}^{2n+1}, & \text{for } 0 \leq l \leq a-2, 0 \leq l' \leq b-1 \text{ if } \char k \nmid a, \char k \nmid b, \\
U_{2l+1,l,l}^{2n+1}, & \text{for } 0 \leq l \leq a-2, 0 \leq l' \leq b-1 \text{ if } \char k \nmid a, \char k \nmid b, \\
\end{cases} \\
\text{ for } 1 \leq l'' \leq n \text{ if } \overline{l} = \overline{al''}, \overline{v} = b(n-l''), \\
\text{ (g)} & \quad \begin{cases} 
U_{2l+1,l,l}^{2n+1}, & \text{for } 0 \leq l \leq a-2, 1 \leq l' \leq b-1 \text{ if } \char k \nmid a, \char k \nmid b, \\
U_{2l+1,l,l}^{2n+1}, & \text{for } 0 \leq l \leq a-2, 0 \leq l' \leq b-1 \text{ if } \char k \nmid a, \char k \nmid b, \\
U_{2l+1,l,l}^{2n+1}, & \text{for } 0 \leq l \leq a-2, 0 \leq l' \leq b-1 \text{ if } \char k \nmid a, \char k \nmid b, \\
\end{cases} \\
\text{ for } 0 \leq l'' \leq n \text{ if } \overline{l} = \overline{al''}+1, \overline{v} = b(n-l''), \\
\end{align*}
\]
(h) $U_{2n,0,l+1}^{2n+1}$ for $0 \leq l' \leq b - 2$ if $\overline{l} = 0$, $\overline{an} \neq 0$,

(i) $U_{2n+1,l',l}^{2n+1}$ for $1 \leq l \leq a - 1$, $0 \leq l' \leq b - 1$ if $\overline{l} = 0$, $\overline{a(n - 1) + 1} \neq 0$,

(j) $U_{2n+1,a-1,l'}^{2n+1}$ for $0 \leq l' \leq b - 1$ if $\overline{l'} = 0$, $\overline{an} \neq 0$,

(k) $U_{2n+1,l,b-1}^{2n+1}$ for $1 \leq l \leq a - 1$ if $\overline{b} = 1$, $\overline{an-l} \neq 0$,

(l) \begin{equation*}
\begin{cases}
U_{2n+1,0,0}^{2n+1} & \text{if } \overline{an} = 0, \text{char } k \mid a, \\
U_{2n+1,0,l'}^{2n+1} & \text{if } 0 \leq l' \leq b - 1 \text{ if } \overline{l'} = 0, \overline{an} = 0, \text{char } k | a,
\end{cases}
\end{equation*}

(m) Additionally in the case of $q = -1$:

i. $U_{0,l,0}^{2n+1}$ for $0 \leq l \leq a - 1$ if $\overline{a(n - 1)} = 0$, $\overline{b} = 0$, $\overline{l} = 1$,

ii. $U_{1,0,0}^{2n+1}$ for $0 \leq l \leq a - 1$ if $\overline{a} = 0$, $\overline{b} = 0$,

iii. $U_{2n+1,0,0}^{2n+1}$ for $0 \leq l \leq b - 1$ if $\overline{a} = 0$, $\overline{b(n - 1)} = 0$, $\overline{l'} = 1$.

In the case $q = 1$, $q$ is a first root of unity. Then $\overline{z} = 0$ for any integer $z$. Hence if $q = 1$ then a basis of $\text{HH}^n(A_q)$ is formed by the elements of (1), (2), (a), (b), (c), (g), (h), (k), (l), (m), and (3) (a), (d), (f), (g), (i), (l).

Next, in the case where $q$ is not a root of unity, we give a basis of $\text{HH}^n(A_q)$ for $n \geq 0$.

**Theorem 3.3** ([16, Theorem 2.2]). If $q$ is not a root of unity and $s, t \geq 2$, then the following elements form a basis of $\text{HH}^n(A_q)$.

1. **Basis of $\text{HH}^0(A_q)$:**
   a. $\mathbf{1}_{A_q} = e_0^0 + \sum_{j=2}^{t} e_{b(j)}^0 + \sum_{i=2}^{s} e_{a(i)}^0$,
   b. $T_{a,l}$ for $1 \leq l \leq a - 1$,
   c. $T_{a,l'}$ for $1 \leq l \leq b - 1$,

2. **Basis of $\text{HH}^{2n+1}(A_q)$ for $n \geq 0$:**
   a. $U_{0,l+1,b-1}^{1}$ for $0 \leq l \leq a - 2$ if $n = 0$,
   b. $U_{0,0,b-1}^{2n+1}$,
   c. $U_{0,0,0}^{2n+1}$ if $n = 0$,
   d. $U_{1,a-1,l'}^{1}$ for $0 \leq l \leq b - 3$ if $n = 0$,
   e. $U_{2n+1,0,a-1,0}^{1}$,
   f. $U_{1,0,0}^{1}$ if $n = 0$,

3. **Basis of $\text{HH}^{2n+2}(A_q)$ for $n \geq 0$:**
   a. $W_{1,1,1}^{2}$ if $n = 0$,
   b. $W_{0,0,b-1}^{2n+2}$,
   c. $W_{2n+2,a-1,0}^{2n+2}$. 


3.3 Hochschild cohomology ring of $A_q$

In this section, we determine the Hochschild cohomology ring of $A_q$ modulo nilpotence. Now we recall the Yoneda product in $HH^*(A)$ (see [8]). For homogeneous elements $\eta \in HH^m(A)$ and $\theta \in HH^n(A)$ represented by $\eta$: $P_m \rightarrow A$ and $\theta$: $P_n \rightarrow A$ respectively, the Yoneda product $\eta \theta \in HH^{m+n}(A)$ is given as follows: There exists a commutative diagram of $A$-bimodules

$$
\begin{array}{cccccc}
\cdots & \rightarrow & P_{m+n} & d_{m+n} & \cdots & \rightarrow & P_{n+1} & d_{n+1} & \cdots & \rightarrow & P_n \\
\downarrow & & \downarrow & \sigma_m & & \downarrow & & \sigma_1 & & \downarrow & & \theta \\
\cdots & \rightarrow & P_m & d_m & \cdots & \rightarrow & P_1 & d_1 & \cdots & \rightarrow & P_0 & \rightarrow & A & \rightarrow & 0,
\end{array}
$$

where $\sigma_i$ are liftings of $\theta$. Here we see that such liftings always exist but are not unique. Then we have $\eta \theta = \eta \sigma_m \in HH^{m+n}(A)$. We note that $\eta \theta$ is independent of the choice of representation $\eta, \theta$ and liftings $\sigma_i$ ($0 \leq i \leq m$). See [16, Proposition 3.1] for the liftings of the basis of $HH^n(A_q)$ ($n \geq 0$). In the case where $q$ is a root of unity, by the liftings given in [16, Proposition 3.1], we see that $HH^{n+2r}(A_q)$ is generated by the elements in $HH^n(A_q)$ and that in $HH^{2r}(A_q)$ for $n \geq 2r$. By corresponding Yoneda product of the basis elements of $HH^*(A_q) := \oplus_{n \geq 0} HH^n(A_q)$ given in Theorem 3.2, we now have the generators of $HH^*(A_q)$. In this paper, we consider the case where $s, t \geq 2$ and $\overline{a}, \overline{b} \neq 0$. In the other cases, we have similar results to the following theorem and corollary. See [16] for the other cases.

**Theorem 3.4** ([16, Theorem 3.2]). In the case where $\overline{a}, \overline{b} \neq 0$, $HH^*(A_q)$ is generated as an algebra by the following generators:

1. **The generators of $HH^*(A_q)$ in degree 0:**
   $1_{A_q}, T_{r, r}, T_{l_1, r}, T_{r, l_1}, T_{l_2, b}, T_{a, l_2}$
   for $1 \leq l_1, l_2 \leq a - 1$, $1 \leq l_1', l_2' \leq b - 1$ if $\overline{l_1} = \overline{l_1'} = 0$.

2. **The generators of $HH^*(A_q)$ in degree 1:**
   - $U_{1,0,l'}^1, U_{1,l,0}^1$ for $0 \leq l \leq a - 1$, $0 \leq l' \leq b - 1$ if $\overline{l} = \overline{l'} = 0$,
   - $\begin{cases}
   U_{0,a-1,l'}^1 & \text{for } 1 \leq l' \leq b - 1 \text{ if } \overline{a} = 1, \overline{l'} \neq 0, \\
   U_{1,a-1,l'}^1 & \text{for } 0 \leq l' \leq b - 1 \text{ if } \overline{a} \neq 1, \overline{l'} = 0,
   \end{cases}$
   - $\begin{cases}
   U_{1,1,b-1}^1 & \text{for } 1 \leq l' \leq a - 1 \text{ if } \overline{l'} \neq 0, \overline{b} = 1, \\
   U_{0,1,b-1}^1 & \text{for } 1 \leq l \leq a - 1 \text{ if } \overline{l} = 0, \overline{b} \neq 1,
   \end{cases}$
   - $U_{1,l,b-1}^1$ for $1 \leq l' \leq b - 1$ if $\overline{l'} = 0, \text{char } k | a$,
   - $U_{0,l,0}^1$ for $1 \leq l \leq a - 1$ if $\overline{l} = 0, \text{char } k | b$.

3. **The generators of $HH^*(A_q)$ in degree 2:**
   - $W_{0,0,l'}^2, W_{0,l_2,0}^2, W_{0,l_2,b'}, W_{2,0,l_2}^2, W_{2,0,b'}^2$
     for $1 \leq l_1, l_2 \leq a - 1$, $1 \leq l_1', l_2' \leq b - 1$ if $\overline{l_1} = \overline{a}, \overline{l_1'} = \overline{b}, \overline{l_2} = \overline{l_2'} = 0$,
   - $\begin{cases}
   W_{0,0,b-1}^2 & \text{for } 0 \leq l \leq a - 1 \text{ if } \overline{l} = 0, \text{char } k | b, \\
   W_{1,l+1,0}^2 & \text{for } 0 \leq l \leq a - 2 \text{ if } \overline{l} = 0, \text{char } k | b,
   \end{cases}$
(4) The generators of $\text{HH}^*(A_q)$ in degree $2n$ for $2 \leq n \leq r$:

\[
\begin{align*}
\{ & W_{2n,a-1,l}^{2n*} \text{ for } 0 \leq l' \leq b - 1 \text{ if } \overline{l'} = 0, \text{char} k \nmid a, \\
& W_{2n,0,l+1}^{2n} \text{ for } 0 \leq l' \leq b - 2 \text{ if } \overline{l'} = 0, \text{char} k \nmid a, \\
& W_{0,a-1,l'}^{2} \text{ for } 0 \leq l' \leq b - 1 \text{ if } \overline{a} = 1, \overline{l'} \neq \overline{b}, \\
& W_{2n,b-1,l}^{2} \text{ for } 0 \leq l \leq a - 1 \text{ if } \overline{b} = 1, \overline{l} \neq \overline{a}.
\end{align*}
\]

\[
\begin{align*}
\{ & W_{2n,0,\overline{bn}}^{2n} \text{ if } \min\{\overline{bn'}|1 \leq n' \leq n - 1\} > \overline{bn}, \\
& W_{2n,0,\overline{bnm}}^{2n} \text{ for } 1 \leq l \leq a - 1 \text{ if } \overline{l} = 0, \overline{bn} \neq 0, \min\{\overline{bn'}|1 \leq n' \leq n - 1\} > \overline{bn}, \\
& W_{2n,0,a-1,\overline{t_{1}^{l_{1}+1}l_{2}^{l_{2}-r}}} \text{ if } \overline{a} = 1, \overline{l_{1}^{l_{1}+1}l_{2}^{l_{2}+1}} \geq r + 1, \overline{l_{1}} = \min\{\overline{bn'}|1 \leq n' \leq n - 1\}, \\
& \overline{l_{2}} \neq b(n - n') \text{ where } n' \text{ is integer such that } 1 \leq n' \leq n - 1 \text{ and } \\
& \overline{bn} \leq \overline{bn''} \text{ for any } 1 \leq n'' \leq n - 1, \\
& W_{2n,0,b-1}^{2n*} \text{ for } 0 \leq l \leq a - 1 \text{ if } \overline{l} = 0, \overline{bn} \neq 0, \overline{b(n-1)+1} \neq 0, \text{char} k \nmid a, \\
& W_{2n,1,l+1,0}^{2n*} \text{ for } 0 \leq l \leq a - 2 \text{ if } \overline{l} = 0, \overline{b(n-1)+1} \neq 0, \text{char} k \nmid b, \\
& W_{2n,1,l+1,1}^{2n} \text{ for } 1 \leq l \leq a - 2 \text{ if } \overline{l} = 0, \overline{b(n-1)} \neq 0, \\
& W_{2n-1,a-1,\overline{l_{1}^{l_{1}+1}l_{2}^{l_{2}+1}}} \text{ if } \overline{a} = 1, \overline{l_{1}^{l_{1}+1}l_{2}^{l_{2}+1}} \geq r + 1, \overline{l_{1}} = \min\{\overline{bn'}|1 \leq n' \leq n - 1\}, \\
& \overline{l_{2}} \neq a(n - n') \text{ where } n' \text{ is integer such that } 1 \leq n' \leq n - 1 \text{ and } \\
& \overline{an'} \leq \overline{an''} \text{ for any } 1 \leq n'' \leq n - 1.
\end{align*}
\]

(5) The generators of $\text{HH}^*(A_q)$ in degree $2n + 1$ for $1 \leq n \leq r - 1$:

\[
\begin{align*}
\{ & U_{2n+1,0,l,\overline{bn}}^{2n+1} \text{ for } 0 \leq l \leq a - 1 \text{ if } \overline{l} = 0, \min\{\overline{bn'}|1 \leq n' \leq n - 1\} \geq \overline{bn}, \\
& U_{2n+1,0,l,b-1}^{2n+1} \text{ for } 0 \leq l \leq a - 1 \text{ if } \overline{l} = 0, \overline{b(n-1)+1} \neq 0, \text{char} k \nmid b, \\
& U_{2n+1,1,l+1,0}^{2n+1} \text{ for } 0 \leq l \leq a - 2 \text{ if } \overline{l} = 0, \overline{bn} \neq 0, \\
& U_{2n+1,2l,\overline{an},0}^{2n+1} \text{ if } \min\{\overline{an'}|1 \leq n' \leq n - 1\} > \overline{an}, \\
& U_{2n+1,2n,\overline{an},l'}^{2n+1} \text{ for } 1 \leq l' \leq b - 1 \text{ if } \overline{l'} = 0, \overline{an} \neq 0, \min\{\overline{an'}|1 \leq n' \leq n - 1\} > \overline{an}, \\
& U_{2n+1,2n,\overline{an},l'}^{2n+1} \text{ if } \overline{b} = 1, \overline{l_{1}^{l_{1}+1}l_{2}^{l_{2}+1}} \geq r + 1, \overline{l_{1}} = \min\{\overline{an'}|1 \leq n' \leq n - 1\}, \\
& \overline{l_{2}} \neq a(n - n') \text{ where } n' \text{ is integer such that } 1 \leq n' \leq n - 1 \text{ and } \\
& \overline{an'} \leq \overline{an''} \text{ for any } 1 \leq n'' \leq n - 1.
\end{align*}
\]
The generators of $\text{HH}^\ast(A_q)$ in degree $2r + 2n + 1$ for $0 \leq n \leq r - 2$:

\[
\begin{align*}
\cdot & \quad U_{2n+1,0,b(n-l')}^{2r+1} \text{ for } 0 \leq l' \leq n - 1, 0 \leq b(n-l') \leq \begin{cases} 
2r - 1 \text{ if } \text{char } k \nmid b, \\
2r - 2 \text{ if } \text{char } k \mid b, 
\end{cases} \\
\cdot & \quad U_{2n-1,0,t'}^{2r+1} \text{ for } 0 \leq t' \leq b - 2 \text{ if } a(n-1) + 1 = 0, t' = b, \\
\cdot & \quad U_{2n,0,b+1}^{2r+1} \text{ for } 0 \leq b - 2 \text{ if } \overline{a} = 0, \overline{a} \neq 0, \\
\cdot & \quad U_{2n+1,0,l'}^{2r+1} \text{ for } 0 \leq l' \leq b - 1 \text{ if } \overline{a} = 0, \text{min}\{\overline{a}l'|1 \leq l' \leq n' \leq n - 1\} \geq \overline{a}n, \\
\cdot & \quad U_{2n+1,a-1,2n+1}^{2r+1} \text{ for } 0 \leq b - 1 \text{ if } \overline{a} = 0, \text{min}\{\overline{a}l'|1 \leq l' \leq n' \leq n - 1\} > \overline{a}n, \text{char } k \mid a.
\end{align*}
\]

It follows from the Theorem 3.4 that $1_{A_q}$, $W_{0,0,0}^{2r}$, and $W_{2r,0,0}^{2r}$ are not nilpotent and the other generators are nilpotent. Thus we have the following corollary.

**Corollary 3.5.** If $s, t \geq 2$ and $\overline{a}, \overline{b} \neq 0$, then the quotient of the Hochschild cohomology ring of $A_q$ modulo nilpotence is isomorphic to the polynomial ring of two variables in all characteristic:

\[
\text{HH}^\ast(A_q)/\mathcal{N} \cong k[W_{0,0,0}^{2r} W_{2r,0,0}^{2r}].
\]

Finally, we consider the ring structure of $\text{HH}^\ast(A_q)$ in the case where $q$ is not a root of unity. It follows from the liftings given in [16] that all basis elements except $1_{A_q}$ of $\text{HH}^n(A_q)$ are nilpotent elements for $n \geq 0$. Thus we have the following results.

**Theorem 3.6.** If $q$ is not a root of unity then $\text{HH}^\ast(A_q)$ is not a finitely generated $k$-algebra.

**Corollary 3.7.** If $q$ is not a root of unity then $\text{HH}^\ast(A_q)/\mathcal{N} \cong k$.

In general, our algebra $A_q$ is not self-injective, monomial or Koszul. Moreover $A_q$ does not have a stratifying ideal. Therefore $A_q$ is new example of a class of algebras for which the Hochschild cohomology ring modulo nilpotence is finitely generated as a $k$-algebra. For example, in the case where $s = 2$, $t = 1$ and $a = b = 2$, our algebra $A_q$ is not self-injective, monomial or Koszul. Moreover $A_q$ does not have a stratifying ideal.

### 3.4 Finiteness conditions for $A_q$

Finally, we show that $A_q$ satisfies the finiteness conditions in the case where $q$ is a root of unity.

Now we consider the case where $q$ is an $r$-th root of unity, $s, t \geq 2$ and $\overline{a}, \overline{b} \neq 0$. In the other case, we see that $A_q$ satisfies the finiteness conditions by the same method. The Yoneda algebra or Ext algebra of $A_q$ is given by $E(A_q) = \oplus_{n \geq 0} \text{Ext}_{A_q}^n(A_q/t, A_q/t)$.
with the Yoneda product. We use the notation $E(A_q)^n = \text{Ext}^n_{A_q}(A_q/\tau, A_q/\tau)$ for the $n$-th graded component of $E(A_q)$. Then it is easy to see that $E(A_q)^n \cong \prod_{i=0}^n ke_i^n \oplus \prod_{j=2}^{i=2} ke_j^n$.

Let $\varphi$: $\text{HH}^*(A_q) \to E(A_q)$ be a homomorphism of graded rings given by $\varphi(\eta) = \eta \otimes_{A_q} A_q/\tau$. Then it is easy to see that the image of $\varphi$ is precisely the graded ring $k[x,y]$ where $x := e_0^{2r} + \sum_{j=2}^{t} e_{b(j)}^{2r}$ and $y := e_2^{2r} + \sum_{i=2}^{s} e_{a(i)}^{2r}$ in degree $2r$.

**Proposition 3.8.** $E(A_q)$ is a finitely generated left $k[x,y]$-module with generators:

- $e_l^{2n}$, $e_{b(j)}^{2n}$, $e_{a(i)}^{2n}$ for $0 \leq l \leq 2n, 2 \leq j \leq t$ and $2 \leq i \leq s$
  
  in degree $2n$ for $0 \leq n \leq r - 1$,

- $e_l^{2n+1}$, $e_{b(j)}^{2n+1}$, $e_{a(i)}^{2n+1}$ for $1 \leq l \leq 2n, 1 \leq j \leq t$ and $1 \leq i \leq s$
  
  in degree $2n+1$ for $0 \leq n \leq r - 1$,

- $e_l^{2r}$ for $1 \leq l \leq 2r - 1$ in degree $2r$,

- $e_l^{2r+2n+1}$ for $2n + 1 \leq l \leq 2r$
  
  in degree $2r + 2n + 1$ for $0 \leq n \leq r - 1$,

- $e_l^{2r+2n+2}$ for $2n + 1 \leq l \leq 2r - 1$
  
  in degree $2r + 2n + 2$ for $0 \leq n \leq r - 2$.

Now we consider the conditions (Fg1) and (Fg2). The element $W_{0,0,0}^{2r} \in \text{HH}^{2r}(A_q)$ is a pre-image of $x$ and the element $W_{2r,0,0}^{2r} \in \text{HH}^{2r}(A_q)$ is a pre-image of $y$. Let $H$ be the graded subalgebra of $\text{HH}^*(A_q)$ generated by $\text{HH}^0(A_q)$, $W_{0,0,0}^{2r}$ and $W_{2r,0,0}^{2r}$, so that $H$ is a pre-image of $k[x,y]$ in $\text{HH}^*(A_q)$. Then we have the following immediate consequence of Proposition 3.8.

**Theorem 3.9.** The conditions (Fg1) and (Fg2) hold for the algebra $A_q$ with respect to the subring $H$ of $\text{HH}^*(A_q)$.

By [2], Theorem 3.6 and 3.9, we have the necessary and sufficient conditions for $A_q$ to satisfy the finiteness conditions.

**Theorem 3.10.** $A_q$ satisfies the finiteness conditions if and only if $q$ is a root of unity.

**References**


