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A BATALIN-VILKOVISKY ALGEBRA STRUCTURE ON THE MOORE SPECTRAL SEQUENCE FOR A POINCARÉ DUALITY SPACE

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ABSTRACT. In this survey article, we introduce a differential Batalin-Vilkovisky algebra structure on the Moore spectral sequence converging the Hochschild cohomology of the singular cochain algebra of a Poincaré duality space.

1. INTRODUCTION AND BACKGROUND

The Hochschild (co)homology of the singular cochain algebra of a space relates to some of important homotopy invariants. In this note, after surveying such interesting and significant results, following Menichi [29], we introduce a Batalin-Vilkovisky (B-V) algebra structure on the Hochschild cohomology of the singular cochain algebra of a Poincaré duality space. Our main theorem, Theorem 2.5, shows that the Moore spectral sequence converging the Hochschild cohomology admits a differential graded B-V algebra structure which is compatible with that of the target in the E_∞-term.

Let $\mathbb{K}$ be an arbitrary field. In what follows, we assume that a space X has the homotopy type of a CW- complex whose cohomology with coefficients in $\mathbb{K}$ is locally finite; that is, $\dim H^i(X;\mathbb{K}) < \infty$ for any i.

We begin by describing why the Hochschild (co)homology of the singular cochain algebra of a space is in our interest.

Let X be a simply-conected space and LX the free loop space, namely, the space $\mathrm{map}(S^1, X)$ of maps from the circle S^1 to X with compact-open topology. We denote by $\mathcal{F}$ the fibre square

$$\begin{array}{ccc} LM & \longrightarrow & \mathrm{map}([0,1],X) \\ {\scriptstyle ev_1}\downarrow & & \downarrow{\scriptstyle ev_0\times ev_1} \\ M & \xrightarrow{\Delta} & M\times M, \end{array}$$

where ev_i denotes the evaluation map at i for $i = 0, 1$ and Δ is the diagonal map. We then have the Eilenberg-Moore spectral sequence (EMSS for short) $\{{}_{EM}E_r^{*,*}, d_r\}$ for the fibre square $\mathcal{F}$ converging to the cohomology $H^*(LM;\mathbb{K})$ with

$${}_{EM}E_2^{*,*} \cong HH_{*,*}(H^*(X;\mathbb{K}); H^*(X;\mathbb{K}))$$

as a bigraded algebra. Observe that the fibre of the evaluation map $ev_1 : LM \to X$ is the based loop space $\Omega X = \mathrm{map}_*(S^1, X)$. It is worth noting that in [34, 21] the problem whether the loop fibration $\Omega X \to LX \overset{ev_1}{\to} X$ is totally non-cohomologous to zero with respect to a given field is considerd drawing on calculations of the

Hoschschild homology of a graded algebra which appears in the EMSS mentioned above.

Recently, the author investigates in [23, 24, 25] new numerical homotopy invariants called the *(co)chain type levels* of maps. The invariants are derived from the notion of the *level* of an object in a triangulated category, which is first introduced by Avramov, Buchweitz, Iyengar and Miller [1].

Consider the category $\mathcal{TOP}_B$ of spaces over a space B. To each object $f : sf \to B$ in $\mathcal{TOP}_B$, the singular cochain complex functor $C^*(\ ;\mathbb{K})$ with coefficients in $\mathbb{K}$ assigns the differential graded (DG) module $C^*(sf;\mathbb{K})$ over the differential graded algebra $C^*(B;\mathbb{K})$. Thus we have a functor

$$C^*(s(\ );\mathbb{K}) : \mathcal{TOP}_B \to \mathrm{D}(\text{Mod-}C^*(B;\mathbb{K}))$$

from the category $\mathcal{TOP}_B$ to the derived category of DG modules over $C^*(B;\mathbb{K})$ † which is a triangulated category; see [19]. Roughly speaking, the level of an object U in a triangulated category $\mathcal{T}$ measures the number of triangles need to build U out of a given object. Since the invariant is defined by using an increasing filtration of a thick subcategory of $\mathcal{T}$, we anticipate that a classification of such subcategories is of use in the study of the level. In order to mention such a classification theorem, we recall the definition of the graded center of a triangulated category.

Definition 1.1. (cf.[2, 3.2], [28, §2]) Let $\mathcal{T}$ be a $\mathbb{K}$-linear triangulated category with suspension functor Σ. The graded center $\mathcal{Z}(\mathcal{T})$ is a graded family whose degree n component $\mathcal{Z}^n(\mathcal{T})$ consists of all natural transformations $\varphi : Id_{\mathcal{T}} \to \Sigma^n$ such that $\varphi\Sigma = (-1)^n\Sigma\varphi$.

Let R be a commutative graded ring and $\Phi : R \to \mathcal{Z}(\mathcal{T})$ a ring homomorphism preserving the degree. Here we ignore set theoretic issues on the graded center. Indeed, the ring homomorphism means that, for each object X in $\mathcal{T}$, one has a homomorphism of graded algebra $\Phi_X : R \to \mathrm{End}^*_{\mathcal{T}}(X)$ such that

$$\Phi_Y(\alpha)\beta = (-1)^{|\alpha||\beta|}\beta\Phi_X(\alpha)$$

for $\alpha \in R$ and $\beta \in \mathrm{Hom}^*_{\mathcal{T}}(X,Y)$. We mention that, using such a ring homomorphism, a classification theorem of thick subcategories, in other words, a theory of stratification of a triangulated category is described in [4, 5] via the theory of support varieties.

Let A be a DG algebra over a field $\mathbb{K}$. Then we have a triangulated category $\mathrm{D}(A)$, which is the derived category of DG modules over A with the shift functor Σ; $(\Sigma N)^n = N^{n+1}$, as the suspension functor. It follows from [3, Proposition 1.1] that the cup product $\smile$ on $HH^*(A;A)$ coincides with the Yoneda product. We then have a ring homomorphism Φ from the Hochschild cohomology ring $HH^*(A;A)$ to the graded center of the triangulated category $\mathrm{D}(A)$. In fact, the homomorphism $\Phi : HH^*(A;A) \to \mathcal{Z}(\mathrm{D}(A))$ is defined by

$$\Phi(f)(M) = \Phi_M(f) = Id_M \otimes_A f : M \to \Sigma^n M$$

†In the rational case, it seems that the study of the derived category $\mathrm{D}(\text{Mod-}C^*(B;\mathbb{Q}))$ is indeed that of fibrewise stable rational homoltopy theory; see [14, Theorem 1.1].

in $\mathrm{D}(A)$ for $f \in HH^n(A, A)$.

Let X be a simply-connected space. The general argument above gives a ring homomorphism

$$HH^*(C^*(X), C^*(X)) \to \mathcal{Z}(\mathrm{D}(C^*(X))).$$

Thus it is expected that the Hochschild homology of the cochain of a space is of great use when studying levels of maps in triangulated categories associated with cochain algebras of spaces. For the stratification of cochains described in topological content, see [33].

One of facts which motivate us to study the Hochschild homology of the singular cochain is also in string topology initiated by Chas and Sullivan [6].

Let M be a closed oriented manifold of dimension d. The main player in string topology is the fee loop space LM. In particular, by lifting the intersection product $H_i(M) \otimes H_j(M) \to H_{i+j-d}(M)$, one can define the so-called *loop product*

$$\bullet : \mathbb{H}_*(LM) \otimes \mathbb{H}_*(LM) \to \mathbb{H}_*(LM)$$

on the shifted homology $\mathbb{H}_*(ML) := H_{*+d}(LM)$. Moreover, a result due to Cohen and Godin [9] asserts that the loop product is regarded as one of string operations arising from a two-dimensional topological quantum field theory with the values in the homology of LM. We refer to a book [10] for a fascinating introduction to this exciting field, string topology.

A considerable result due to Cohen and Jones enables us to find the Hochschold cohomology of the singular cochain in large realm of string topology.

Theorem 1.2. [11, 8] *Suppose that M is a simply-connected closed oriented manifold. Then there is a morphism of graded algebras between the loop homology $(\mathbb{H}_*(LM), \bullet)$ and $(HH^*(C^*(M); C^*(M)), \smile)$ provided the underlying coefficients are in a field.*

These backgrounds explain the reasons why we are interested in the Hochschild cohomology of the cochains of spaces.

One might be strongly interested in explicit calculations of the Hochschild cohomology. The following table summarizes spectral sequences in string topology which compute the Hochschild cohomology of the cochains of spaces and the Chas-Sullivan loop homology. We assume that the underlying ring is a field $\mathbb{K}$.

The homological Leray-Serre type	The cohomological Eilenberg-Moore type
$E^2_{-p,q} = H^p(M; H_q(\Omega M)) \Rightarrow \mathbb{H}_{-p+q}(LM)$, where M is a simply-connected closed oriented manifold; see [12].	$E_2^{p,q} = HH^{p,q}(H^*(M); H^*(M)) \Rightarrow \mathbb{H}_{-p-q}(LM)$, where M is a simply-connected Poincaré duality space; see [26].
$E^2_{p,q} = H^{-p}(M) \otimes \mathrm{Ext}^{-q}_{C^*(M)}(\mathbb{K}, \mathbb{K}) \Rightarrow HH^{-p-q}(C^*(M); C^*(M))$, where M is a simply-connected space whose cohomology is of finite dimension; see [32].	$E_2^{p,q} = HH^{p,q}(H^*(M); H^*(M)) \Rightarrow HH^{p+q}(C^*(M); C^*(M))$, where M is a simply-connected space whose cohomology is of finite dimension; see [22].

Tabel 1

Observe that all spectral sequences in Tabel 1 converge to the target as algebras. We refer the reader to the papers cited in Table 1 for explicit calculations and applications of the spectral sequences; see also [7, 27] for other spectral sequences which appear in string topology.

We would like to end this section with comments on a class of spaces to which string topology is applicable.

Let M be a $\mathbb{K}$-Gorenstein space (simply, Gorenstein space) of dimension d; that is, M is simply-connected and satisfies the condition that

$$\dim \operatorname{Ext}^*_{C^*(M;\mathbb{K})}(\mathbb{K}, C^*(M;\mathbb{K})) = \begin{cases} 0 & \text{if } * \neq d, \\ 1 & \text{if } * = d. \end{cases}$$

For example, a simply-connected closed oriented manifold, more generally, a Poincaré duality space M is a Gorenstein space of dimension $\dim M$. We see that the classifying space of a connected Lie group G is also a Gorenstein space of dimension $-\dim G$. Moreover for a simply-connected Poincaré duality space M with an action of a connected Lie group G, the Borel construction $EG \times_G M$ is a Gorenstein space of dimension $\dim M - \dim G$, see [30]. we remark that the dmension of a Gorenstein space may be negative. For more details of Gorenstein spaces, see [13].

Let $LM \times_M LM$ be the space which fits into the pullback diagram

$$\begin{tikzcd} LM \times_M LM \arrow[r, "q"] \arrow[d, "ev_0"'] & LM \times LM \arrow[d, "ev_0 \times ev_0"] \\ M \arrow[r, "\Delta"] & M \times M. \end{tikzcd}$$

In [15], it is proved that for a simply-connected Gorenstein space M of dimension d, a shriek map $q^! : C^*(LM \times_M LM) \to C^*(LM \times LM)$ of degree d is defined and that it gives rise to the dual to the loop product

$$Dlp : H^*(LM) \to (H^*(LM) \otimes H^*(LM))^{*+d}$$

which coincides with the original loop product on $H_*(LM)$ by dualizing it if M is a manifold. This remarkable result due to Félix and Thomas implies that the range of applications of string topology extends to a more large class of Gorenstein spaces. Observe that $\mathbb{H}_*(LM)$ in the right-upper-hand square in Table 1 is considered the shifted homology algebra together with the loop product in the wide sense.

2. A differential B-V algebra structure on the Moore spectral sequence

Our discussion on the Hochschild cohomology below focusses on the Moore spectral sequence in the right-lower-hand side square in Table 1. In particular, we show that the spectral sequence comes equipped with the Batalin-Vilkovisky operators.

We recall the spectral sequence more precisely. Let $\mathbb{K}$ be a field. Unless stated otherwise, coefficients of the singular cochain algebra of a space are in $\mathbb{K}$. Let M and N be connected spaces and $f : N \to M$ a map. The singular cochain algebra $C^*(N)$ is regarded as a $C^*(M)$-bimodule via the map $f^* : C^*(M) \to C^*(N)$ induced by f. Then it follows that the cup product gives rise to a $C^*(M)\otimes C^*(M)^{op}$-module map $C^*(N)\otimes_{C^*(M)} C^*(N)\to C^*(N)$.

Theorem 2.1. ([22, Theorem 3.1], *cf.* [16, 1 Proposition]) *Under the above hypothesis, we assume further that $H^*(N)$ is of finite dimension. Then there exists a right-half plane cohomological spectral sequence $\{E_r^{*,*}, d_r\}$ converging to the Hochschild cohomology $HH^*(C^*(M); C^*(N))$ as an algebra such that*

$$E_2^{p,q} \cong HH^{p,q}(H^*(M); H^*(N))$$

as a bigraded algebra.

The spectral sequence in Theorem 2.1 is called the *Moore spectral sequence.*

Remark 2.2. Let S be a complement of the vector subspace generated by cycles of $C^d(N)$, where $d=\sup\{n \mid H^*(N)\neq 0\}$. We define I to be the two-sided ideal generated by $C^{>d}(N)\oplus S$. Then the projection $C^*(N)\to C^*(N)/I$ is a quasi-isomorphism of A-bimodules. We define a decreasing filtration $\{F^p\mathbf{C}^*\}_{p\geq 0}$ of the Hochschild cochain complex $\mathbf{C}^*=\{\mathrm{Hom}_{A\otimes A^{op}}(\mathbb{B}_*(A;A;A), C^*(N)/I)\}_{n\in\mathbb{Z}}$ by

$$F^p\mathbf{C}^n = \prod_{s\geq p} \mathrm{Hom}^n_{A\otimes A^{op}}(\mathbb{B}_s(A;A;A), C^*(N)/I),$$

where $A := C^*(M)$, $\mathbb{B}(A;A;A)$ denotes the normalized bar resolution of A as $A\otimes A^{op}$ module and $\mathbb{B}_s(A;A;A) = A\otimes s\overline{A}^{\otimes s}\otimes A$. We see that the filtration $\{F^p\mathbf{C}^*\}_{p\geq 0}$ is bounded; that is, for any n, there exists $p(n)$ such that $F^p\mathbf{C}^n=0$ for $p>p(n)$; see the proof of [22, Theorem 3.1]. This implies the the Moore spectral sequence converges strongly to the target.

Before describing our main theorem on the Hochschild cohomology, we recall here the definition of the Batalin-Vilkovisky algebra.

Definition 2.3. A graded commutative algebra A^* is a Batalin-Vilkovisky algebra if A^* is equipped with an operation $\Delta : A^*\to A^{*-1}$ such that $\Delta^2\equiv 0$ and for $a,b,c\in A^*$,

$$\begin{aligned}\Delta(abc) &= \Delta(ab)c + (-1)^{|a|}a\Delta(bc) + (-1)^{(|a|-1)|b|}b\Delta(ac)\\ &\quad -(\Delta a)bc - (-1)^{|a|}a(\Delta b)c - (-1)^{|a|+|b|}ab(\Delta c).\end{aligned}$$

The map Δ is called the B-V operator.

We move on to the definition of the B-V operator defined on the Hochschild cohomology of the singular cochain algebra of a space by Menichi [29]. Let M be a simply-connected Poincaré duality space of formal dimension d. By definition, the space M is equipped with an *orientation class* $[M]\in H_m(M;\mathbb{K})$ such that the cap product

$$-\cap[M] : H^*(M;\mathbb{K})\to H_{m-*}(M;\mathbb{K})$$

is an isomorphism. The *fundamental class* of M is the element ω_M such that $\langle\omega_M, [M]\rangle = 1$, where $\langle\ ,\ \rangle$ denotes the Kronecker product.

Let A stand for the singular cochain algebra $C^*(M;\mathbb{K})$. Let $\mathbb{B}$ denote the normalized bar complex $\mathbb{B}(A;A;A)$. We define an isomorphism of complexes

$$\iota : \mathrm{Hom}(A \otimes_{A\otimes A^{op}} \mathbb{B}, \mathbb{K}) \xrightarrow{\cong} \mathrm{Hom}_{A\otimes A^{op}}(\mathbb{B}, A^\vee)$$

by $\iota(f)(\alpha)(a) = (-1)^{|a||\alpha|}f(a\otimes\alpha)$ for $\alpha \in \mathbb{B}$ and $a \in A$. Here the A-bimodule structure of $A^\vee$ is defined by $\langle f\cdot\alpha\cdot g; h\rangle = (-1)^{|f|}\langle \alpha; ghf\rangle$ for $f,g,h \in A$ and $\alpha \in A^\vee$. Then one obtains an isomorphism

$$\iota^* : \mathrm{Hom}(H(A \otimes_{A\otimes A^{op}} \mathbb{B}), \mathbb{K})) \underset{\cong}{\xleftarrow{\kappa}} H(\mathrm{Hom}(A \otimes_{A\otimes A^{op}} \mathbb{B}, \mathbb{K})) \underset{\cong}{\xrightarrow{H(\iota)}} HH^*(A;A^\vee),$$

where κ denotes the Künneth isomorphism. Observe that the source of the map ι^* is the dual $HH_*(A;A)^\vee$ to the Hochschild homology $HH_*(A;A)$ of A. We also recall the quasi-isomorphism $J : A \otimes_{A\otimes A^{op}} \mathbb{B} \to C^*(LM)$ of differential graded modules due to Jones [18]. We observe that J is defined by the composite

$$J : C^*(M)^{\otimes k+1} \xrightarrow{f_k^*} C^*(\Delta^k \times LM) \xrightarrow{\int_{\Delta^k}} C^{*-k}(LM),$$

where $f_k(0 \le t_1 \le \cdots \le t_k \le 1, \gamma) = (ev_1(\gamma), \gamma(t_1), ..., \gamma(t_k))$ and $\int_{\Delta^k}$ denotes the slant product; see also [31] in which a generalization of Jones' map J is described. Then it follows that this quasi-isomorphism fits into the commutative diagram

$$\begin{tikzcd}
C^*(LM) & & A \otimes_{A\otimes A^{op}} \mathbb{B} \arrow[ll, "J"', "\simeq"] \\
& C^*(M), \arrow[ul, "ev_1^*"] \arrow[ur, "\eta'"'] &
\end{tikzcd}$$

where η' is the chain map defined by $\eta'(a) = a\otimes 1$. Therefore we obtain a commutative diagram
(2.1)

$$\begin{tikzcd}
H^*(LM)^\vee \arrow[r, "H(J)^\vee", "\cong"'] \arrow[dr, "H(ev)^\vee"'] & HH_*(A;A)^\vee \arrow[d, "H(\eta')^\vee"] & H(\mathrm{Hom}(A \otimes_{A\otimes A^{op}} \mathbb{B}, \mathbb{K})) \arrow[l, "\kappa"', "\cong"] \arrow[r, "H(\iota)", "\cong"'] \arrow[d, "H(\eta^\vee)"] & HH^*(A;A^\vee) \arrow[d, "HH(\eta,1)"] \\
& H^*(M)^\vee & H(\mathrm{Hom}(A,\mathbb{K})) \arrow[l, "\kappa"', "\cong"] \arrow[r, "H(\iota)", "\cong"'] & HH^*(\mathbb{K};A^\vee),
\end{tikzcd}$$

where $\eta : \mathbb{K} \to A$ stands for the unit. It is readily seen that a section $s : M \to LM$ of the evaluation map ev_1 induces a section $H(s)^\vee$ of the map $H(ev_1)^\vee$. Let B be the Connes boundary map on $A\otimes T(s\bar{A}) \cong A \otimes_{A\otimes A^{op}} \mathbb{B}$; see [17]. By definition, we see that $B(a_0[a_1|a_2|...|a_k]) = \sum_{i=0}^k (-1)^{(\varepsilon_i+1)(\varepsilon_{k+1}-\varepsilon_i)}1[a_i|...|a_k|a_0|...|a_{i-1}]$. Here $\varepsilon_i = |a| + \sum_{j<i}(|sa_j|)$. We then have

Proposition 2.4. ([29, Propositions 11 and 12])(i) *Let $\omega_A^\vee \in H(A)^\vee$ be the dual base of the fundamental class of M. Define an element $[m] \in HH^{-d}(A, A^\vee)$ by $[m] = \iota^* H(J)^\vee H(s)^\vee(\omega_A^\vee)$. Then the product $- \smile [m]$ induces an isomorphism*

$$\theta : HH^p(A;A) \to HH^{p-d}(A;A^\vee).$$

(ii) *The Hochschild cohomology ring $HH^*(A;A)$ is a Batalin-Vilkovisky algebra equipped with the B-V operator Δ of degree -1 defined by the composite*

$$\begin{tikzcd} HH^p(A;A) \arrow[r, "\theta", "\cong"'] & HH^{p-d}(A;A^\vee) \arrow[r, "\iota^{*-1}"] & HH_{-p+d}(A;A)^\vee \arrow[d, "H(B)^\vee"] \\ HH^{p-1}(A;A) \arrow[r, "\cong", "\theta"'] & HH^{p-d-1}(A;A^\vee) & HH_{-p+d+1}(A;A)^\vee. \arrow[l, "\iota^*"] \end{tikzcd}$$

Our main theorem allows us to give the Moore spectral sequence a B-V algebra structure.

Theorem 2.5. ([22, Theorem 4.3]) *Let M be a simply-connected Poincaré duality space. Then the Moore spectral sequence $\{E_r^{*,*}, d_r\}$ converging to $HH(C^*(M);C^*(M))$ admits the structure of a differential Batalin-Vilkovisky bigraded algebra, in the sense that each term $E_r^{*,*}$ is endowed with the B-V operator $\Delta_r : E_r^{p,q} \to E_r^{p-1,q}$ such that $d_r\Delta_r + \Delta_r d_r = 0$, $H(\Delta_r) = \Delta_{r+1}$ and $E_\infty^{*,*}$ is isomorphic to $\mathrm{Gr}HH^*(C^*(M);C^*(M))$ as bigraded Batalin-Vilkovisky algebras.*

Sketch of Proof. Let $\widehat{\mathbf{C}}$ stand for the Hochschild cochains $\mathrm{Hom}_{A\otimes A^{op}}(\mathbb{B}, A^\vee)$. Let $m \in F^0\widehat{\mathbf{C}}^{-d}$ be a cocycle representing the element $[m] \in HH^{-d}(A, A^\vee)$ described in Proposition 2.4. Then it follows from [20, Lemma 2.1] that $\{m\}$ is a permanent cycle. The cup product

$$\smile : \mathrm{Hom}_{A\otimes A^{op}}(\mathbb{B}_*, A) \otimes \mathrm{Hom}_{A\otimes A^{op}}(\mathbb{B}_*, A^\vee) \to \mathrm{Hom}_{A\otimes A^{op}}(\mathbb{B}_*, A^\vee)$$

respects the filtrations; that is, $F^s\mathbf{C}^n \smile F^t\widehat{\mathbf{C}}^m \subset F^{s+t}\widehat{\mathbf{C}}^{n+m}$. Therefore the product with the element $\{m\} \in \widehat{E}_2^{0,-d} \cong \widehat{E}_r^{0,-d}$ induces a morphism

$$E(m)_r := - \smile \{m\} : E_r^{p,q} \to \widehat{E}_r^{p,q-d}$$

of spectral sequences. We can show that

$$E(m)_2 : HH^*(H^*(M);H^*(M)) \to HH^*(H^*(M);H^*(M)^\vee)$$

is nothing but the cup product with $\omega_M^\vee$, namely the map induced by the Poincaré duality isomorphism $H^*(M) \to H^*(M)^\vee$. Thus $E(m)_2$ is an isomorphism and hence so is $E(m)_r$ for $2 \le r \le \infty$. This yields the result. □

Remark 2.6. It is important to mention that if A is a symmetric algebra, then the Connes boundary map on $HH^*(A;A^\vee)$ defines a structure of Batalin-Vilkovisky algebra on the Gerstenharber algebra $HH^*(A;A)$; see [29, Theorem 18] and [36].

3. Computations

In this section by applying Theorem 2.5, we give computational examples of the Hochschild cohomology with the B-V algebra structure.

Proposition 3.1. (A particular version of [22, Proposition 3.2]). *Let M be a simply-connected Poincaré duality space. Let $\{{}_{EM}E_r^{*,*}, d_r\}$ be the Eilenberg-Moore spectral sequence mentioned in Section 1 and $\{E_r^{*,*}, d_r\}$ the Moore spectral sequence converging to $HH^*(C^*(M);C^*(M))$. Then $\{{}_{EM}E_r^{*,*}, d_r\}$ collapses at the E_2-term if and only if so does $\{E_r^{*,*}, d_r\}$.*

Theorem 3.2. ([22, Theorem 1.3])*Let M be a simply-connected space whose mod 2 cohomology is an exterior algebra, say $H^*(M;\mathbb{Z}/2) \cong \wedge(y_1, y_2, ..., y_l)$. Suppose further that the operation Sq^1 vanishes on the cohomology. Then as a bigraded Batalin-Vilkovisky algebra,*

$$\mathrm{Gr}HH^*(C^*(M;\mathbb{Z}/2);C^*(M;\mathbb{Z}/2)) \cong \wedge(y_1, y_2, ..., y_l) \otimes \mathbb{Z}/2[\nu_1^*, \nu_2^*, ..., \nu_l^*]$$

in which $\Delta(y_j) = 0$, $\Delta(\nu_i^*) = 0$, $\Delta(y_i y_j) = 0$, $\Delta(\nu_i^*\nu_j^*) = 0$ for $1 \leq i,j \leq l$ and $\Delta(y_i\nu_j^*) = \delta_{ij}\cdot 1$, where bideg $y_j = (0, \deg y_j)$ *and* bideg $\nu_j^* = (1, -\deg y_j)$ *for $1 \leq j \leq l$.*

Sketch of Proof. By assumption, the operation Sq^1 vanishes. Then the main theorem in [35] due to Smith yields that $\{{}_{EM}E_r^{*,*}, d_r\}$ converging to $H^*(LM)$ collapses at the E_2-term. Thus it follows from Proposition 3.1 that the Moore spectral sequence does collapse at E_2-term. Therefore we see that, as bigraded algebras,

$$\begin{aligned}\mathrm{Gr}HH^*(C^*(M;\mathbb{Z}/2);C^*(M;\mathbb{Z}/2)) \cong E_\infty^{*,*} \cong E_2^{*,*} &\cong HH^{*,*}(H^*(M);H^*(M)) \\ &\cong \wedge(y_1, ..., y_l) \otimes \mathbb{Z}/2[\nu_1^*, ..., \nu_l^*].\end{aligned}$$

The last isomorphism follows from the explicit calculation of the Hochschild cohomology of the exterior algebra; see for example [22, Proposition 2.4]. As mentioned in the proof of Theorem 2.5, the Poincaré duality gives rise to the isomorphism $E(m)_2 : HH^*(H^*(M);H^*(M)) \xrightarrow{\cong} HH^*(H^*(M);H^*(M)^\vee)$. By using the fact, we determine the B-V operator on the E_∞-term. □

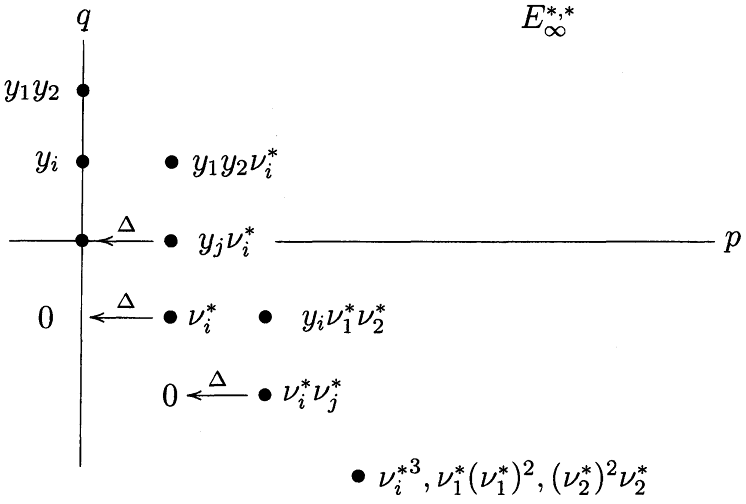
The following corollary illustrates that the Moore spectral sequence is reliable when calculating explicitly the Hochschild cohomology of the singular cochain on a space.

Corollary 3.3. ([22, Corollary 4.6]) *Let M be a simply-connected mod 2 Poincaré duality space whose mod 2 cohomology is isomorphic to an exterior algebra of the form $\wedge(y_1, y_2)$, where* $\deg y_1 = \deg y_2 = n$. *Suppose that $n > 4$. Then as a Batalin-Vilkovisky algebra*

$$HH^*(C^*(M;\mathbb{Z}/2);C^*(M;\mathbb{Z}/2)) \cong \wedge(y_1, y_2) \otimes \mathbb{Z}/2[\nu_1^*, \nu_2^*]$$

in which $\Delta(y_j) = 0$, $\Delta(y_j y_j) = 0$, $\Delta(\nu_i^*) = 0$, $\Delta(\nu_i^*\nu_j^*) = 0$ for $1 \leq i,j \leq l$ and $\Delta(y_i\nu_j^*) = \delta_{ij}\cdot 1$, where $\deg y_j = n$ *and* $\deg \nu_j^* = -n+1$ *for $j = 1$ and 2.*

Sketch of Proof. We can solve extension problems on the product and on the B-V operator. Since there exists no nonzero element in $E_\infty^{p,q}$ for $p \geq 1$ and $p + q = 2n$, it follows that $y_i^2 = 0$ for $i = 1$ and 2; see the figure displayed below.



For dimensional reasons, we see that there is no extension problem on the B-V operator. Then we have the result.

Remark 3.4. As mentioned in Section 1, these explicit computations may be of use in the study of levels and of thick subcategories of the triangulated category $\mathrm{D}(\text{Mod-}C^*(B;\mathbb{K}))$. However the author does not have any result concerning the levels of a maps by applying Hochschild cohomology. Behavior of the B-V algebra structure of the Hochshcild cohomology in representation theory is also obscure.

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